On the analytic form of the discrete Kramer sampling theorem

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Abstract

The classical Kramer sampling theorem is, in the subject of self–adjoint boundary value problems, one of the richest sources to obtain sampling expansions. Also it has become very fruitful in connection with discrete Sturm–Liouville problems. In this note a discrete version of the analytic Kramer sampling theorem is proved. Orthogonal polynomials arising from indeterminate Hamburger moment problems provide examples of Kramer analytic kernels. The same occurs for the second kind polynomials associated with them.

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1 Introduction

The classical Kramer sampling theorem provides a method for obtaining orthogonal sampling theorems [9, 11, 14]. The statement of this result is as follows: *Let $K(\omega,\lambda)$ be a function, defined for all λ in a suitable subset D of $\mathbb{R}$ such that, as a function of ω, $K(\cdot,\lambda)\in L^2(I)$ for every number $\lambda\in D$, where I is an interval of the real line. Assume that there exists a sequence of distinct real numbers $\{\lambda_n\}\subset D$, with n belonging to an indexing set $\mathbb{I}$ contained in $\mathbb{Z}$, such that $\{K(\omega,\lambda_n)\}$ is a complete orthogonal sequence of functions of $L^2(I)$. Then for any F of the form*

$$F(\lambda)=\int_I f(\omega)K(\omega,\lambda)\,d\omega\,, \tag{1}$$

where $f\in L^2(I)$, we have

$$F(\lambda)=\lim_{N\to\infty}\sum_{|n|\le N}F(\lambda_n)S_n(\lambda)\,, \tag{2}$$

with

$$S_n(\lambda) = \frac{\int_I K(\omega,\lambda)\overline{K(\omega,\lambda_n)}\,d\omega}{\int_I |K(\omega,\lambda_n)|^2\,d\omega}\,. \tag{3}$$

The series in (2) *converges absolutely and uniformly wherever* $\|K(\cdot,\lambda)\|_{L^2(I)}$ *is bounded.*

In [6] it is proved an extension of the Kramer sampling theorem to the case when the kernel is analytic in the sampling parameter λ. Assume that the Kramer kernel K is an entire function for any fixed $\omega \in I$, and that the function $h(\lambda) = \int_I |K(\omega,\lambda)|^2 d\omega$ is locally bounded on $\mathbb{C}$. Then any function F defined by (1) is an entire function and so are as well the sampling (interpolatory) functions (3). A kernel K satisfying the above additional conditions is called a *Kramer analytic kernel.*

A straightforward discrete version of Kramer's theorem can be obtained [2, 8]. To this end, let $K(n,\lambda)$ be a kernel such that, as a function of the discrete variable $n \in \mathbb{I} \subset \mathbb{Z}$, the kernel $K(\cdot,\lambda)$ is in $\ell^2(\mathbb{I})$ for any fixed $\lambda \in D \subset \mathbb{R}$. Assume that, for a suitable sequence $\{\lambda_n\}$ in D, $\{K(\cdot,\lambda_n)\}$ is an orthogonal basis for $\ell^2(\mathbb{I})$. Then any function of the form $F(\lambda) = \sum_{n\in\mathbb{I}} f(n)K(n,\lambda)$, where $f \in \ell^2(\mathbb{I})$, can be expanded by means of the sampling series

$$F(\lambda) = \sum_n F(\lambda_n)S_n(\lambda)\,,$$

where the sampling functions are given by

$$S_n(\lambda) = \frac{1}{\|K(\cdot,\lambda_n)\|^2}\sum_{m\in\mathbb{I}} \overline{K(m,\lambda_n)}K(m,\lambda)\,.$$

The main aim in this note is to prove the analytic version of the Kramer sampling theorem for the discrete case. The additional conditions required are in close parallel with the ones assumed in [6] for the continuous case. In the last section, we propose discrete Kramer analytic kernels arising either from orthogonal polynomials associated with indeterminate Hamburger moment problems, or from the second kind orthogonal polynomials associated with the former ones.

2 The discrete analytic Kramer sampling theorem

In this section we state the discrete version of the analytic Kramer sampling theorem. We use the index set $\mathbb{Z}$ both for the discrete variable and for the sampling sequence $\{\lambda_n\}$. When the index set is $\mathbb{N}$ or $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the results are essentially the same after some minor changes in the hypotheses.

Theorem 1 *Let $K : \mathbb{Z} \times \mathbb{C} \to \mathbb{C}$ be a mapping satisfying the following properties and conditions:*

1. $K(\cdot, \lambda) \in \ell^2(\mathbb{Z})$ *for each* $\lambda \in \mathbb{C}$
2. $K(n, \cdot)$ *is an entire function for each* $n \in \mathbb{Z}$
3. *There exists a sequence of real numbers* $\{\lambda_m\}_{m\in\mathbb{Z}}$ *satisfying*
   (a) $\lambda_m < \lambda_{m+1}$ *for all* $m \in \mathbb{Z}$
   (b) $\lim_{m\to\pm\infty} \lambda_m = \pm\infty$
   (c) *The sequence* $\{K(\cdot, \lambda_m)\}_{m\in\mathbb{Z}}$ *is an orthogonal basis in* $\ell^2(\mathbb{Z})$
4. *The function* $h(\lambda) = \|K(\cdot, \lambda)\|^2_{\ell^2(\mathbb{Z})}$ *is locally bounded on* $\mathbb{C}$

Let $\mathcal{H}$ *be the set of functions* $F : \mathbb{C} \to \mathbb{C}$ *determined by*

$$F(\lambda) = \sum_{n=-\infty}^{\infty} K(n, \lambda) f(n) , \quad \lambda \in \mathbb{C} , \tag{4}$$

where $f \in \ell^2(\mathbb{Z})$. *Then for all* $F \in \mathcal{H}$ *the following results hold:*

i. F *is an entire function*

ii. *The functions* $S_m : \mathbb{C} \to \mathbb{C}$ *defined by*

$$S_m(\lambda) = \frac{1}{\|K(\cdot, \lambda_m)\|^2_{\ell^2(\mathbb{Z})}} \langle K(\cdot, \lambda), K(\cdot, \lambda_m)\rangle_{\ell^2(\mathbb{Z})} , \tag{5}$$

are entire functions

iii. *Every* $F \in \mathcal{H}$ *admits the sampling expansion*

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n) S_n(\lambda) , \tag{6}$$

where the convergence of the series is absolute and uniform on compact subsets of $\mathbb{C}$.

Proof. First we prove that any function defined by (4) is an entire function. Consider the function

$$F_m(\lambda) = \sum_{n=-\infty}^{\infty} K(n, \lambda) f_m(n) ,$$

where $f_m(n) = f(n)$ if $|n| \leq m$ and $f_m(n) = 0$ if $|n| > m$. Clearly, F_m is an entire function for each m. Moreover, by using the Cauchy–Schwarz inequality we obtain

$$|F(\lambda)-F_m(\lambda)| \leq \sum_{n=-\infty}^{\infty} |K(n,\lambda)||f(n)-f_m(n)| \leq \|K(\cdot,\lambda)\|_{\ell^2(\mathbb{Z})}\|f-f_m\|_{\ell^2(\mathbb{Z})} .$$

Hence, given an arbitrary compact subset Ω of $\mathbb{C}$ we get

$$|F(\lambda) - F_m(\lambda)| \leq C\|f - f_m\|_{\ell^2(\mathbb{Z})} ,$$

for every $\lambda \in \Omega$ and every $m \in \mathbb{N}$. Then, F_m converges to F uniformly on compact sets and consequently, F is an entire function. The proof that $S_n(\lambda)$ is an entire function for each n goes much in the same manner.

Now, expanding $K(\cdot,\lambda)$ with respect to the orthonormal basis

$$\left\{\phi_k(n) = \frac{K(n,\lambda_k)}{\|K(\cdot,\lambda_k)\|_{\ell^2(\mathbb{Z})}}\right\}_{k\in\mathbb{Z}} ,$$

we have $K(n,\lambda) = \sum_{k=-\infty}^{\infty} a_k(\lambda)\phi_k(n)$, where

$$a_k(\lambda) = \langle K(\cdot,\lambda), \phi_k\rangle_{\ell^2(\mathbb{Z})} = \|K(\cdot,\lambda_k)\|_{\ell^2(\mathbb{Z})}S_k(\lambda) , \quad k \in \mathbb{Z} . \tag{7}$$

Therefore, for each $\lambda \in \mathbb{C}$, the series $\sum_{k=-\infty}^{\infty} S_k(\lambda)K(n,\lambda_k)$ converges to $K(n,\lambda)$ in $\ell^2(\mathbb{Z})$. Now we prove the pointwise convergence in (6). Indeed, by using the Cauchy–Schwarz inequality

$$\begin{aligned}\left|F(\lambda) - \sum_{m=-M}^{M} F(\lambda_m)S_m(\lambda)\right| &\leq \sum_{n=-\infty}^{\infty} \left|K(n,\lambda) - \sum_{m=-M}^{M} K(n,\lambda_m)S_m(\lambda)\right|\big|f(n)\big| \\ &\leq \left\|K(n,\lambda) - \sum_{m=-M}^{M} K(n,\lambda_m)S_m(\lambda)\right\|_{\ell^2(\mathbb{Z})}\big\|f\big\|_{\ell^2(\mathbb{Z})} .\end{aligned}$$

Hence, for each $\lambda \in \mathbb{C}$, we have $F(\lambda) = \sum_{m=-\infty}^{\infty} F(\lambda_m)S_m(\lambda)$.

By expanding f with respect to the orthonormal basis $\{\phi_n\}$ we obtain, from (6), that $\{F(\lambda_n)/\|K(\cdot,\lambda_n)\|_{\ell^2(\mathbb{Z})}\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$. In the same way, from (7), $\{\|K(\cdot,\lambda_n)\|_{\ell^2(\mathbb{Z})}S_n(\lambda)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$. As a consequence, by using the Cauchy–Schwarz inequality in

$$\sum_{n=-\infty}^{\infty} \big|F(\lambda_n)S_n(\lambda)\big| = \sum_{n=-\infty}^{\infty} \frac{|F(\lambda_n)|}{\|K(\cdot,\lambda_n)\|_{\ell^2(\mathbb{Z})}} \big(\|K(\cdot,\lambda_n)\|_{\ell^2(\mathbb{Z})}\big|S_n(\lambda)\big|\big) , \tag{8}$$

we obtain that the series in (6) converges absolutely for each $\lambda \in \mathbb{C}$. Finally, by proceeding as in (8), we have

$$\left|F(\lambda) - \sum_{|m|\leq M} F(\lambda_m)S_m(\lambda)\right| = \left|\sum_{|m|>M} F(\lambda_m)S_m(\lambda)\right|$$
$$\leq \left\{\sum_{|m|>M} \left|\frac{F(\lambda_m)}{\|K(\cdot,\lambda_m)\|_{\ell^2(\mathbb{Z})}}\right|^2\right\}^{1/2} \|K(\cdot,\lambda)\|_{\ell^2(\mathbb{Z})}\,.$$

Since $\|K(\cdot,\lambda)\|_{\ell^2(\mathbb{Z})}$ is locally bounded on $\mathbb{C}$, we obtain the uniform convergence in (6) on compact subsets of $\mathbb{C}$. ■

As a straightforward consequence of the above theorem, we derive the following interpolation result:

Corollary 1 *Assuming all conditions in theorem 1, let $\{c_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers such that*

$$\sum_{n=-\infty}^{\infty} \left|\frac{c_n}{\|K(\cdot,\lambda_n)\|_{\ell^2(\mathbb{Z})}}\right|^2 < \infty\,. \tag{9}$$

Then there exists a unique $f \in \ell^2(\mathbb{Z})$ such that the corresponding $F(\lambda)$ given by (4) satisfies

$$F(\lambda_n) = c_n\,,\ \textit{for each } n \in \mathbb{Z}\,.$$

Proof. Since $\{\phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$, by using (9) and the Riesz–Fischer theorem, there exists a unique sequence h in $\ell^2(\mathbb{Z})$ such that

$$\bar{c}_n = \langle h, K(\cdot,\lambda_n)\rangle_{\ell^2(\mathbb{Z})} = \sum_{k=-\infty}^{\infty} h(k)\overline{K(k,\lambda_n)}\,.$$

Therefore, $c_n = \overline{\langle h, K(\cdot,\lambda_n)\rangle}_{\ell^2(\mathbb{Z})} = \langle \bar{h}, \overline{K(\cdot,\lambda_n)}\rangle_{\ell^2(\mathbb{Z})}$. By defining $f = \bar{h}$, we have the desired result for the function $F(\lambda) = \sum_{k\in\mathbb{Z}} K(k,\lambda)f(k)$. ■

3 Orthogonal polynomials as discrete analytic Kramer kernels

We consider an indeterminate Hamburger moment sequence $s = \{s_n\}_{n=0}^{\infty}$ and we denote by V_s the set of positive Borel measures satisfying it, i.e.,

$$V_s = \left\{\mu \geqslant 0\ :\ s_n = \int_{-\infty}^{\infty} x^n d\mu(x)\,,\ n \geqslant 0\right\}\,.$$

Let us consider the associated polynomials of first and second kind, $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$. The sequence $\{P_n\}_{n=0}^{\infty}$ forms an orthonormal system with respect to the inner product given by $\langle x^n, x^m \rangle = \int_{-\infty}^{\infty} x^{n+m} d\mu(x)$, where $\mu \in V_s$. The P_n's are uniquely determined by the additional condition that their leading coefficients are positive [1]. The sequence $\{Q_n\}_{n=0}^{\infty}$ is given by

$$Q_n(x) = \int_{-\infty}^{\infty} \frac{P_n(x) - P_n(t)}{x - t} d\mu(t) \,,$$

where μ is any measure in V_s.

In the case of an indeterminate moment problem, the series

$$\sum_{n=0}^{\infty} |P_n(z)|^2 \quad \text{and} \quad \sum_{n=0}^{\infty} |Q_n(z)|^2 \tag{10}$$

converge uniformly on compact subsets of the complex plane.

Moreover, in [3, 7, 8] it is proved the existence of sequences $\{\lambda_m\}$ of distinct real numbers such that

$$\{P_0(\lambda_m),\, P_1(\lambda_m),\, P_2(\lambda_m),\, P_3(\lambda_m), \cdots\cdots\}_{m=0}^{\infty}$$

is an orthogonal basis for $\ell^2(\mathbb{N}_0)$. The same occurs for the sequence of polynomials of second kind $\{Q_n\}$ by considering the so–called shifted moment problem [8, 12]. As a consequence, we can state the following result:

Let $\{P_n\}$ and $\{Q_n\}$ be the sequences of first and second kind polynomials associated with an indeterminate Hamburger moment problem. Then by defining either $K(n, \lambda) = P_n(\lambda)$ or $K(n, \lambda) = Q_n(\lambda)$ we obtain discrete Kramer analytic kernels.

The existence of the sequences $\{\lambda_m\}$ can be explained as follows: The set V_s of solutions to an indeterminate Hamburger moment problem can be parametrized with the one–point compactification $\mathcal{P} \cup \{\infty\}$ of the Pick (or Herglotz) functions set $\mathcal{P}$ [1, 13]. When the parameter is restricted to constant functions taking values in $\mathbb{R} \cup \{\infty\}$, we obtain the set of N–extremal measures $\{\mu_t\}$ which satisfy

$$\int_{-\infty}^{\infty} \frac{d\mu_t(x)}{x - z} = -\frac{A(z)t - C(z)}{B(z)t - D(z)} \,, \quad z \in \mathbb{C} \setminus \mathbb{R} \,, \tag{11}$$

where A, B, C, D are the entire functions forming the so–called Nevanlinna matrix associated with the moment problem. See [4, 13] for explicit formulas of the Nevanlinna matrix. It is known that, for each $t \in \mathbb{R} \cup \{\infty\}$, μ_t is the discrete measure $\mu_t = \sum_{z \in Z_t} m_z \delta_z$ where

$$Z_t = \begin{cases} \{z \in \mathbb{C} \mid B(z)t - D(z) = 0\} & \text{if } t \in \mathbb{R}\,, \\ \{z \in \mathbb{C} \mid B(z) = 0\} & \text{if } t = \infty\,, \end{cases}$$

and

$$m_z = \frac{A(z)t - C(z)}{B'(z)t - D'(z)}\,, \quad \text{for } z \in Z_t\,.$$

Recall that the zeros of the entire function $B(z)t - D(z)$ or $B(z)$ are real and simple, and they form, precisely, one of sequences $\{\lambda_m^t\}$ we are looking for. The N–extremal measures are characterized as those measures $\mu \in V_s$ for which the polynomials are dense in $L^2(\mu)$. In the above cases we can identify the set $\mathcal{H}$ determined by (4) with the space $L^2(\mu_t)$.

Another equivalent formulation can be given either in terms of the self–adjoint extensions of a semi–infinite Jacobi matrix, or in terms of the discrete Weyl's limit point/limit circle theory for the self–adjoint extensions of a discrete Sturm–Liouville boundary value problem. Given the sequence $\{s_n\}_{n=0}^{\infty}$ of Hamburger moments, we can find two sequences $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ of real and positive numbers respectively, namely $b_n = \int x P_n^2(x) d\mu(x)$ and $a_n = \int x P_n(x) P_{n+1}(x) d\mu(x)$, so that the moment problem is associated to self–adjoint extensions of the semi-infinite Jacobi matrix

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \ddots \\ 0 & 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

with its domain $D(\mathcal{A})$ being the set of sequences of finite support. The uniqueness of the Hamburger problem depends on the existence of a unique self-adjoint extension of the operator defined by $\mathcal{A}$ [13, p.86]. Associated with the Jacobi matrix $\mathcal{A}$ is the three–term recurrence relation

$$\lambda X_n = a_n X_{n+1} + b_n X_n + a_{n-1} X_{n-1}\,, \quad n \geqslant 0 \tag{12}$$

By taking initial conditions $X_{-1} = 0$ and $X_0 = 1$ we obtain the sequence $\{P_n(\lambda)\}_{n=0}^{\infty}$, and for $X_{-1} = -1$ and $X_0 = 0$ we obtain the second kind associated polynomials $\{Q_n(\lambda)\}_{n=0}^{\infty}$. Following B. Simon [13, p.94], the indeterminacy of the Hamburger moment problem can be thought as the discrete

Weyl's limit point/limit circle theory for the self-adjoint extensions of the discrete Sturm-Liouville problem associated with the former three–term recurrence relation. When the moment problem is indeterminate we are in the limit–circle at infinity, and the Weyl–Titchmarsh functions $m^t_\infty(z)$ are given precisely by (11). Its poles, i.e., the zeros of the denominator in (11) are the eigenvalues of the self–adjoint extension, and $\{P_0(\lambda^t_m)\,,P_1(\lambda^t_m)\,,\cdots\}_{m=0}^\infty$ the corresponding sequence of complete orthogonal eigenvectors in $\ell^2(\mathbb{N}_0)$.

Furthermore, in [5, 8] it is proved that the sampling (interpolatory) functions in this particular case are nothing more than analytic interpolation functions, i.e., they are of the form

$$S_m(\lambda) = \frac{G(\lambda)}{G'(\lambda_m)(\lambda-\lambda_m)}\,,$$

where G is an entire function having its only simple zeros at the sequence $\{\lambda_m\}$. The corresponding sampling expansion can be written as a Lagrange–type interpolation series

$$F(\lambda) = \sum_{m=0}^{\infty} F(\lambda_m)\frac{G(\lambda)}{G'(\lambda_m)(\lambda-\lambda_m)}\,,$$

where the convergence of the series is absolute and uniform on compact subsets of the complex plane.

Finally, in [10] one can find an example, the q-Hermite polynomials, where all the necessary ingredients for the analytic discrete Kramer sampling theorem are explicity computed.

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