Applicable Analysis
Vol. 82, No. 12, December 2003, pp. 1145–1158

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Sampling in the Functional Hilbert Space Induced by a Hilbert Space Valued Kernel

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Communicated by R.P. Gilbert

(Received 10 June 2002; In final form 20 October 2002)

This work focuses on the study of sampling formulas for the range space $\mathcal{H}$ of a linear transform defined on a Hilbert space by means of a suitable kernel. The sampling property consists of the reconstruction of any function in $\mathcal{H}$ through its values on an appropriate sequence of points by means of a sampling expansion involving these values. In our case, the sampling property is derived by assuming some requirements on the kernel of the linear transform. A converse result shows the generality of the required conditions in the Riesz bases setting. Finally, we deal with sampling formulas in the case where samples of a related function, the derivative for instance, are allowed in order to recover the initial function.

Keywords: Reproducing kernel Hilbert spaces; Sampling formulas; Biorthonormal Riesz bases

AMS Subject Classifications: 94A20; 44A05; 46E22

1. STATEMENT OF THE PROBLEM

Let $\mathbb{H}$ be a separable Hilbert space, and Ω a fixed subset of $\mathbb{R}$. Given a $\mathbb{H}$-valued function $K:\Omega\to\mathbb{H}$ for $x\in\mathbb{H}$ the function $f(t):=\langle x,K(t)\rangle_{\mathbb{H}}$ is well-defined as a function $f:\Omega\to\mathbb{C}$. We denote by $\mathcal{H}$ the set of functions obtained in this way and by T the linear transform

$$T:\mathbb{H}\ni x\mapsto f\in\mathcal{H} \tag{1}$$

Hereafter we refer the function K as the *kernel* of the transform T. Note that the continuity of K implies that the functions in $\mathcal{H}$ are continuous in Ω. If we define in $\mathcal{H}$ the norm $\|f\|_{\mathcal{H}}=\inf\{\|x\|_{\mathbb{H}}: f=T(x)\}$ we obtain a reproducing kernel Hilbert space (RKHS hereafter) whose reproducing kernel is given by, cf. [11, p. 21],

$$k(t,s):=\langle K(s),K(t)\rangle_{\mathbb{H}} \tag{2}$$

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ISSN 0003-6811 print: ISSN 1563-504X online © 2003 Taylor & Francis Ltd
DOI: 10.1080/0003681031000063748

i.e., for each $s \in \Omega$ the function k_s defined as $k_s(t) := k(t, s)$ belongs to $\mathcal{H}$, and the reproducing property

$$f(s) = \langle f, k_s \rangle_{\mathcal{H}} = \langle f, k(\cdot, s) \rangle_{\mathcal{H}}, \quad s \in \Omega, \ f \in \mathcal{H} \tag{3}$$

holds. Recall that the Moore–Aronszajn procedure [1] leads to the same RKHS via the *positive definite* (*or positive matrix*) function k. Under these circumstances it is known that the linear operator T is one-to-one if and only if T is an isometry between $\mathbb{H}$ and $\mathcal{H}$, or, equivalently, if and only if the set of functions $\{K(t)\}_{t \in \Omega}$ is complete in $\mathbb{H}$ [11]. The RKHS $\mathcal{H}$ has been largely studied in the mathematical literature (see the superb monograph [11] and references therein).

In this article we are interested in the *sampling property* for the RKHS $\mathcal{H}$, i.e., the existence of a sequence of points $\{t_n\}_{n=1}^{\infty}$ in Ω and a sequence of functions $\{S_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ with the interpolatory property $S_n(t_m) = \delta_{n,m}$, where $\delta_{n,m}$ stands for the Kronecker delta, such that the sampling series

$$f(t) = \sum_{n=1}^{\infty} f(t_n) S_n(t) \tag{4}$$

is valid for every $f \in \mathcal{H}$, where the convergence of the series is at least absolute and uniform on compact subsets of Ω. The sampling series (4) might also contain samples from a function related with f, e.g., its derivative.

Perhaps the most important examples of RKHS $\mathcal{H}$ with the sampling property are the classical Paley-Wiener spaces of bandlimited functions, i.e., square integrable functions in $\mathbb{R}$ such that their Fourier transforms are zero outside a bounded set in $\mathbb{R}$. For instance, any function of the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x) e^{itx}\, dx, \quad F \in L^2[-\pi, \pi],$$

can be expanded as the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc}(t-n),$$

where sinc stands for cardinal sine function (or sinc function) defined as $\operatorname{sinc} t = \sin \pi t / \pi t$ for $t \neq 0$ and $\operatorname{sinc} 0 = 1$. The eminent British mathematician Hardy was the first who noticed that the above cardinal series was an orthogonal expansion in an appropriate Hilbert space. Thus, to obtain a sampling expansion (4) for $\mathcal{H}$ it is enough to search for a sequence $\{t_n\}_{n=1}^{\infty}$ in Ω such that the sequence $\{k(\cdot, t_n)\}_{n=1}^{\infty}$ forms an orthogonal basis in $\mathcal{H}$. Expanding any function $f \in \mathcal{H}$ in the orthonormal basis $\{k(\cdot, t_n)/\sqrt{k(t_n, t_n)}\}_{n=1}^{\infty}$, one easily obtains (see [9,11]) the sampling expansion

$$f(t) = \sum_{n=1}^{\infty} f(t_n) \frac{k(t, t_n)}{k(t_n, t_n)}$$

Pointwise convergence properties in the above series emerge from the RKHS setting as we will see below. Orthogonal sampling formulas for a RKHS $\mathcal{H}$ as stated here have been studied in [3] where $\mathbb{H}$ is a L^2-space, or in [4] where $\mathbb{H}$ is a ℓ^2-space.

In the present work we deal with sampling expansions (4) in the range space $\mathcal{H}$ of a linear transform (1) which are nonorthogonal expansions following Riesz bases in $\mathcal{H}$. To this end, we assume that, for any fixed $t \in \Omega$, the kernel of (1) evaluated at t, $K(t)$ has a particular expansion in a suitable Riesz basis of $\mathbb{H}$. Recall that a Riesz basis $\{x_n\}_{n=1}^{\infty}$ for a separable Hilbert space $\mathbb{H}$ is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n=1}^{\infty}$ has a unique biorthonormal (dual) Riesz basis $\{x_n^*\}_{n=1}^{\infty}$, i.e., $\langle x_n, x_m^* \rangle_{\mathbb{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle_{\mathbb{H}} x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle_{\mathbb{H}} x_n^*$$

hold for every $x \in \mathbb{H}$ (see [13] for more details and proofs). In [7] one can find a sampling principle associated with the linear transforms T by using more general bases or frames. We confine ourselves to the case of Riesz bases because these bases are the only bases which ensure stable sampling (see [6, p. 103]). Roughly speaking, stable sampling means that errors in the output sampling are bounded by errors in the input, i.e., the samples.

The approach proposed here has, from our viewpoint, two major advantages. The first one (also pointed out in [7]) is to exhibit the intimate relationship between the Hilbert spaces $\mathcal{H}$ and $\mathbb{H}$; T will be a unitary operator between both spaces. The second one is to show the generality of setting proposed here by proving a converse result in Section 3. The sampling series obtained here may also contain samples from a transformed version of f, as the derivative, for instance. This topic will be the goal of Section 4. Classical sampling results, taken as particular examples, illustrate the method followed in the article.

2. NONORTHOGONAL SAMPLING FORMULAS

Let $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ be a pair of biorthonormal Riesz bases for a Hilbert space $\mathbb{H}$. Assume that, for each fixed $t \in \Omega$, $K(t)$ can be written as

$$K(t) = \sum_{n=1}^{\infty} \overline{S_n(t)} x_n^* \tag{5}$$

where the functions $S_n : \Omega \to \mathbb{C}$ satisfy, for some fixed sequence $\{t_n\}_{n=1}^{\infty}$ in Ω, the interpolatory condition:

$$S_n(t_m) = a_n \delta_{n,m}, \qquad \{a_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}. \tag{6}$$

Note that, for each $t \in \Omega$, the sequence $\{S_n(t)\}_{n=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$. The sequence $\{K(t_n)\}_{n=1}^{\infty} = \{\overline{a_n} x_n^*\}_{n=1}^{\infty}$ is a complete sequence in $\mathbb{H}$. Therefore, the linear transform (1) is a bijective isometry (unitary operator) between the Hilbert spaces $\mathbb{H}$ and $\mathcal{H}$. As a consequence, we obtain the following sampling theorem for functions in $\mathcal{H}$.

THEOREM 1 *Assume that, for each fixed $t \in \Omega$, the kernel K of the linear transform* (1) *has an expansion like* (5), *where the coefficients satisfy the interpolatory condition* (6). *Then, the sequence $\{S_n\}_{n=1}^{\infty}$ forms a Riesz basis for the RKHS $\mathcal{H}$. Expanding any $f \in \mathbb{H}$ in this Riesz basis we obtain the nonorthogonal sampling expansion*

$$f(t) = \sum_{n=1}^{\infty} f(t_n) \frac{S_n(t)}{a_n} \tag{7}$$

The series converges in the $\mathcal{H}$-norm sense and also, absolutely and uniformly on subsets of Ω where $k(t,t) = \|K(t)\|_{\mathbb{H}}^2$ is bounded.

Proof Since $T(x_m) = S_m$ and T is a bijective isometry we obtain that $\{S_m\}_{m=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ whose unique biorthonormal basis $\{S_m^*\}_{m=1}^{\infty}$ is given by $S_m^* = T(x_m^*)$. Expanding any $f \in \mathcal{H}$ in this Riesz basis, we have $f = \sum_{n=1}^{\infty} \langle f, S_n^* \rangle_{\mathcal{H}} S_n$, in the $\mathcal{H}$-norm sense and, consequently, pointwise in Ω since $\mathcal{H}$ is a RKHS [11]. Moreover, having in mind that T is an isometry, and that $K(t_n) = \overline{a_n} x_n^*$ we obtain

$$\langle f, S_n^* \rangle_{\mathcal{H}} = \langle T(x), T(x_n^*) \rangle_{\mathcal{H}} = \langle x, x_n^* \rangle_{\mathbb{H}} = f(t_n)/a_n$$

and hence the sampling expansion (7). Since a Riesz basis is an unconditional basis (any orthonormal basis is an unconditional basis by the Parseval equality), the sampling series (7) is pointwise unconditionally convergent for each $t \in \Omega$ and hence pointwise absolutely convergent. The uniform convergence, on subsets of Ω where $k(t,t)$ is bounded, follows from the inequality $|f(t)| \le \|f\|_{\mathcal{H}} \sqrt{k(t,t)}$, obtained by using Cauchy–Schwarz inequality in the reproducing formula (3). ■

In the particular case when $\{x_n^*\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathbb{H}$, it is self-dual and we have the following result:

COROLLARY 1 *Whenever the sequence $\{x_n^*\}_{n=1}^{\infty}$ in* (5) *is an orthonormal basis for $\mathbb{H}$, the sequence $\{S_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and the sampling expansion* (7) *is an orthonormal expansion in $\mathcal{H}$ having the same pointwise covergence properties.*

2.1 Classical WSK and PWL Theorems

Now we illustrate the sampling theory explained above with two important sampling examples for the classical Paley–Wiener space $PW_{\pi\sigma}$ of bandlimited functions to the interval $[-\pi\sigma, \pi\sigma]$, i.e.,

$$PW_{\pi\sigma} := \{f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subseteq [-\pi\sigma, \pi\sigma]\}$$

where $\hat{f}$ stands for the Fourier transform. This space can also be expressed, by using the classical Paley–Wiener theorem [13, p. 100], as

$$PW_{\pi\sigma} = \{f \in \mathcal{H}(\mathbb{C}) : |f(z)| \le A e^{\pi\sigma|z|}, \quad f|_{\mathbb{R}} \in L^2(\mathbb{R})\}$$

i.e., entire functions of exponential type at most $\pi\sigma$ such that their restriction to the real axis are square integrable. Note that any function f in $PW_{\pi\sigma}$ can be written as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\sigma}^{\pi\sigma} F(x) e^{itx}\, dx = \left\langle F, \frac{e^{-itx}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi\sigma, \pi\sigma]}$$

with $F \in L^2[-\pi\sigma, \pi\sigma]$. Thus, in this case, the kernel K (the Fourier kernel) is given by

$$K : \mathbb{R} \to L^2[-\pi\sigma, \pi\sigma]$$
$$t \to K(t) = \frac{e^{-itx}}{\sqrt{2\pi}}.$$

Consider the orthonormal basis $\{e^{-inx/\sigma}/\sqrt{2\pi\sigma}\}_{n\in\mathbb{Z}}$ in $L^2[-\pi\sigma, \pi\sigma]$. Expanding $K(t) \in L^2[-\pi\sigma, \pi\sigma]$ in this basis, we obtain

$$K(t) = \frac{1}{2\pi\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \left\langle e^{-itx}, e^{-inx/\sigma} \right\rangle_{L^2[-\pi\sigma,\pi\sigma]} \frac{e^{-inx/\sigma}}{\sqrt{2\pi\sigma}} = \sqrt{\sigma} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)} \frac{e^{-inx/\sigma}}{\sqrt{2\pi\sigma}}.$$

Therefore, taking $S_n(t) = \sqrt{\sigma}(\sin \pi(\sigma t - n)/\pi(\sigma t - n))$, $t_n = n/\sigma$, $n \in \mathbb{Z}$ and $a_n = \sqrt{\sigma}$ we obtain the Whittaker–Shannon–Kotel'nikov sampling theorem which reads:

Corollary 2 (WSK Theorem) *Any function f in the Paley-Wiener space $PW_{\pi\sigma}$ can be expanded as the cardinal series*

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\sigma}\right) \frac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)} \tag{8}$$

where the convergence in the series is absolute and uniform on $\mathbb{R}$ since the reproducing kernel $k_{\pi\sigma}(t,s) = \sigma$ sinc $\sigma(t-s)$ satisfies $k_{\pi\sigma}(t,t) = \sigma$ for all $t \in \mathbb{R}$.

On the other hand, consider a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of real numbers satisfying Kadec's condition, i.e., $D := \sup_{n\in\mathbb{Z}} |t_n - n| < 1/4$. As a consequence of Kadec's 1/4-theorem [13, p. 42], $\{e^{-it_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi, \pi]$. It can be proved [2,10] that, for any fixed $t \in \mathbb{R}$, we can expand $K(t)$, the Fourier kernel evaluated at $t \in \mathbb{R}$, in $L^2[-\pi, \pi]$ as

$$K(t) = \sum_{n=-\infty}^{\infty} \frac{G(t)}{(t - t_n)G'(t_n)} \frac{e^{-it_n x}}{\sqrt{2\pi}}$$

where G stands for the infinite product of the sequence $\{t_n\}_{n\in\mathbb{Z}}$

$$G(t) = (t - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{t}{t_n}\right)\left(1 - \frac{t}{t_{-n}}\right)$$

Taking $S_n(t) = G(t)/((t - t_n)G'(t_n))$ and the sampling points $\{t_n\}_{n\in\mathbb{Z}}$, the expansion (7) is nothing more than the Paley–Wiener–Levinson irregular sampling theorem in PW_π (for notational ease we take $\sigma = 1$) which reads:

Corollary 3 (PWL Theorem) *Any $f \in PW_\pi$ can be recovered from its sample values $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange type interpolation series*

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{G'(t_n)(t - t_n)}$$

The series converges absolutely and uniformly on $\mathbb{R}$.

3. A CONVERSE RESULT

The method proposed in Section 2 is limited to those linear transforms T whose kernel K, evaluated at each $t \in \Omega$, can be written as (5) and where the interpolatory condition (6) is fulfilled. However, we prove that, under plausible hypotheses, the kernel K adopts the required form. Namely, consider a linear transform T as in (1), and let $\mathcal{H}$ be its range space, a RKHS whose reproducing kernal is given by (2). Assume also that T is one-to one, although as we will point out at the end of the section this assumption can be dropped so as to get a similar result. Under these circumstances we prove the following converse result which was originally proved in [5] as a converse result of the Kramer sampling theorem (see [6,14]).

THEOREM 2 *Let $\mathcal{H}$ be the range of a linear transform T as in* (1) *considered as a RKHS with reproducing kernel* (2). *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ such that $\{S_n(t)\}_{n=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$ for each $t \in \Omega$. Suppose that the following conditions are fulfilled*:

(i) $\sum_{n=1}^{\infty} \sigma_n S_n(t) = 0$ *for all $t \in \Omega$ and $\{\sigma_n\}_{n=1}^{\infty}$ $\ell^2(\mathbb{N})$ implies $\sigma_n = 0$ for all n.*
(ii) *There exist sequences $\{t_n\}_{n=1}^{\infty}$ in Ω and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$ such that*

$$\left\{\frac{f(t_n)}{a_n}\right\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \quad \textit{and} \quad f(t) = \sum_{n=1}^{\infty} f(t_n)\frac{S_n(t)}{a_n}, \qquad \textit{for any } f \in \mathcal{H},$$

where the sampling series is pointwise convergent in Ω.

Then, $\{S_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ and the kernel K of the linear transform T evaluated at $t \in \Omega$ can be expressed as $K(t) = \sum_{n=1}^{\infty} \overline{S_n(t)} x_n^*$, where $\{x_n^*\}_{n=1}^{\infty}$ is the dual basis of the Riesz basis $\{x_n = T^{-1}(S_n)\}_{n=1}^{\infty}$ in $\mathbb{H}$.

Proof By defining $\tilde{k}(t,s) := \sum_{n=1}^{\infty} S_n(t)\overline{S_n(s)}$, we obtain a positive definite function which defines a RKHS $\tilde{\mathcal{H}}$, such that $\tilde{\mathcal{H}} \subseteq \mathcal{H}$. Condition (i) implies that the sequence $\{S_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\tilde{\mathcal{H}}$ (see [12]).

Now we prove that $\tilde{\mathcal{H}} = \mathcal{H}$ and that the identity mapping $\tilde{\mathcal{H}} \hookrightarrow \mathcal{H}$ is continuous. Take $f \in \mathcal{H}$, by Condition (ii), the sequence $\{f(t_n)a_n^{-1}\}_{n=1}^{\infty}$ is in $\ell^2(\mathbb{N})$. As a consequence, the series $\sum_{n=1}^{\infty} f(t_n)a_n^{-1}S_n$ converges in the norm of $\tilde{\mathcal{H}}$. By the reproducing kernel property, we have that the series $\sum_{n=1}^{\infty} f(t_n)a_n^{-1}S_n$ is pointwise convergent. Comparing this with what we get from the sampling for f we deduce that $f = \sum_{n=1}^{\infty} f(t_n)a_n^{-1}S_n$, where the convergence is in $\tilde{\mathcal{H}}$ and, consequently, $f \in \tilde{\mathcal{H}}$.

Next we show the continuity of the identity mapping by application of the closed graph theorem. Indeed, let $\{f_n\}_{n=1}^{\infty}$ be a sequence such that $f_n \to f$ in $\tilde{\mathcal{H}}$ and $f_n \to g$ in $\mathcal{H}$ as $n \to \infty$. Using the reproducing property in both $\mathcal{H}$ and $\tilde{\mathcal{H}}$, for $t \in \Omega$ we have

$$|f_n(t) - f(t)| \le \|f_n - f\|_{\tilde{\mathcal{H}}}\sqrt{\tilde{k}(t,t)}$$
$$|f_n(t) - g(t)| \le \|f_n - g\|_{\mathcal{H}}\sqrt{k(t,t)},$$

and therefore, $\lim_{n\to\infty} f_n(t) = f(t) = g(t)$ for each $t \in \Omega$, and hence $f = g$.

Since it is also surjective, we infer that the norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\tilde{\mathcal{H}}}$ are equivalent from the open mapping theorem. As a consequence, the orthonormal basis $\{S_n\}_{n=1}^{\infty}$ in $\tilde{\mathcal{H}}$ is a Riesz basis for $\mathcal{H}$.

Finally, consider $\{x_n = T^{-1}(S_n)\}_{n=1}^{\infty}$, a Riesz basis in $\mathbb{H}$, and denote by $\{x_n^*\}_{n=1}^{\infty}$ its biorthonormal basis. Expanding $K(t)$ with respect to $\{x_n^*\}_{n=1}^{\infty}$, for each fixed $t \in \Omega$ we obtain

$$K(t) = \sum_{n=1}^{\infty} \langle K(t), x_n \rangle_{\mathbb{H}} x_n^* = \sum_{n=1}^{\infty} \overline{\langle T^{-1}(S_n), K(t) \rangle} x_n^* = \sum_{n=1}^{\infty} \overline{S_n(t)} x_n^*,$$

i.e., the required expansion for $K(t)$.

Notice that the interpolatory condition (6) comes out of a direct application of Condition (ii) to S_n, followed by Condition (i). ■

As to the case when, *a priori*, T is not known to be one-to-one, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{H}$ with $P(x_n) \neq 0$ for all n, where P denotes the orthogonal projection onto the closed subspace $(\operatorname{Ker} T)^{\perp}$. Consider $S_n = T(x_n) \in \mathcal{H}$, and suppose that these functions satisfy the hypotheses in Theorem 2. In this case, $\{S_n\}_{n=1}^{\infty}$ is a Riesz basis in $\mathcal{H}$. Consequently, since $S_n = T[P(x_n)]$ and $T|_{P(\operatorname{Ker} T)} = 0$, we obtain that $\{P(x_n)\}_{n=1}^{\infty}$ is a Riesz basis in $P(\mathbb{H}) = (\operatorname{Ker} T)^{\perp}$. The result comes out taking into account the orthogonal sum $\mathbb{H} = (\operatorname{Ker} T)^{\perp} \oplus (\operatorname{Ker} T)$.

4. SAMPLING USING SAMPLES OF A RELATED FUNCTION

Throughout this section the sequences $\{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty} \cup \{y_n^*\}_{n=1}^{\infty}$ will denote a pair of biorthonormal Riesz bases for the separable Hilbert space $\mathbb{H}$. Consider $\mathcal{H}$, the range space of a linear transform T as in (1) with kernel K.

Assume that, for each fixed $t \in \Omega$, we have the following expansion for $K(t)$

$$K(t) = \sum_{n=1}^{\infty} \overline{S_n(t)} x_n^* + \sum_{n=1}^{\infty} \overline{T_n(t)} y_n^*,$$

where the function $S_n, T_n : \Omega \mapsto \mathbb{C}$ satisfy the interpolatory condition: there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in Ω such that

$$S_n(t_m) = a_n \delta_{n,m}; \quad T_n(t_m) = b_n \delta_{n,m}, \quad \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C} \tag{9}$$

Assume also that there exists another $\mathbb{H}$-valued kernel $\tilde{K}$ defining the linear transform $\tilde{T}$ in $\mathbb{H}$ such that for each $x \in \mathbb{H}$, $\tilde{T}(x) = \tilde{f}$ where $\tilde{f}(t) := \langle x, \tilde{K}(t) \rangle_{\mathbb{H}}$ for $t \in \Omega$. Now, for each fixed $t \in \Omega$, assume the following expansion for $\tilde{K}(t)$

$$\tilde{K}(t) = \sum_{n=1}^{\infty} \overline{U_n(t)} x_n^* + \sum_{n-1}^{\infty} \overline{V_n(t)} y_n^*,$$

where the functions $U_n, V_n : \Omega \to \mathbb{C}$ satisfy, for the sequence $\{t_n\}_{n=1}^{\infty}$ in Ω, the interpolatory condition:

$$U_n(t_m) = c_n \delta_{n,m}; \quad V_n(t_m) = d_n \delta_{n,m}, \quad \{c_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty} \subset \mathbb{C}. \tag{10}$$

The interpolatory conditions (9) and (10) satisfy the condition

$$\Delta_n := a_n d_n - b_n c_n \neq 0 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

Finally, suppose that the linear transforms T and $\tilde{T}$ are related in the sense that $\mathrm{Ker}T \subseteq \mathrm{Ker}\tilde{T}$, i.e., there is a unique $\tilde{f}$ associated with each $f \in \mathcal{H}$.

Under the above hypotheses we prove a sampling result for $\mathcal{H}$ allowing to recover any function f in $\mathcal{H}$ from the sequence of samples $\{f(t_n)\}_{n=1}^{\infty} \cup \{\tilde{f}(t_n)\}_{n=1}^{\infty}$.

Theorem 3 *Suppose that the linear transforms T and $\tilde{T}$ defined on $\mathbb{H}$ satisfy* $\mathrm{Ker}T \subseteq \mathrm{Ker}\tilde{T}$*, and their respective kernels the interpolatory conditions* (9) *and* (10) *together with condition* (11). *Then, T is a bijective isometry between $\mathbb{H}$ and the RKHS $\mathcal{H}$, and consequently, the sequence $\{S_n\}_{n=1}^{\infty} \cup \{T_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$. Moreover, any function $f = T(x)$ in $\mathcal{H}$, $x \in \mathbb{H}$, can be recovered from the its samples $\{f(t_n)\}_{n=1}^{\infty}$ and the samples $\{\tilde{f}(t_n)\}_{n=1}^{\infty}$ of its related function $\tilde{f} = \tilde{T}(x)$, by means of the sampling formula*

$$f(t) = \sum_{n=1}^{\infty} \left[f(t_n) \frac{d_n S_n(t) - c_n T_n(t)}{\Delta_n} + \tilde{f}(t_n) \frac{a_n T_n(t) - b_n S_n(t)}{\Delta_n} \right]. \tag{12}$$

The convergence of the series is absolute and uniform on subsets of Ω where $k(t,t)$ is bounded.

Proof Regarding that T is a bijective isometry, we prove that T in one-to-one. Indeed, suppose that $f_x = f_y$ where we are using the notation $f_x := T(x)$. Then $f_x - f_y = f_{x-y} = 0$ and also $\tilde{f}_{x-y} = 0$. In particular,

$$f_{x-y}(t_n) = \langle x - y, K(t_n) \rangle_{\mathbb{H}} = a_n \langle x - y, x_n^* \rangle_{\mathbb{H}} + b_n \langle x - y, y_n^* \rangle_{\mathbb{H}} = 0$$
$$\tilde{f}_{x-y}(t_n) = \langle x - y, \tilde{K}(t_n) \rangle_{\mathbb{H}} = c_n \langle x - y, x_n^* \rangle_{\mathbb{H}} + d_n \langle x - y, y_n^* \rangle_{\mathbb{H}} = 0$$

for all $n \in \mathbb{N}$. Condition (11) and the completeness of the Riesz basis $\{x_n^*\}_{n=1}^{\infty} \cup \{y_n^*\}_{n=1}^{\infty}$ implies $x = y$. Consequently, $T : \mathbb{H} \to \mathcal{H}$ is a bijective isometry and $\{S_n = T(x_n)\}_{n=1}^{\infty} \cup \{T_n = T(y_n)\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$, whose biorthonormal Riesz basis is given by the sequence $\{S_n^* = T(x_n^*)\}_{n=1}^{\infty} \cup \{T_n^* = T(y_n^*)\}_{n=1}^{\infty}$. Expanding $f = T(x)$ in $\mathcal{H}$ with respect to the above Riesz basis we obtain

$$f = \sum_{n=1}^{\infty} [\langle f, S_n^* \rangle_{\mathcal{H}} S_n + \langle f, T_n^* \rangle_{\mathcal{H}} T_n]. \tag{13}$$

From the equalities

$$f(t_n) = \langle x, K(t_n) \rangle_{\mathbb{H}} = a_n \langle x, x_n^* \rangle_{\mathbb{H}} + b_n \langle x, y_n^* \rangle_{\mathbb{H}}$$
$$\tilde{f}(t_n) = \langle x, \tilde{K}(t_n) \rangle_{\mathbb{H}} = c_n \langle x, x_n^* \rangle_{\mathbb{H}} + d_n \langle x, y_n^* \rangle_{\mathbb{H}},$$

we get

$$\langle x, x_n^* \rangle_{\mathbb{H}} = \frac{d_n f(t_n) - b_n \tilde{f}(t_n)}{\Delta_n} \quad \text{and} \quad \langle x, y_n^* \rangle_{\mathbb{H}} = \frac{a_n \tilde{f}(t_n) - c_n f(t_n)}{\Delta_n}.$$

Since T is an isometry, $\langle x, x_n^* \rangle_{\mathbb{H}} = \langle f, S_n^* \rangle_{\mathcal{H}}$ and $\langle x, y_n^* \rangle_{\mathbb{H}} = \langle f, T_n^* \rangle_{\mathcal{H}}$. Substituting in (13) and after calculations we finally deduce formula (12). Convergence properties come out of the RKHS setting. ■

Sampling formulas for $\mathcal{H}$ using samples of several related functions can be obtained in a similar manner. Given $f \in \mathcal{H}$, one candidate to play the role of $\tilde{f}$ is its derivative f'. We go into more detail in the next section.

4.1 Sampling with Derivatives

Assume that the kernel $K : \Omega \to \mathbb{H}$ has a derivative K' in Ω. In this case, any function f in $\mathcal{H}$ is differentiable in Ω and its derivative is given by $f'(t) = \langle x, K'(t) \rangle_{\mathbb{H}}$. If $K(t)$ admits the expansion

$$K(t) = \sum_{n=1}^{\infty} \overline{S_n(t)} x_n^* + \sum_{n=1}^{\infty} \overline{T_n(t)} y_n^*,$$

then

$$K'(t) = \sum_{n=1}^{\infty} \overline{S_n'(t)} x_n^* + \sum_{n=1}^{\infty} \overline{T'_n(t)} y_n^*.$$

Indeed, the coefficients in the expansion of $K'(t)$ with respect to the Riesz basis $\{x_n^*\}_{n=1}^{\infty} \cup \{y_n^*\}_{n=1}^{\infty}$ are $\langle K'(t), x_n \rangle_{\mathcal{H}} = \overline{S_n'(t)}$ and $\langle K'(t), y_n \rangle_{\mathcal{H}} = \overline{T'_n(t)}$ since $S_n = T(x_n)$ and $T_n = T(y_n)$ respectively.

Assume also that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in Ω such that the interpolatory conditions

$$\begin{aligned} S_n(t_m) &= a_n \delta_{n,m}; \quad T_n(t_m) = b_n \delta_{n,m}, \quad \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C} \\ S_n'(t_m) &= c_n \delta_{n,m}; \quad T'_n(t_m) = d_n \delta_{n,m}, \quad \{c_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty} \subset \mathbb{C} \end{aligned}$$

hold with $\Delta_n \neq 0$ for $n \in \mathbb{N}$. Then, as a consequence of Theorem 3, we obtain the following sampling formula with derivatives for $\mathcal{H}$.

COROLLARY 4 *Under the above hypotheses on the kernels K and K', any function f in the RKHS $\mathcal{H}$ can be recovered from its samples $\{f(t_n)\}_{n=1}^{\infty}$ and those of its derivative $\{f'(t_n)\}_{n=1}^{\infty}$ by means of the sampling formula*

$$f(t) = \sum_{n=1}^{\infty} \left[f(t_n) \frac{d_n S_n(t) - c_n T_n(t)}{\Delta_n} + f'(t_n) \frac{a_n T_n(t) - b_n S_n(t)}{\Delta_n} \right]. \tag{14}$$

The convergence of the series is absolute and uniform on subsets of Ω where $k(t,t)$ is bounded.

Now we put to use this result in order to obtain, as a straightforward consequence, the sampling formula with derivatives in the Paley–Wiener space PW_π. To this end, note that the Fourier kernel

$$\begin{aligned} K &: \mathbb{R} \to L^2[-\pi, \pi] \\ t &\to K'(t) = \frac{e^{-itx}}{\sqrt{2\pi}} \end{aligned}$$

is differentiable in $\mathbb{R}$ and its derivative is given by

$$K' : \mathbb{R} \to L^2[-\pi, \pi]$$
$$t \to K'(t) = -ix \frac{e^{-itx}}{\sqrt{2\pi}}.$$

Now we consider the following Riesz basis in $L^2[-\pi, \pi]$ (see [6] for a proof)

$$\left\{ x_n = \frac{1}{\sqrt{\pi}} e^{-2inx} \right\}_{n \in \mathbb{Z}} \cup \left\{ y_n = \frac{1}{\sqrt{\pi}} ixe^{-2inx} \right\}_{n \in \mathbb{Z}} \tag{15}$$

whose biorthonormal basis is given by

$$\left\{ x_n^* = \frac{1}{\sqrt{\pi}} \left(1 - \frac{|x|}{\pi} \right) e^{-2inx} \right\}_{n \in \mathbb{Z}} \cup \left\{ y_n^* = \frac{i \text{ sgn } x}{\pi \sqrt{\pi}} e^{-2inx} \right\}_{n \in \mathbb{Z}}$$

Expanding the function $K(t) = (e^{-itx})/\sqrt{2\pi}$ in $L^2[-\pi, \pi]$ with respect to the first basis and using special Fourier transforms, we obtain

$$\begin{aligned} K(t) &= \sum_{n=-\infty}^{\infty} \langle K(t), x_n^* \rangle x_n + \sum_{n=-\infty}^{\infty} \langle K(t), y_n^* \rangle y_n \\ &= \sum_{n=-\infty}^{\infty} \left\langle \frac{e^{-itx}}{\sqrt{2\pi}}, \left(1 - \frac{|x|}{\pi} \right) \frac{e^{-2inx}}{\sqrt{\pi}} \right\rangle x_n + \sum_{n=-\infty}^{\infty} \left\langle \frac{e^{-itx}}{\sqrt{2\pi}}, \frac{i \text{ sgn } x}{\pi \sqrt{\pi}} e^{-2inx} \right\rangle y_n \\ &= \sum_{n=-\infty}^{\infty} \frac{\sqrt{2}}{2} \left(\frac{\sin(\pi/2)(t-2n)}{(\pi/2)(t-2n)} \right)^2 x_n - \sum_{n=-\infty}^{\infty} \frac{\sqrt{2}}{\pi} \frac{\sin^2(\pi/2)(t-2n)}{(\pi/2)(t-2n)} y_n. \end{aligned}$$

Taking the functions

$$S_n(t) = \frac{\sqrt{2}}{2} \left(\frac{\sin(\pi/2)(t-2n)}{(\pi/2)(t-2n)} \right)^2 \quad \text{and} \quad T_n(t) = -\frac{\sqrt{2}}{\pi} \frac{\sin^2(\pi/2)(t-2n)}{(\pi/2)(t-2n)},$$

and the sequence $\{t_n = 2n\}_{n \in \mathbb{Z}}$, it is easy to check that the following interpolatory conditions hold

$$\begin{aligned} S_n(2m) &= \frac{\sqrt{2}}{2} \delta_{n,m}; \quad T_n(2m) = 0, \\ S_n'(2m) &= 0; \quad T_n'(2m) = -\frac{\sqrt{2}}{2} \delta_{n,m}. \end{aligned}$$

Hence, following (14) we obtain:

COROLLARY 5 *Any function f in PW_π can be expanded as the derivative sampling series*

$$f(t) = \sum_{n=-\infty}^{\infty} f(2n)\left(\frac{\sin(\pi/2)(t-2n)}{(\pi/2)(t-2n)}\right)^2 + \sum_{n=-\infty}^{\infty} f'(2n)\frac{2}{\pi}\left(\frac{\sin^2(\pi/2)(t-2n)}{(\pi/2)(t-2n)}\right)$$
$$= \sum_{n=-\infty}^{\infty} [f(2n) + (t-2n)f'(2n)]\left(\frac{\sin(\pi/2)(t-2n)}{(\pi/2)(t-2n)}\right)^2.$$

The convergence of the series is absolute and uniform in $\mathbb{R}$.

For an account of derivative sampling in PW_π, including historical notes and different mathematical techniques, see [8, pp. 56–77].

4.2 Hermite-type Interpolation Series

Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of distinct real numbers such that $\sum_n 1/|t_n|^2 < \infty$. There exists an entire function $P(t)$ with simple zeros at the sequence $\{t_n\}_{n=1}^{\infty}$. Specifically, the function $P(t)$ is given by the canonical product

$$P(t) = \begin{cases} \prod_{n=1}^{\infty}\left(1-\frac{t}{t_n}\right)\exp(t/t_n) & \text{if } \sum_{n=1}^{\infty}|t_n|^{-1} = \infty \\ \prod_{n=1}^{\infty}\left(1-\frac{t}{t_n}\right) & \text{if } \sum_{n=1}^{\infty}|t_n|^{-1} < \infty \end{cases}$$

whenever $t_n \neq 0$ for all $n \in \mathbb{N}$, or by

$$P(t) = \begin{cases} t\prod_{n=2}^{\infty}\left(1-\frac{t}{t_n}\right)\exp(t/t_n) & \text{if } \sum_{n=2}^{\infty}|t_n|^{-1} = \infty \\ t\prod_{n=2}^{\infty}\left(1-\frac{t}{t_n}\right) & \text{if } \sum_{n=1}^{\infty}|t_n|^{-1} < \infty \end{cases}$$

in the case when, for instance, $t_1 = 0$ (see [13, p. 55] for the details).

Consider $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$, a pair of biorthonormal Riesz bases for a separable Hilbert space $\mathbb{H}$. Define $K(t)$ as in (5), where $S_n(t) = P(t)/t - t_n$. Since $S_n(t_m) = P'(t_n)\delta_{n,m}$, any function f defined for $t \in \mathbb{R}$ by $f(t) = \langle x, K(t)\rangle_{\mathbb{H}}$, where $x \in \mathbb{H}$, can be expanded as the *Lagrange-type interpolation series*.

$$f(t) = \sum_{n=1}^{\infty} f(t_n)\frac{P(t)}{(t-t_n)P'(t_n)}.$$

For an account of Lagrange interpolation and sampling results, see [15].

Next we show that we can obtain RKHS $\mathcal{H}$ such that any function in $\mathcal{H}$ can be recovered from the samples $\{f(t_n)\}_{n=1}^{\infty}$ and $\{f'(t_n)\}_{n=1}^{\infty}$, taken at the above sequence $\{t_n\}_{n=1}^{\infty}$, by means of Hermite-type interpolation series. Indeed, consider $Q(t) = P^2(t)$ (which has double zeros at $\{t_n\}_{n=1}^{\infty}$), and take the functions

$$S_n(t) = \frac{Q(t)}{(t-t_n)^2} \quad \text{and} \quad T_n(t) = \frac{Q(t)}{t-t_n}.$$

Define the kernel $K(t)$, $t \in \mathbb{R}$ by

$$K(t) = \sum_{n=1}^{\infty} \left[\frac{Q(t)}{(t-t_n)^2} x_n^* + \frac{Q(t)}{t-t_n} y_n^* \right],$$

where $\{x_n^*\}_{n=1}^{\infty} \cup \{y_n^*\}_{n=1}^{\infty}$ denotes the biorthonormal Riesz basis of a Riesz basis $\{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}$ for a separable Hilbert space $\mathbb{H}$ (this basis might be obtained by partitioning a given Riesz basis into two arbitrary sequences). It is easy to check the interpolatory conditions:

$$S_n(t_m) = \frac{Q''(t_n)}{2} \delta_{n,m}; \qquad T_n(t_m) = 0,$$
$$S_n'(t_m) = \frac{Q'''(t_n)}{6} \delta_{n,m}; \qquad T_n'(t_m) = \frac{Q''(t_n)}{2} \delta_{n,m}.$$

Taking into account that $Q''(t_n) \neq 0$ for all $n \in \mathbb{N}$, any function f defined by $t \in \mathbb{R}$ by $f(t) = \langle x, K(t) \rangle_{\mathbb{H}}$, where $x \in \mathbb{H}$, can be expanded, following (12), as the *Hermite-type interpolation series*

$$f(t) = \sum_{n=1}^{\infty} \left\{ f(t_n) \left[1 - \frac{Q'''(t_n)}{3Q''(t_n)} (t-t_n) \right] + f'(t_n)(t-t_n) \right\} \frac{2Q(t)}{Q''(t_n)(t-t_n)^2}.$$

Taking the sampling points $\{t_n = 2n\}_{n \in \mathbb{Z}}$ and the Riesz basis (15) in $L^2[-\pi, \pi]$, the above construction leads to the derivative sampling in PW_π stated in Corollary 5.

4.3 Sampling with Hilbert Transforms

As another application of Theorem 3, we see in this section that any function f in the Paley–Wiener space PW_π can be recovered from its samples $\{f(2n)\}_{n \in \mathbb{Z}}$ and the samples $\{\tilde{f}(2n)\}_{n \in \mathbb{Z}}$ of its Hilbert transform $\tilde{f}$ which is given by the formula

$$\tilde{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x)(-i \operatorname{sgn} x) e^{itx} dx = \left\langle F, \frac{i \operatorname{sgn} x}{\sqrt{2\pi}} e^{-itx} \right\rangle_{L^2[-\pi,\pi]}$$

whenever f is expressed as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x) e^{itx} dx = \left\langle F, \frac{e^{-itx}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]} \qquad F \in L^2[-\pi, \pi].$$

Hence, the kernel $\tilde{K}$ associated with the Hilbert transform in $L^2[-\pi,\pi]$ is

$$\begin{aligned}\tilde{K}:\mathbb{R}&\to L^2[-\pi,\pi]\\ t&\to \tilde{K}(t)=\frac{i\ \mathrm{sgn}\ x}{\sqrt{2\pi}}e^{-itx}.\end{aligned}$$

Expanding, for a fixed $t\in\mathbb{R}$, the Fourier kernel $K(t)=(e^{-itx})/(\sqrt{2\pi})$ with respect to the orthonormal basis of $L^2[-\pi,\pi]$ given by

$$\left\{x_n=\frac{1}{\sqrt{2\pi}}e^{-2inx}\right\}_{n\in\mathbb{Z}}\cup\left\{y_n=\frac{1}{\sqrt{2\pi}}(-i\ \mathrm{sgn}\ x)e^{-2inx}\right\}_{n\in\mathbb{Z}},$$

(see [6, p. 134] for a proof) we obtain

$$\begin{aligned}K(t)&=\sum_{n=-\infty}^{\infty}\langle K(t),x_n\rangle x_n+\sum_{n=-\infty}^{\infty}\langle K(t),y_n\rangle y_n\\ &=\sum_{n=-\infty}^{\infty}\frac{\sin\pi(t-2n)}{\pi(t-2n)}x_n+\sum_{n=-\infty}^{\infty}\frac{\sin^2(\pi/2)(t-2n)}{(\pi/2)(t-2n)}y_n.\end{aligned}$$

For the sequence $\{t_n=2_n\}_{n\in\mathbb{Z}}$ the functions

$$S_n(t)=\frac{\sin\pi(t-2n)}{\pi(t-2n)}\quad\text{and}\quad T_n(t)=\frac{\sin^2(\pi/2)(t-2n)}{(\pi/2)(t-2n)}$$

satisfy the interpolatory condition: $S_n(2m)=\delta_{n,m}$ and $T_n(2m)=0$. On the other hand, an easy calculation shows that

$$\tilde{K}(t)=\sum_{n=-\infty}^{\infty}T_n(t)x_n-\sum_{n=-\infty}^{\infty}S_n(t)y_n,$$

and hence, $U_n(t_m)=T_n(2m)=0$ and $V_n(t_m)=-S_n(2m)=-\delta_{n,m}$. Therefore, as a consequence of Theorem 3 we obtain

COROLLARY 6 *Any function in PW_π can be expanded as the sample series*

$$\begin{aligned}f(t)&=\sum_{n=-\infty}^{\infty}f(2n)\frac{\sin\pi(t-2n)}{\pi(t-2n)}-\sum_{n=-\infty}^{\infty}\tilde{f}(2n)\frac{\sin^2(\pi/2)(t-2n)}{(\pi/2)(t-2n)}\\ &=\sum_{n=-\infty}^{\infty}[f(2n)\cos(\pi/2)(t-2n)-\tilde{f}(2n)\sin(2/\pi)(t-2n)]\frac{\sin(\pi/2)(t-2n)}{(\pi/2)(t-2n)}.\end{aligned}$$

The convergence of the series is absolute and uniform in $\mathbb{R}$.

Acknowledgments

The authors wish to thank the referee for her/his valuable and constructive comments. This work has been supported by the grant BFM2000–0029 from the D.G.I. of the Spanish Ministerio de Ciencia y Tecnología.

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