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Hypercircle Inequalities and Sampling Theory

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In this article the well-known hypercircle inequality is extended to the Riesz bases setting. A natural application for this new inequality is given by the estimation of the truncation error in nonorthogonal sampling formulas. Examples including the estimation of the truncation error for wavelet sampling expansions or for nonorthogonal sampling formulas in Paley–Wiener spaces are exhibited.

Keywords: Reproducing kernel Hilbert spaces; Biorthonormal Riesz bases; Hypercircle inequality; Sampling formulas; Multiresolution Analysis; Frames

AMS Subject Classifications: 94A20; 44A05; 46E22

1. INTRODUCTION

The classical hypercircle inequality estimates the error when we are evaluating a bounded linear functional on a hypercircle C_r of a Hilbert space $\mathbb{H}$, i.e., the intersection of a hyperplane of finite codimension P and the closed ball of ratio r, B_r. The approximate value is just the evaluation of the functional in the nearest point to the origin in the hyperplane P. To be precise,

Let P be a hyperplane of co-dimension N in a Hilbert space $\mathbb{H}$, and let w be the element of P nearest to the origin. Then, for any x in the hypercircle $C_r = P \cap B_r$ and any bounded linear functional L in $\mathbb{H}$ we have

$$\left|L(x) - L(w)\right|^2 \leq \left(r^2 - \|w\|^2\right) \sum_{k=N+1}^{\infty} \left|L(x_k)\right|^2,$$

where $\{x_k\}_{k=1}^{\infty}$ denotes an orthonormal basis in $\mathbb{H}$ such that P is given by the equations: $\langle x, x_i \rangle = a_i$, $i = 1, \ldots, N$ and $w = \sum_{k=1}^{N} a_i\, x_i$.

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One can find the proof of this result in [3,7], or [9], and an account of applications of the hypercircle inequality to numerical analysis in [7]. Also, the hypercircle inequality has been used to estimate the truncation error in the Whittaker–Shannon–Kotel'nikov sampling formula [8, 9]. This well-known sampling result reads as follows: *Any function f band-limited to the interval* $[-\pi\sigma, \pi\sigma]$ *in the classical sense, i.e.*,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\sigma}^{\pi\sigma} F(w)\, e^{iwt}\, dw,$$

where $F \in L^2[-\pi\sigma, \pi\sigma]$, *can be expanded as the cardinal series*

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\sigma}\right) \frac{\sin \pi(\sigma t - n)}{\pi(\sigma t - n)} = \sum_{n=-\infty}^{\infty} f(n/\sigma)\, \mathrm{sinc}(\sigma t - n).$$

Assuming that $\|f\| \leq r$, the hypercircle inequality shows that the truncation error in the WSK sampling formula satisfies the inequality

$$\left|f(t) - f_N(t)\right|^2 \leq \sigma r^2 - \sum_{n=-N}^{N} \left|f(n/\sigma)\right|^2,$$

where $f_N(t) := \sum_{n=-N}^{N} f(n/\sigma)\, \mathrm{sinc}(\sigma t - n)$.

By extending the hypercircle inequality in the Riesz bases (frames) setting we can estimate the truncation error for nonorthogonal sampling expansions. The obtained estimation uses only the samples and the norm of the function together with other quantities, intrinsically related both with the sampling formula and with the underlying functional space. Besides, as pointed out in [7], the use of a nonoptimal approximation including known data, as the samples here, can be sometimes preferred to the best approximation, which may have rather cumbersome coefficients.

The article is organized as follows: In Section 2 we prove the version of the hypercircle inequality in the Riesz bases setting. In Section 3 we put to work this new hypercircle inequality to obtain estimations of the truncation error for nonorthogonal sampling formulas in reproducing kernel Hilbert spaces. We particularize the method in two important cases: The first one concerns the wavelet sampling expansion; the second one the derivative sampling expansion in Paley–Wiener spaces. Finally, the last section is devoted to prove a version of the hypercircle inequality in the frame setting.

2. THE HYPERCIRCLE INEQUALITY IN THE RIESZ BASES SETTING

Let $\mathbb{H}$ be a separable Hilbert space. Given N independent elements $x_1, x_2, \ldots, x_N$ in $\mathbb{H}$, and N constants $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}$, the elements $y \in \mathbb{H}$ satisfying

$$\langle y, x_i \rangle_{\mathbb{H}} = \alpha_i, \quad 1 \leq i \leq N,$$

constitute a hyperplane P of codimension N. The hypercircle C_r is defined as the intersection $P \cap B_r$ of this hyperplane P and the ball $B_r = \{x \in \mathbb{H}\colon \|x\|_{\mathbb{H}} \leq r\}$.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ be a pair of dual Riesz bases in $\mathbb{H}$. Recall that a Riesz basis $\{x_n\}_{n=1}^{\infty}$ for a separable Hilbert space $\mathbb{H}$ is the image of an orthonormal basis through a bounded invertible operator. Any Riesz basis $\{x_n\}_{n=1}^{\infty}$ has a unique biorthonormal (dual) Riesz basis $\{x_n^*\}_{n=1}^{\infty}$, i.e., $\langle x_n, x_m^*\rangle_{\mathbb{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^{\infty}\langle x, x_n^*\rangle_{\mathbb{H}}\, x_n = \sum_{n=1}^{\infty}\langle x, x_n\rangle_{\mathbb{H}}\, x_n^*$$

hold for every $x \in \mathbb{H}$ (see [11] for more details and proofs). Hence, there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in a suitable Hilbert space $\widehat{\mathbb{H}}$ (maybe $\mathbb{H}$ itself) and an invertible bounded operator $\mathcal{T} : \widehat{\mathbb{H}} \longrightarrow \mathbb{H}$ such that $\mathcal{T}(e_n) = x_n$, for each $n \in \mathbb{N}$. Under these circumstances we prove the following version of the hypercircle inequality:

THEOREM 1 *Given the vector* $w_N = \sum_{k=1}^{N} \alpha_k x_k$ *and the subspace* $K = \{y \in \mathbb{H}\colon \langle y, x_k^*\rangle_{\mathbb{H}} = 0,\quad 1 \le k \le N\}$, *consider the hyperplane* $P := w_N + K$ *and the hypercircle* $C_r = P \cap B_r$. *Then, for any* $x \in C_r$ *and any bounded linear functional* L *in* $\mathbb{H}$ *we have*

$$\left|L(x) - L(w_N)\right|^2 \le \left(\left\|\mathcal{T}^{-1}\right\|^2 r^2 - \sum_{k=1}^{N} |\alpha_k|^2\right) \sum_{k=N+1}^{\infty} \left|L(x_k)\right|^2. \tag{1}$$

Proof: For $x \in P$, we have

$$x = w_N + (x - w_N) = w_N + \sum_{k=1}^{\infty}\left\langle x - w_N, x_k^*\right\rangle_{\mathbb{H}} x_k = w_N + \sum_{k=N+1}^{\infty} \left\langle x, x_k^*\right\rangle_{\mathbb{H}} x_k,$$

since $x - w_N \in K$ and $w_N = \sum_{k=1}^{N} \alpha_k x_k$. Hence,

$$\begin{aligned}
\left|L(x) - L(w_N)\right|^2 = \left|L(x - w_N)\right|^2 &= \left|\sum_{k=N+1}^{\infty} \langle x, x_k^*\rangle_{\mathbb{H}} L(x_k)\right|^2 \\
&\le \sum_{k=N+1}^{\infty} \left|\langle x, x_k^*\rangle_{\mathbb{H}}\right|^2 \sum_{k=N+1}^{\infty} \left|L(x_k)\right|^2 \\
&= \left\|\mathcal{T}^{-1}(x - w_N)\right\|^2 \sum_{k=N+1}^{\infty} \left|L(x_k)\right|^2,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the Parseval equality with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Notice that w_N is not orthogonal to $x - w_N \in K$, but their inverse images by $\mathcal{T}$ are orthogonal, hence

$$\left|L(x) - L(w_N)\right|^2 \le \left(\left\|\mathcal{T}^{-1}(x)\right\|^2 - \left\|\mathcal{T}^{-1}(w_N)\right\|^2\right) \sum_{k=N+1}^{\infty} \left|L(x_k)\right|^2$$

$$= \left(\left\| \mathcal{T}^{-1}(x) \right\|^2 - \sum_{k=1}^{N} |\alpha_k|^2 \right) \sum_{k=N+1}^{\infty} \left| L(x_k) \right|^2$$

$$\leq \left(\left\| \mathcal{T}^{-1} \right\|^2 \|x\|^2 - \sum_{k=1}^{N} |\alpha_k|^2 \right) \sum_{k=N+1}^{\infty} \left| L(x_k) \right|^2.$$

Since $x \in C_r$ we finally obtain

$$\left| L(x) - L(w_N) \right|^2 \leq \left(\left\| \mathcal{T}^{-1} \right\|^2 r^2 - \sum_{k=1}^{N} |\alpha_k|^2 \right) \sum_{k=N+1}^{\infty} \left| L(x_k) \right|^2.$$

■

3. AN APPLICATION: ESTIMATION OF THE TRUNCATION ERROR FOR NONORTHOGONAL SAMPLING FORMULAS

Let $\mathcal{H}$ be a reproducing kernel Hilbert space (RKHS hereafter) of complex-valued functions defined on a fixed subset Ω of $\mathbb{R}$ (or $\mathbb{C}$). There exists a function $k : \Omega \times \Omega \to \mathbb{C}$, its reproducing kernel, such that for each $s \in \Omega$ the function k_s defined as $k_s(t) := k(t, s)$ belongs to $\mathcal{H}$, and the reproducing property

$$f(s) = \langle f, k_s \rangle_{\mathcal{H}} = \langle f, k(\cdot, s) \rangle_{\mathcal{H}}, \quad s \in \Omega, \ f \in \mathcal{H} \tag{2}$$

holds.

Assume that we have a nonorthogonal sampling formula in a reproducing kernel Hilbert space $\mathcal{H}$, i.e., for any $f \in \mathcal{H}$ we have

$$f(t) = \sum_{n=1}^{\infty} f(t_n)\, S_n(t), \tag{3}$$

where the sequence of sampling functions $\{S_n\}_{n=1}^{\infty}$ forms a Riesz basis in $\mathcal{H}$.

By using the Cauchy–Schwarz inequality in (2), the point-evaluation functional $\mathcal{E}_t(f) := f(t)$ is bounded in the RKHS $\mathcal{H}$. If we set $w_N := f_N = \sum_{n=1}^{N} f(t_n)\, S_n \in \mathcal{H}$, the truncation error in (3) is given by

$$\left| \mathcal{E}_t(f) - \mathcal{E}_t(f_N) \right| = \left| f(t) - f_N(t) \right|.$$

Therefore, in this case the hypercircle inequality (1) reads:

$$\left| f(t) - f_N(t) \right|^2 \leq \left(\left\| \mathcal{T}^{-1} \right\|^2 r^2 - \sum_{n=1}^{N} \left| f(t_n) \right|^2 \right) \sum_{n=N+1}^{\infty} \left| S_n(t) \right|^2,$$

where $\mathcal{T}$ denotes the invertible bounded operator which maps an orthonormal basis onto the Riesz basis $\{S_n\}_{n=1}^{\infty}$.

As a consequence, the key point is to compute (estimate) $\|\mathcal{T}^{-1}\|$, and to compute (estimate) the sum $\sum_{n=1}^{\infty} |S_n(t)|^2$. The sampling formula (3) might also contain samples from a function related with f, e.g., its derivative [6]. In this case, the same technique for estimating the truncation error applies. Next, we illustrate the method in two important examples.

3.1 Truncation Error for Wavelet Sampling Expansions

Let ϕ be a real-valued scaling function defined on $\mathbb{R}$ such that $\phi \in \mathcal{C}^r(\mathbb{R})$ and

$$\left|\phi^{(k)}(t)\right| \le C_{p,k}(1+|t|)^{-p},$$

for $0 \le k \le r$, $p \in \mathbb{N}$ and $t \in \mathbb{R}$, which is associated with a multiresolution analysis $\{V_m\}_{m\in\mathbb{Z}}$ of $L^2(\mathbb{R})$, i.e., a nested sequence of closed subspaces such that

(1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$.
(2) $f \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}$.
(3) $\cap_{m\in\mathbb{Z}} V_m = \{0\}$ and $\overline{\cup_{m\in\mathbb{Z}} V_m} = L^2(\mathbb{R})$.
(4) $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0.

We define

$$k(t,s) = \sum_{n\in\mathbb{Z}} \phi(t-n)\phi(s-n).$$

For each $m \in \mathbb{Z}$, it is known that V_m is a reproducing Kernel Hilbert space whose reproducing kernel is given by

$$k_m(t,s) = 2^m k(2^m t, 2^m s).$$

Assume that the function G, defined as

$$G(w) := \sum_{n\in\mathbb{Z}} \phi(n)e^{-iwn}, \quad w \in \mathbb{R},$$

has no zeros in $\mathbb{R}$. Then, $\{k(t,n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of V_0 whose biorthogonal basis is given by $\{S_n(t) = S(t-n)\}_{n\in\mathbb{Z}}$, where the Fourier transform of S satisfies

$$\mathcal{F}S(w) := \widehat{S}(w) = \frac{\widehat{\phi}(w)}{G(w)}.$$

Furthermore, for any $f \in V_0$, the following sampling formula holds

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), \tag{4}$$

where the convergence of the series is absolute and uniform in $\mathbb{R}$. See [10] for more details and proofs.

Next, we use the hypercircle inequality (1) to estimate the truncation error in the wavelet sampling expansion (4). Indeed, consider the bounded invertible operator $T : V_0 \longrightarrow V_0$, which maps the orthonormal basis $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ of V_0 into the Riesz basis $\{k(t,n)\}_{n\in\mathbb{Z}}$ of the same space. The operator $\widehat{T} := \mathcal{F}T\mathcal{F}^{-1}$ maps $\{\mathcal{F}(\phi(\cdot-n))\}_{n\in\mathbb{Z}}$ into $\{\mathcal{F}(k(\cdot,n))\}_{n\in\mathbb{Z}}$. Taking into account that

$$k(t,m) = \sum_{n\in\mathbb{Z}} \phi(t-n)\phi(m-n) = \sum_{n\in\mathbb{Z}} \phi(t-m-n)\phi(-n) = k(t-m,0),$$

we obtain that

$$\begin{aligned}\widehat{T}(\widehat{\phi}e^{-inx})(w) &= \mathcal{F}k(x,n)(w) = \mathcal{F}k(x-n,0)(w) = \mathcal{F}\left(\sum_{k\in\mathbb{Z}} \phi(x-n-k)\phi(-k)\right)(w)\\ &= e^{-inw}\widehat{\phi}(w)\sum_{k\in\mathbb{Z}}\phi(-k)e^{-ikw} = e^{-inw}\widehat{\phi}(w)\overline{G}(w).\end{aligned}$$

The bounded invertible operator which maps $\phi(t-n)$ into $S(t-n)$ is $(T^{-1})^* = (T^*)^{-1}$, where T^* stands for the adjoint operator of T. Then, for $f \in V_0$ with $\|f\| \le r$ the hypercircle inequality (1) implies

$$|f(t) - f_N(t)|^2 \le \left(\|T^*\|^2 r^2 - \sum_{|n|\le N} |f(n)|^2\right) \sum_{|n|>N} |S(t-n)|^2, \tag{5}$$

where $f_N(t) := \sum_{|n|\le N} f(n)S(t-n)$.

Using the reproducing property in V_0,

$$\begin{aligned}\sum_{|n|>N} |S(t-n)|^2 \le \sum_{n\in\mathbb{Z}} |S(t-n)|^2 &= \sum_{n\in\mathbb{Z}} |\langle S(\cdot-n), k(\cdot,t)\rangle|^2\\ &= \sum_{n\in\mathbb{Z}} |\langle (T^{-1})^*\phi(\cdot-n), k(\cdot,t)\rangle|^2\\ &= \sum_{n\in\mathbb{Z}} |\langle \phi(\cdot-n), T^{-1}k(\cdot,t)\rangle|^2.\end{aligned}$$

Since $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ constitutes an orthonormal basis for V_0, the Parseval equality gives

$$\sum_{|n|>N} |S(t-n)|^2 \le \sum_{n\in\mathbb{Z}} |S(t-n)|^2 = \|T^{-1}k(\cdot,t)\|^2 \le \|T^{-1}\|^2\, k(t,t).$$

Substituting in (5) we obtain

$$\left| f(t) - f_N(t) \right|^2 \leq \left(\|T\|^2 r^2 - \sum_{|n| \leq N} \left| f(n) \right|^2 \right) \left\| T^{-1} \right\|^2 k(t,t). \tag{6}$$

Our last goal is giving an estimation for both $\|T\|^2$ and $\|T^{-1}\|^2$. Consider $F \in \mathcal{F}V_0$. Then,

$$\begin{aligned} \|\widehat{T}F\| &= \left\| \widehat{T}\left(\sum_{n\in\mathbb{Z}} \langle F, \widehat{\phi e^{-in\cdot}} \rangle \widehat{\phi e^{-in\cdot}} \right) \right\| \\ &= \left\| \sum_{n\in\mathbb{Z}} \langle F, \widehat{\phi e^{-in\cdot}} \rangle \widehat{\phi e^{-in\cdot}} \overline{G} \right\| \\ &\leq \|G\|_\infty \|F\|. \end{aligned}$$

Since the Fourier transform $\mathcal{F}$ is a unitary operator in $L^2(\mathbb{R})$, we have $\|T\| = \|\widehat{T}\| \leq \|G\|_\infty$. On the other hand, taking again $F \in \mathcal{F}V_0$, we have

$$\begin{aligned} \left\| (\widehat{T}^{-1})^* F \right\| &= \left\| (\widehat{T}^{-1})^* \left(\sum_{n\in\mathbb{Z}} \langle F, \widehat{\phi e^{-in\cdot}} \rangle \widehat{\phi e^{-in\cdot}} \right) \right\| = \left\| \sum_{n\in\mathbb{Z}} \langle F, \widehat{\phi e^{-in\cdot}} \rangle (\widehat{T}^{-1})^* \left(\widehat{\phi e^{-in\cdot}} \right) \right\| \\ &= \left\| \sum_{n\in\mathbb{Z}} \langle F, \widehat{\phi e^{-in\cdot}} \rangle \widehat{S e^{-in\cdot}} \right\| = \left\| \sum_{n\in\mathbb{Z}} \langle F, \widehat{\phi e^{-in\cdot}} \rangle \widehat{\phi e^{-in\cdot}} \frac{1}{\overline{G}} \right\| \\ &\leq \left\| 1/G \right\|_\infty \|F\|. \end{aligned}$$

As a consequence, $\|T^{-1}\| = \|(T^{-1})^*\| = \|(\widehat{T^{-1}})^*\| \leq \|1/G\|_\infty$.

Finally, inequality (6) can be written as

$$\left| f(t) - f_N(t) \right|^2 \leq \left(\|G\|_\infty^2 r^2 - \sum_{|n| \leq N} \left| f(n) \right|^2 \right) \left\| 1/G \right\|_\infty^2 k(t,t).$$

Closing the section, it is worth mentioning that the truncation error on wavelet sampling expansions has been estimated in [1] by imposing some extra conditions on the function f.

3.2 Truncation Error for the Derivative Sampling Formula in Paley–Wiener Spaces

Consider the Hilbert Space $\widehat{\mathbb{H}} := L^2[0,\pi] \oplus L^2[0,\pi]$ endowed with its usual norm $\|(\phi,\varphi)\|^2_{\widehat{\mathbb{H}}} = \|\phi\|^2_{L^2[0,\pi]} + \|\varphi\|^2_{L^2[0,\pi]}$. It is easy to check that the sequence $\{w_n\}_{n\in\mathbb{Z}}$, given by

$$\begin{cases} w_{2n} = (e^{-ix2n}/\sqrt{\pi}, \mathbf{0}) \\ w_{2n+1} = (\mathbf{0}, e^{-ix2n}/\sqrt{\pi}) \end{cases}, \quad n \in \mathbb{Z},$$

constitutes an orthonormal basis for the space $\widehat{\mathbb{H}}$.

For each function ϕ in $L^2[0,\pi]$, we denote by ϕ^p its π-periodic extension. Any function in $L^2[-\pi,\pi]$ can be expressed as $f = f^{①} + f^{②}$, where $f^{①} = f\chi_{[-\pi,0]}$ and $f^{②} = f\chi_{[0,\pi]}$.

Define the operator

$$\begin{array}{rccl} \mathcal{S}: & \widehat{\mathbb{H}} & \longrightarrow & L^2[-\pi,\pi] \\ & (\phi_1,\phi_2) & \longmapsto & \left(1-\dfrac{|x|}{\pi}\right)\phi_1^p(x) + \dfrac{i\operatorname{sgn} x}{\pi}\phi_2^p(x), \end{array} \tag{7}$$

which transforms the orthonormal basis $\{w_n\}_{n\in\mathbb{Z}}$ in $\widehat{\mathbb{H}}$ into the sequence $\{x_n\}_{n\in\mathbb{Z}}$ given by

$$\begin{cases} x_{2n} = \dfrac{1}{\sqrt{\pi}}\left(1-\dfrac{|x|}{\pi}\right)e^{-ix2n} \\ x_{2n+1} = \dfrac{i\operatorname{sgn} x}{\pi\sqrt{\pi}}e^{-ix2n} \end{cases}, \quad n \in \mathbb{Z}.$$

In order to simplify the argument, we consider the matrix

$$\mathcal{M}(x) := \frac{1}{\pi}\begin{pmatrix} x & -i \\ \pi - x & i \end{pmatrix}, \quad x \in [0,\pi]. \tag{8}$$

Observe that

$$\mathcal{M}(x)\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} x\phi_1(x) - i\phi_2(x) \\ (\pi - x)\phi_1(x) + i\phi_2(x) \end{pmatrix}.$$

On the other hand, if $x \in [0,\pi]$, then, using [7], we have

$$\mathcal{S}(\phi_1,\phi_2)(x-\pi) = \frac{x}{\pi}\phi_1(x) - \frac{i}{\pi}\phi_2(x)$$

and

$$\mathcal{S}(\phi_1,\phi_2)(x) = \left(1-\frac{x}{\pi}\right)\phi_1(x) + \frac{i}{\pi}\phi_2(x),$$

so we can evaluate the bounded operator $\mathcal{S}$ by means of the matrix $\mathcal{M}(x)$. Since $\mathcal{M}(x)$ is nonsingular in $[0,\pi]$, we can deduce the invertibility of the operator $\mathcal{S}$ by giving an explicit expression of $\mathcal{S}^{-1}$ in terms of $\mathcal{M}^{-1}(x) := [\mathcal{M}(x)]^{-1}$. In fact, since

$$\mathcal{M}^{-1}(x) = \begin{pmatrix} 1 & 1 \\ -i(x-\pi) & -ix \end{pmatrix},$$

we obtain that

$$\mathcal{S}^{-1}(f) = \mathcal{M}^{-1}(x)\begin{pmatrix} f^{①}(-\pi + x) \\ f^{②}(x) \end{pmatrix} = \begin{pmatrix} f^{①}(x-\pi) + f^{②}(x) \\ -i(x-\pi)f^{①}(x-\pi) - ixf^{②}(x) \end{pmatrix}.$$

Therefore, the sequence $\{x_n\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2[-\pi,\pi]$ whose dual Riesz basis, $\{x_n^*\}_{n\in\mathbb{Z}}$, is given by

$$\begin{cases} x_{2n}^* = e^{-ix2n}/\sqrt{\pi} \\ x_{2n+1}^* = ixe^{-ix2n}/\sqrt{\pi} \end{cases}, \quad n \in \mathbb{Z}.$$

Next, we estimate $\|\mathcal{S}^{-1}\|$. Indeed,

$$\begin{aligned}
\|\mathcal{S}^{-1}f\|^2 &= \|f^{①}(x-\pi) + f^{②}(x)\|^2 + \| - i(x-\pi)f^{①}(x-\pi) - ixf^{②}(x)\|^2 \\
&= \int_0^\pi |f^{①}(x-\pi)|^2 dx + \int_0^\pi |f^{②}(x)|^2 dx \\
&\quad + \int_0^\pi (x-\pi)^2 |f^{①}(x-\pi)|^2 dx + \int_0^\pi x^2 |f^{②}(x)|^2 dx \\
&\quad + 2\int_0^\pi (x^2 - \pi x + 1)\Re\big(f^{①}(x-\pi)\overline{f^{②}(x)}\big) dx \\
&= 2\int_0^\pi (x^2 - \pi x + 1)\Re\big(f^{①}(x-\pi)\overline{f^{②}(x)}\big) dx \\
&\quad + \int_0^\pi (x^2 - \pi x + 1)\big(|f^{①}(x-\pi)|^2 + |f^{②}(x)|^2\big) dx \\
&\quad + \pi^2 \int_0^\pi |f^{①}(x-\pi)|^2 dx - \pi \int_0^\pi x|f^{①}(x-\pi)|^2 dx + \pi \int_0^\pi x|f^{②}(x)|^2 dx \\
&= \int_0^\pi (x^2 - \pi x + 1)|f^{①}(x-\pi) + f^{②}(x)|^2 dx \\
&\quad + \pi^2 \int_0^\pi |f^{①}(x-\pi)|^2 dx + \pi \int_0^\pi x|f^{②}(x)|^2 dx - \pi \int_0^\pi x|f^{①}(x-\pi)|^2 dx.
\end{aligned}$$

Using the fact that $x^2 - \pi x + 1 \leq 1$ for $x \in [0, \pi]$ and the parallelogram law, one gets

$$\| \mathcal{S}^{-1} f \|^2 \leq 2 \int_0^\pi \left| f^{①}(x-\pi) \right| a^2 + \left| f^{②}(x) \right|^2 dx \\ + \pi^2 \int_0^\pi \left| f^{①}(x-\pi) \right|^2 dx + \pi \int_0^\pi x \left| f^{②}(x) \right|^2 dx - \pi \int_0^\pi x \left| f^{①}(x-\pi) \right|^2 dx,$$

from which

$$\| \mathcal{S}^{-1} f \|^2 \leq 2 \| f \|^2 + \pi^2 \| f \|^2 = (2 + \pi^2) \| f \|^2.$$

Let us consider PW_π, the space of functions which are band-limited to $[-\pi, \pi]$, i.e.,

$$PW_\pi := \left\{ f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}),\ \operatorname{supp} \widehat{f} \subseteq L^2[-\pi, \pi] \right\}.$$

It is known that the Fourier transform $\mathcal{F}$ is a unitary operator in $L^2(\mathbb{R})$, which maps PW_π onto $L^2[-\pi, \pi]$. Any $F \in L^2[-\pi, \pi]$ can be expanded, using the dual Riesz bases $\{x_n\}_{n \in \mathbb{Z}}$ and $\{x_n^*\}_{n \in \mathbb{Z}}$, as follows:

$$F = \sum_{n \in \mathbb{Z}} \langle F, x_n^* \rangle x_n = \sum_{n \in \mathbb{Z}} \langle F, e^{-ix2n} / \sqrt{\pi} \rangle \, x_{2n} + \sum_{n \in \mathbb{Z}} \langle F, ixe^{-ix2n} / \sqrt{\pi} \rangle \, x_{2n+1}.$$

If we denote $f = \mathcal{F}^{-1}(F)$, taking into account that

$$\left\langle F, e^{-ix2n} / \sqrt{\pi} \right\rangle = \sqrt{2} f(2n)$$

and

$$\left\langle F, ixe^{-ix2n} / \sqrt{\pi} \right\rangle = -\sqrt{2} f'(2n),$$

we have

$$F = \sqrt{2} \sum_{n \in \mathbb{Z}} \left[f(2n) x_{2n} - f'(2n) x_{2n+1} \right].$$

Taking the inverse Fourier transform $\mathcal{F}^{-1}$ and using special Fourier transforms we obtain the derivative sampling expansion in PW_π

$$f(t) = \sum_{n \in \mathbb{Z}} [f(2n) + (t - 2n) f'(2n)] \operatorname{sinc}^2 \frac{1}{2}(t - 2n). \tag{9}$$

Our next goal is to estimate the truncation error in the above formula using the hypercircle inequality. In this case, the operator which transforms an orthonormal basis onto the Riesz basis $\{x_n\}_{n \in \mathbb{Z}}$ is $\mathcal{T} = \mathcal{F}^{-1} \mathcal{S}$. Since $\mathcal{F}$ is a unitary operator,

$$\| \mathcal{T}^{-1} \| = \| \mathcal{S}^{-1} \mathcal{F} \| = \| \mathcal{S}^{-1} \| \leq \sqrt{2 + \pi^2}.$$

On the other hand, the evaluation functional $\mathcal{E}_t(f) := f(t), f \in PW_\pi$, is continuous for all $t \in \mathbb{R}$ since

$$|f(t)| = \left|\left\langle F, e^{-ixt}/\sqrt{2\pi}\right\rangle\right| \le \|F\| \left\|e^{-ixt}/\sqrt{2\pi}\right\| = \|f\|.$$

Hence, for any $\|f\| \le r$, the hypercircle inequality (1) gives

$$|f(t) - f_N(t)|^2 \le \left((2+\pi^2)r^2 - 2\sum_{|n|\le N} |f(2n)|^2 + |f'(2n)|^2\right) \times \sum_{|n|>N}\left[\left|\mathcal{E}_t(\mathcal{F}^{-1}x_{2n})\right|^2 + \left|\mathcal{E}_t(\mathcal{F}^{-1}x_{2n+1})\right|^2\right]. \tag{10}$$

Moreover,

$$\mathcal{E}_t(\mathcal{F}^{-1}x_n) = \left\langle x_n, e^{-itx}/\sqrt{2\pi}\right\rangle = \left\langle \mathcal{S}w_n, e^{-itx}/\sqrt{2\pi}\right\rangle = \left\langle w_n, \mathcal{S}^*(e^{-itx}/\sqrt{2\pi})\right\rangle,$$

where $\mathcal{S}^*$ denotes the adjoint operator of $\mathcal{S}$. From the Parseval equality with respect to the orthonormal basis $\{w_n\}_{n\in\mathbb{Z}}$ we obtain

$$\sum_{|n|>N}\left[\left|\mathcal{E}_t(\mathcal{F}^{-1}x_{2n})\right|^2 + \left|\mathcal{E}_t(\mathcal{F}^{-1}x_{2n+1})\right|^2\right] \le \sum_{n\in\mathbb{Z}}\left|\mathcal{E}_t(\mathcal{F}^{-1}x_n)\right|^2 = \left\|\mathcal{S}^*(e^{-itx}/\sqrt{2\pi})\right\|^2.$$

To compute $\|\mathcal{S}^*(e^{-itx}/\sqrt{2\pi})\|$ we express the adjoint operator $\mathcal{S}^*$ in terms of the matrix $\mathcal{M}(x)$ given by (8), which represents the operator $\mathcal{S}$. Indeed, for $(\phi_1, \phi_2) \in \widehat{\mathbb{H}}$ we have that

$$\mathcal{M}(x)\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} = \begin{pmatrix}\mathcal{S}(\phi_1,\phi_2)^{①}(x-\pi)\\ \mathcal{S}(\phi_1,\phi_2)^{②}(x)\end{pmatrix},$$

a.e. in $[0,\pi]$. Consequently, for $(\phi_1, \phi_2) \in \widehat{\mathbb{H}}$ and $g \in L^2[-\pi,\pi]$ we have

$$\begin{aligned}\langle \mathcal{S}(\phi_1,\phi_2), g\rangle &= \left\langle \mathcal{S}(\phi_1,\phi_2)^{①}, g^{①}\right\rangle + \left\langle \mathcal{S}(\phi_1,\phi_2)^{②}, g^{②}\right\rangle\\ &= \int_0^\pi (\mathcal{S}(\phi_1,\phi_2)^{①}(x-\pi), \mathcal{S}(\phi_1,\phi_2)^{②}(x))\begin{pmatrix}\overline{g^{①}(x-\pi)}\\ \overline{g^{②}(x)}\end{pmatrix} dx\\ &= \int_0^\pi (\phi_1(x),\phi_2(x))\mathcal{M}^T(x)\begin{pmatrix}\overline{g^{①}(x-\pi)}\\ \overline{g^{②}(x)}\end{pmatrix} dx,\end{aligned}$$

where $\mathcal{M}^T(x)$ stands for the transpose matrix of $\mathcal{M}(x)$. Therefore, $\mathcal{S}^*g = (\varphi_1, \varphi_2)$ is given by

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \overline{\mathcal{M}^T(x)} \begin{pmatrix} g^{①}(x-\pi) \\ g^{②}(x) \end{pmatrix},$$

a.e. in $[0, \pi]$. Finally, in the Paley–Wiener space PW_π we have

$$\left\| \mathcal{S}^*\left(e^{-itx}/\sqrt{2\pi}\right) \right\| = \frac{1}{\sqrt{2\pi}} \left\| \overline{\mathcal{M}^T(x)} \begin{pmatrix} e^{-it(x-\pi)} \\ e^{-itx} \end{pmatrix} \right\|_{\widehat{\mathbb{H}}}.$$

In this case,

$$\overline{\mathcal{M}^T(x)} = \frac{1}{\pi} \begin{pmatrix} x & \pi - x \\ i & -i \end{pmatrix}.$$

Hence,

$$\begin{aligned} \mathcal{S}^*\left(e^{-itx}/\sqrt{2\pi}\right) &= \frac{1}{\pi\sqrt{2\pi}} \begin{pmatrix} x & \pi - x \\ i & -i \end{pmatrix} \begin{pmatrix} e^{-it(x-\pi)} \\ e^{-itx} \end{pmatrix} \\ &= \frac{1}{\pi\sqrt{2\pi}} \begin{pmatrix} xe^{-it(x-\pi)} + (\pi - x)e^{-itx} \\ ie^{-it(x-\pi)} - ie^{-itx} \end{pmatrix}, \end{aligned}$$

a.e. in $[0, \pi]$. Its norm in $\widehat{\mathbb{H}}$ is

$$\left\| \mathcal{S}^*\left(e^{-itx}/\sqrt{2\pi}\right) \right\|^2 = \frac{(\pi^2 - 6)\cos \pi t + 2(\pi^2 + 3)}{6\pi^2},$$

whose global maximum value is 1/2. Substituting in (10) we obtain the following estimation for the truncation error in the derivative sampling formula:

For any $f \in PW_\pi$ with $\|f\| \le r$, the truncation error in the derivative sampling formula (9) *satisfies the inequality*

$$|f(t) - f_N(t)|^2 \le \left(1 + \frac{\pi^2}{2}\right) r^2 - \sum_{n=-N}^{N} \left[|f(2n)|^2 + |f'(2n)|^2 \right].$$

One can find in [5] estimations of the truncation error for sampling formulas in the Paley–Wiener space PW_π obtained through the use of Fourier multipliers in $L^2[-\pi, \pi]$.

4. THE HYPERCIRCLE INEQUALITY IN THE FRAME SETTING

A sequence $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathbb{H}$ is said to be a frame if there exist constants $0 < A \leq B$, called the frame bounds, such that for any $x \in \mathbb{H}$ the frame inequality

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B\|x\|^2$$

holds. The following results concerning frames will be needed.

The frame operator defined as $S(x) := \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$ is a bounded positive operator in $\mathbb{H}$ with $AI \leq S \leq BI$, where I denotes the identity operator in $\mathbb{H}$. Recall that T is a positive operator in $\mathbb{H}$, and we denote $T \geq 0$, if $\langle T(x), x \rangle \geq 0$ for all $x \in \mathbb{H}$. We will write $T \leq S$ whenever $S - T \geq 0$.

The inverse S^{-1} exists and is positive in $\mathbb{H}$, and $B^{-1}I \leq S^{-1} \leq A^{-1}I$. Also $\{S^{-1}(x_n)\}_{n=1}^{\infty}$ is a frame, called the dual frame, with frame bounds B^{-1} and A^{-1}.

Any $x \in \mathbb{H}$ can be written in terms of the dual frame as

$$x = \sum_{n=1}^{\infty} \langle x, S^{-1}(x_n) \rangle x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle S^{-1}(x_n). \tag{11}$$

See, for instance, [2] for more details and proofs of the statements above.

From now on $\{x_n\}_{n=1}^{\infty}$ will denote a frame in $\mathbb{H}$ with frame bounds A and B. Its dual frame will be denoted by $\{x_n^*\}_{n=1}^{\infty}$.

THEOREM 2 *For* $w_N = \sum_{k=1}^{N} \alpha_k x_k$ *consider the hyperplane* $P = \{y \in \mathbb{H}\colon \langle y, x_k^* \rangle_{\mathbb{H}} = \alpha_k, k = 1, \ldots, N\}$, *and the hypercircle* $C_r = P \cap B_r$. *Then, for any* $x \in C_r$ *and any bounded linear functional* L *in* $\mathbb{H}$ *we have*

$$\left|L(x) - L(w_N)\right|^2 \leq \left(\frac{r^2}{A} - \sum_{k=1}^{N} |\alpha_k|^2\right) \sum_{k=N+1}^{\infty} \left|L(x_k)\right|^2. \tag{12}$$

Proof For $x \in C_r$ we have

$$x = w_N + \sum_{k=N+1}^{\infty} \langle x, x_k^* \rangle x_k.$$

Then,

$$|L(x) - L(w_N)|^2 = \left| \sum_{k=N+1}^{\infty} \langle x, x_k^* \rangle L(x_k) \right|^2.$$

Using the Cauchy–Schwarz inequality, and taking into account that $x \in P$ we obtain

$$\left|L(x) - L(w_N)\right|^2 \leq \sum_{k=N+1}^{\infty} \left|\langle x, x_k^* \rangle\right|^2 \sum_{k=N+1}^{\infty} |L(x_k)|^2$$

$$= \left(\sum_{k=1}^{\infty} \left| \langle x, x_k^* \rangle \right|^2 - \sum_{k=1}^{N} |\alpha_k|^2 \right) \sum_{k=N+1}^{\infty} \left| L(x_k) \right|^2$$

$$\leq \left(\frac{r^2}{A} - \sum_{k=1}^{N} |\alpha_k|^2 \right) \sum_{k=N+1}^{\infty} \left| L(x_k) \right|^2,$$

which concludes the proof. ■

Finally, we put to work inequality (12) estimating the truncation error for a frame expansion in a RKHS $\mathcal{H}$ with reproducing kernel k. Indeed, assume that $\{k(\cdot, t_k)\}_{k=1}^{\infty}$ is the dual frame of the frame $\{S_k\}_{k=1}^{\infty}$ with frame bounds A and B. For any $f \in \mathcal{H}$, the frame representation (11) gives $f = \sum_{k=1}^{\infty} f(t_k)\, S_k$. Consider $w_N := f_N = \sum_{k=1}^{N} f(t_k)\, S_k$, and the point-evaluation functional $\mathcal{E}_t(f) := f(t)$ in $\mathcal{H}$. Taking into account that

$$\sum_{k=1}^{\infty} \left| \mathcal{E}_t(S_k) \right|^2 = \sum_{k=1}^{\infty} \left| S_k(t) \right|^2 = \sum_{k=1}^{\infty} \left| \langle S_k, k(\cdot, t) \rangle \right|^2 \leq B \left\| k(\cdot, t) \rangle \right\|^2 = B\, k(t, t)$$

and using (12) we obtain

$$\left| f(t) - f_N(t) \right|^2 \leq \left(\frac{r^2}{A} - \sum_{k=1}^{N} \left| f(t_k) \right|^2 \right) B\, k(t, t). \tag{13}$$

Notice that f belongs to the hyperplane P given by the equations: $\langle f, k(\cdot, t_k) \rangle = f(t_k)$ for $k = 1, \ldots, N$ although $f_N \notin P$ in general.

In the particular case of a tight frame, i.e., $A = B$, we have that $S = AI$ and the frame representation reads

$$x = \frac{1}{A} \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n,$$

for every $x \in \mathbb{H}$.

It is known that $\{\sigma \operatorname{sinc} \sigma(t - n)\}_{n \in \mathbb{Z}}$ is a tight frame with bound $A = 1$ for every $\sigma < 1$ in the corresponding Paley–Wiener space $PW_{\pi\sigma}$ [4]. Then, for any $f \in PW_{\pi\sigma}$ the oversampled sampling formula

$$f(t) = \sigma \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc} \sigma(t - n) \tag{14}$$

holds. As a consequence of (13), and taking into account that the reproducing kernel is $k(t, s) = \sigma \operatorname{sinc} \sigma(t - s)$, we obtain the following result: For any $f \in PW_{\pi\sigma}$ with $\|f\| \leq r$ the truncation error in (14) satisfies the inequality

$$\left| f(t) - f_N(t) \right|^2 \leq \sigma \left(r^2 - \sum_{n=-N}^{N} \left| f(n) \right|^2 \right).$$

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