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On analytic sampling theory[☆]

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Abstract

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}})$ be a complex, separable Hilbert space with orthonormal basis $\{x_n\}_{n=1}^{\infty}$ and let Ω be a domain in \mathbb{C} , the field of complex numbers. Suppose K is a \mathbb{H} -valued function defined on Ω . For each $x \in \mathbb{H}$, define $f_x(z) = \langle K(z), x \rangle_{\mathbb{H}}$ and let \mathcal{H} denote the collection of all such functions f_x . In this paper, we endow \mathcal{H} with a structure of a reproducing kernel Hilbert space. Furthermore, we show that each element in \mathcal{H} is analytic on Ω if and only if K is analytic on Ω or, equivalently, if and only if $\langle K(z), x_n \rangle$ is analytic for each $n \in \mathbb{N}$ and $\|K(\cdot)\|_{\mathbb{H}}$ is bounded on all compact subsets of Ω . In this setting, an abstract version of the analytic Kramer theorem is exhibited. Some examples considering different \mathbb{H} spaces are given to illustrate these new results. © 2004 Published by Elsevier B.V.

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1. Introduction

For the past few years a significant mathematical literature on the topic of sampling theorems associated with differential or difference problems has flourished. As a small sample of these works we can cite, among others, [4,7,16] and the references therein. In its turn, we might consider the Weiss–Kramer theorem as the *leitmotiv* of all these sampling results [12]. Roughly speaking, the common situation for these sampling problems is the following:

[☆] The authors are very pleased to dedicate this paper to Professor W.N. Everitt on the occasion of his 80th birthday. Professor Everitt's research and mentorship have, over the years, inspired and influenced many mathematicians throughout the world; we are fortunate to be two of these mathematicians.

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Let f be a function defined on \mathbb{C} by $f(z) = \int_I F(x)K(x, z) dx$, $F \in L^2(I)$, (or $f(z) = \sum_n F(n)K(n, z)$, $F \in \ell^2$). The kernel K , which belongs to $L^2(I)$ (or ℓ^2) for each fixed $z \in \mathbb{C}$, satisfies the differential (difference) equation appearing in a differential (difference) problem (P) which has the sequence of eigenvalues $\{z_n\}$, and such that whenever we substitute the spectral parameter z by $\{z_n\}$ we obtain the sequence of orthogonal eigenfunctions associated with (P) which constitutes an orthogonal basis for $L^2(I)$ (ℓ^2). Under these circumstances, f is an entire function which can be recovered from its samples $\{f(z_n)\}$ by means of a sampling formula $f(z) = \sum_n f(z_n)S_n(z)$, where the sampling functions $\{S_n\}$ are given by $S_n(z) = \|K(\cdot, z_n)\|^{-2} \langle K(\cdot, z), K(\cdot, z_n) \rangle$ (the inner product in $L^2(I)$ or ℓ^2).

Often, the sampling functions can be written as Lagrange-type interpolation functions $S_n(z) = G(z)/((z - z_n)G'(z_n))$, where G is an entire function having simple zeros at $\{z_n\}$. In this case, for a fixed $z \in \mathbb{C}$, the expansion of the kernel $K(x, z)$ ($K(n, z)$) in the orthogonal basis $\{K(x, z_n)\}$ ($\{K(m, z_n)\}_n$) has the form

$$K(x, z) = \sum_n \frac{A_n G(z)}{z - z_n} \frac{K(x, z_n)}{\|K(\cdot, z_n)\|} \left(K(m, z) = \sum_n \frac{A_n G(z)}{z - z_n} \frac{K(m, z_n)}{\|K(\cdot, z_n)\|} \right),$$

where the coefficients are entire functions (the A_n are constants), and $\sum_n |A_n G(z)/(z - z_n)|^2 = \|K(\cdot, z)\|^2$ is bounded on compact sets of \mathbb{C} (see, for instance, [7,16]). Roughly speaking, the form assumed for the coefficients comes out, in general, by applying the Green’s formula (or Lagrange’s formula) associated with the differential or difference problem.

As it can be easily checked in the mathematical literature, these claims are a common pattern for a large class of sampling theorems associated with differential or difference problems.

In this paper we consider functions defined on a domain $\Omega \subseteq \mathbb{C}$ as $f_x(z) = \langle K(z), x \rangle_{\mathbb{H}}$, where K is a kernel defined on Ω and taking values in a separable Hilbert space \mathbb{H} , and $x \in \mathbb{H}$. It is straightforward to endow the linear space \mathcal{H} of all the functions f obtained in this way when x ranges over \mathbb{H} with a structure of a reproducing kernel Hilbert space (see Section 2.1). Next, we characterize \mathcal{H} as a Hilbert space of analytic functions on Ω : a necessary and sufficient condition is that the kernel K be analytic on Ω . Taking advantage from the examples mentioned above, we give another characterization in terms of the coefficients of the expansion of $K(z)$ with respect to an orthonormal basis in \mathbb{H} . Furthermore, whenever these coefficients satisfy an interpolatory condition at a sequence $\{z_n\}$ in Ω , then there exists a sampling expansion for functions in \mathcal{H} . Finally, we illustrate the theory with examples which use different Hilbert spaces \mathbb{H} .

2. General setting for anti-linear analytic transforms

Let \mathbb{H} be a complex, separable Hilbert space and let Ω be a domain in the complex plane \mathbb{C} . We consider a function K defined on Ω and taking values in \mathbb{H} , i.e.,

$$K : \Omega \rightarrow \mathbb{H} \\ z \rightarrow K(z).$$

Using this function K we define a mapping between \mathbb{H} and the set \mathbb{C}^Ω of all functions between Ω and \mathbb{C} as follows:

$$T : \mathbb{H} \ni x \mapsto T(x) = f_x \quad \text{such that} \quad f_x(z) := \langle K(z), x \rangle_{\mathbb{H}} \quad \text{for} \quad z \in \Omega. \tag{2.1}$$

The mapping T is anti-linear, i.e., $T(\alpha x + \beta y) = \bar{\alpha}T(x) + \bar{\beta}T(y)$ for $x, y \in \mathbb{H}$ and $\alpha, \beta \in \mathbb{C}$. In the sequel we omit the subscript in f_x except it is strictly necessary.

We denote by \mathcal{H}_K the linear space of all functions $f : \Omega \rightarrow \mathbb{C}$ in the range space of T , i.e., $\mathcal{H}_K = T(\mathbb{H})$. For notational ease we will denote $\mathcal{H}_K := \mathcal{H}$ throughout the paper. From now on, we refer the function K as the kernel of the anti-linear mapping T .

2.1. The Hilbert space structure of \mathcal{H}

Next, we endow \mathcal{H} with the structure of a Hilbert space. Concretely, we will see that \mathcal{H} is a reproducing kernel Hilbert space, RKHS hereafter. Although most of these results are well established in the mathematical literature we include them here for the sake of completeness (see for instance the superb Ref. [13] or the classical references therein).

Consider in \mathbb{H} the set $M := \{x \in \mathbb{H} \mid T(x) = 0\}$. It is easy to check that M is a closed subspace of \mathbb{H} . We endow \mathcal{H} with the norm of the quotient space $\mathbb{H} = \mathbb{H}/M$: For $f \in \mathcal{H}$ we set

$$\|f\|_{\mathcal{H}} := \|\bar{x}\|_{\mathbb{H}} = \inf_{m \in M} \{\|x + m\|_{\mathbb{H}}\} = \inf \{\|x\|_{\mathbb{H}} : f = T(x)\},$$

where \bar{x} denotes the coset of x , i.e., $\bar{x} = \{x + m \mid m \in M\}$.

In fact, the infimum is actually reached: There exists $\tilde{x} \in \mathbb{H}$ such that $f = T(\tilde{x})$ and

$$\|\bar{x}\|_{\mathbb{H}} = \|\tilde{x}\|_{\mathbb{H}} = \|f\|_{\mathcal{H}}.$$

Indeed, given $x, x' \in \bar{x}$ we can write

$$x = P_M(x) + P_{M^\perp}(x); \quad x' = P_M(x') + P_{M^\perp}(x'),$$

where P_M and P_{M^\perp} denote the orthogonal projection onto M and M^\perp , respectively. Since $x - x' \in M$, it implies that $P_{M^\perp}(x) = P_{M^\perp}(x') = \tilde{x}$ for all elements in the coset \bar{x} . Therefore, for any $x \in \bar{x}$ we have $\|x\|^2 = \|P_M(x)\|^2 + \|\tilde{x}\|^2$ and, as a consequence, $\|\bar{x}\|_{\mathbb{H}} = \inf_{x \in \bar{x}} \|x\|_{\mathbb{H}} = \|\tilde{x}\|_{\mathbb{H}}$.

By using the polarization identity it is easy to prove that $\langle f, g \rangle_{\mathcal{H}} = \langle \tilde{x}, \tilde{y} \rangle_{\mathbb{H}}$ whenever $\|f\|_{\mathcal{H}} = \|\tilde{x}\|_{\mathbb{H}}$ and $\|g\|_{\mathcal{H}} = \|\tilde{y}\|_{\mathbb{H}}$.

Consequently, the anti-linear mapping

$$\tilde{T} : M^\perp \ni \tilde{x} \mapsto f \in \mathcal{H} \quad \text{such that} \quad f(z) := \langle K(z), \tilde{x} \rangle_{\mathbb{H}} \text{ for } z \in \Omega,$$

is a bijective isometry. Hence $T : \mathbb{H} \rightarrow \mathcal{H}$ is injective if and only if the anti-linear mapping T is an isometry, or equivalently, if and only if the set $\{K(z)\}_{z \in \Omega}$ is complete in \mathbb{H} . In particular, if there exists a sequence $\{z_n\}_{n=1}^\infty$ in Ω such that $\{K(z_n)\}_{n=1}^\infty$ is an orthogonal basis for \mathbb{H} , then T is an anti-linear isometry from \mathbb{H} onto \mathcal{H} .

The function space \mathcal{H} is a RKHS. Indeed, for fixed $z \in \Omega$, for each $f \in \mathcal{H}$ we have that $f(z) = \langle K(z), \tilde{x} \rangle_{\mathbb{H}}$. By using the Cauchy–Schwarz inequality we obtain

$$|f(z)| \leq \|K(z)\|_{\mathbb{H}} \|\tilde{x}\|_{\mathbb{H}} = C_z \|f\|_{\mathcal{H}},$$

that is, all the evaluation functionals $E_z(f) = f(z)$, $f \in \mathcal{H}$, are bounded. As a consequence, convergence in the norm $\|\cdot\|_{\mathcal{H}}$ implies pointwise convergence which will be uniform on subsets of Ω where $\|K(\cdot)\|_{\mathbb{H}}$ is bounded.

Its reproducing kernel is given by $k(z, \omega) = \langle K(z), K(\omega) \rangle_{\mathbb{H}}$. Indeed,

$$\begin{aligned} \langle f, k(\cdot, \omega) \rangle_{\mathcal{H}} &= \langle T(x), T(K(\omega)) \rangle_{\mathcal{H}} = \langle \overline{P_{M^\perp}(x)}, \overline{P_{M^\perp}(K(\omega))} \rangle_{\mathbb{H}} \\ &= \langle \tilde{x}, \overline{P_{M^\perp}(K(\omega))} \rangle_{\mathbb{H}} = \langle \tilde{x}, \overline{K(\omega)} \rangle_{\mathbb{H}} = f(\omega), \end{aligned}$$

for each $\omega \in \Omega$.

2.2. Analyticity of the elements in \mathcal{H}

On the other hand, concerning whether \mathcal{H} is a RKHS of analytic functions on Ω , we give the following result.

Theorem 2.1. *\mathcal{H} is a RKHS of analytic functions in Ω if and only if the kernel K is analytic in Ω .*

Proof. It is a straightforward consequence of Theorem 1.1 in [14, p. 266]. Essentially, this theorem says that a function $F : \Omega \subseteq \mathbb{C} \rightarrow X$, where X is a complex Banach space, is analytic in Ω if and only if for each $x' \in X'$, its topological dual space, the function $x' \circ F : \Omega \rightarrow \mathbb{C}$ is analytic in Ω . \square

Notice that, for fixed $\omega \in \Omega$, the function $k_\omega(\cdot) := k(\cdot, \omega)$ is analytic in Ω . Hence, the function $\overline{k_\omega(\bar{z})}$ is analytic for $z \in \bar{\Omega} := \{z \in \mathbb{C} \mid \bar{z} \in \Omega\}$. This means that, for fixed $\omega \in \Omega$, the function $k(\omega, \bar{z})$ is analytic in $\bar{\Omega}$ since $k(\omega, \bar{z}) = \overline{k(\bar{z}, \omega)} = \overline{k_\omega(\bar{z})}$.

Next, suppose that an orthonormal basis $\{x_n\}_{n=1}^\infty$ for \mathbb{H} is given. Expanding $K(z)$, where $z \in \Omega$ is fixed, in this basis we obtain

$$K(z) = \sum_{n=1}^\infty \langle K(z), x_n \rangle_{\mathbb{H}} x_n,$$

where the coefficients $S_n(z) := \langle K(z), x_n \rangle_{\mathbb{H}}$ are in \mathcal{H} . The following result holds.

Theorem 2.2. *\mathcal{H} is a RKHS of analytic functions in Ω if and only if the functions $\{S_n\}_{n=1}^\infty$ are analytic in Ω and $\|K(\cdot)\|_{\mathbb{H}}$ is bounded on compact sets of Ω .*

Proof. The necessary condition is obvious. The sufficient condition is a straightforward consequence of Montel’s theorem. Indeed, K will be analytic in Ω if and only if the function $f_x(z) := \langle K(z), x \rangle_{\mathbb{H}}$ is analytic in Ω for each $x \in \mathbb{H}$. By using the continuity of the inner product we get

$$f_x(z) = \langle K(z), x \rangle_{\mathbb{H}} = \left\langle \sum_{n=1}^\infty S_n(z) x_n, x \right\rangle_{\mathbb{H}} = \sum_{n=1}^\infty S_n(z) \langle x_n, x \rangle_{\mathbb{H}}. \tag{2.2}$$

Applying the Cauchy–Schwarz inequality we get

$$\left| \sum_{n=1}^N S_n(z) \langle x_n, x \rangle_{\mathbb{H}} \right|^2 \leq \left(\sum_{n=1}^N |S_n(z)|^2 \right) \left(\sum_{n=1}^N |\langle x_n, x \rangle_{\mathbb{H}}|^2 \right) \leq \sum_{n=1}^\infty |S_n(z)|^2 \|x\|^2.$$

Since $\|K(z)\|_{\mathbb{H}}^2 = \sum_{n=1}^\infty |S_n(z)|^2$, the partial sums in the series of (2.2) are uniformly bounded on compact sets of Ω . Consequently, Montel’s theorem assures the existence of a subsequence which

will converge to an analytic function in Ω which necessarily coincides with f_x . Hence, f_x is analytic in Ω . \square

Moreover, Theorem 2.2 still remains true when one considers sequences $\{x_n\}_{n=1}^\infty$ more general than orthonormal bases: namely, frames in \mathbb{H} . Recall that a sequence $\{x_n\}_{n=1}^\infty$ in a separable Hilbert space \mathbb{H} is a *frame* if there exist two positive constants A and B such that

$$A \|x\|^2 \leq \sum_{n=1}^\infty |\langle x, x_n \rangle_{\mathbb{H}}|^2 \leq B \|x\|^2,$$

for all $x \in \mathbb{H}$. In terms of the *dual frame* $\{S^{-1}x_n\}_{n=1}^\infty$, for each $x \in \mathbb{H}$ we have the representation

$$x = \sum_{n=1}^\infty \langle x, x_n \rangle_{\mathbb{H}} S^{-1}x_n = \sum_{n=1}^\infty \langle x, S^{-1}x_n \rangle_{\mathbb{H}} x_n.$$

Recall that S^{-1} is the inverse of the frame operator $S : \mathbb{H} \rightarrow \mathbb{H}$ which is defined by

$$Sx = \sum_{n=1}^\infty \langle x, x_n \rangle_{\mathbb{H}} x_n \quad \text{for each } x \in \mathbb{H}.$$

See, for instance, [2, pp. 464–474] for more details about frames.

In order to characterize \mathcal{H} as a RKHS of analytic functions in Ω in terms of the coefficients of the expansion of $K(z)$ with respect to a frame $\{x_n\}_{n=1}^\infty$ in \mathbb{H} :

$$K(z) = \sum_{n=1}^\infty \langle K(z), x_n \rangle_{\mathbb{H}} S^{-1}x_n = \sum_{n=1}^\infty S_n(z) S^{-1}x_n,$$

the following result holds.

Theorem 2.3. *Let $\{x_n\}_{n=1}^\infty$ be a frame for \mathbb{H} . Then, \mathcal{H} is a RKHS of analytic functions in Ω if and only if the functions $S_n(z) := \langle K(z), x_n \rangle_{\mathbb{H}}$ are analytic in Ω and $\|K(\cdot)\|_{\mathbb{H}}$ is bounded on compact sets of Ω .*

Proof. The proof is similar to the one given in Theorem 2.2 using the inequalities

$$A \|K(z)\|^2 \leq \sum_{n=1}^\infty |S_n(z)|^2 \leq B \|K(z)\|^2, \quad z \in \Omega. \quad \square$$

A final remark is in order concerning the cases of $\mathbb{H} = L^2(I)$, where I denotes an interval of the real axis, or $\mathbb{H} = \ell^2(\mathbb{I})$, where \mathbb{I} denotes a denumerable set of indexes in \mathbb{Z} . In the first case, given an integral kernel $\mathcal{K}(z, x)$ such that, for each fixed $z \in \Omega$, $\mathcal{K}(z, \cdot) \in L^2(I)$, any function f defined as

$$f(z) = \int_I F(x) \mathcal{K}(z, x) dx, \quad z \in \Omega, \quad F \in L^2(I),$$

can be written as $f(z) = \langle K(z), \tilde{F} \rangle_{L^2(I)}$, where $[K(z)](\cdot) := \mathcal{K}(z, \cdot)$. In [5] it is proved that the analyticity of $\mathcal{K}(z, x)$ with respect to the first variable and the local boundedness of $\|\mathcal{K}(z, \cdot)\|_{L^2(I)}$ implies that f is analytic on Ω . However, this condition is not necessary for the analyticity of f as one can see in [3] where a counterexample is given.

The same occurs when $f(z) = \sum_{n \in \mathbb{N}} F(n) \mathcal{K}(z, n)$, $z \in \Omega$ and $F \in \ell^2(\mathbb{N})$, and we set $f(z) = \langle K(z), \bar{F} \rangle_{\ell^2(\mathbb{N})}$ where $K(z) \in \ell^2(\mathbb{N})$ is defined as $[K(z)](n) := \mathcal{K}(z, n)$ (see [8] for the details).

3. Analytic sampling theory

For our sampling purposes we will suppose the existence of a sequence of points $\{z_n\}_{n=1}^\infty$ in Ω such that $K(z_n) = a_n x_n$, $n \in \mathbb{N}$, for some nonzero constants $\{a_n\}_{n=1}^\infty$, where $\{x_n\}_{n=1}^\infty$ denotes an orthonormal basis for \mathbb{H} . This is equivalent to saying that the sequence of functions $\{S_n\}_{n=1}^\infty$, where $S_n(z) := \langle K(z), x_n \rangle_{\mathbb{H}}$, satisfy at $\{z_n\}_{n=1}^\infty$ the interpolation property:

$$S_n(z_m) = a_n \delta_{n,m}. \tag{3.1}$$

In this case, the following sampling result holds:

Theorem 3.1. *Let $K : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{H}$ be an analytic kernel on an unbounded domain Ω . Assume that the interpolation property (3.1) holds for some sequences $\{z_n\}_{n=1}^\infty$ in Ω and $\{a_n\}_{n=1}^\infty$ in $\mathbb{C} \setminus \{0\}$. Let \mathcal{H} be the corresponding RKHS of analytic functions. Then any $f \in \mathcal{H}$ can be recovered from its samples $\{f(z_n)\}_{n=1}^\infty$ by means of the sampling series*

$$f(z) = \sum_{n=1}^\infty f(z_n) \frac{S_n(z)}{a_n}. \tag{3.2}$$

The series converges absolutely and uniformly on compact subsets of Ω .

Proof. First notice that $\lim_{n \rightarrow \infty} |z_n| = +\infty$ except in the trivial case when any f is identically zero in Ω , which is already excluded by assumption (3.1). The anti-linear mapping T is a bijective isometry between \mathbb{H} and \mathcal{H} . As a consequence, $\{S_n = T(x_n)\}_{n=1}^\infty$ will be an orthonormal basis for \mathcal{H} . Expanding any $f = T(x)$ in \mathcal{H} in this basis we obtain

$$f(z) = \sum_{n=1}^\infty \langle f, S_n \rangle_{\mathcal{H}} S_n(z).$$

Moreover,

$$\langle f, S_n \rangle_{\mathcal{H}} = \overline{\langle x, x_n \rangle_{\mathbb{H}}} = \left\langle \frac{K(z_n)}{a_n}, x \right\rangle_{\mathbb{H}} = \frac{f(z_n)}{a_n}.$$

Since an orthonormal basis is an unconditional basis, the sampling series will be pointwise unconditionally convergent and hence, absolutely convergent. The uniform convergence is a standard result in the setting of the RKHS theory since $\|K(\cdot)\|_{\mathbb{H}}$ is bounded on compact subsets of Ω (see Section 2.1). \square

Corollary 3.1. *Assuming all conditions in Theorem 3.1, let $\{c_n\}_{n=1}^\infty$ be a sequence of complex numbers such that $\{c_n/a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$. Then there exists a unique function $f \in \mathcal{H}$ which satisfies $f(z_n) = c_n$ for each $n \in \mathbb{N}$.*

Proof. Since $\{S_n\}_{n=1}^\infty$ is an orthonormal basis for \mathcal{H} , by using the Riesz-Fischer theorem there exists a unique $f \in \mathcal{H}$ such that $f(z) = \sum_{n=1}^\infty (c_n/a_n) S_n(z)$. Hence, $f(z_m) = c_m$ for each $m \in \mathbb{N}$. \square

Whenever $\{x_n\}_{n=1}^\infty$ is a Riesz basis for \mathbb{H} , a similar sampling theorem for \mathcal{H} holds. In this case (3.2) is not an orthonormal expansion in \mathcal{H} : It is an expansion with respect to a Riesz basis in \mathcal{H} . Recall that a Riesz basis $\{x_n\}_{n=1}^\infty$ for a separable Hilbert space \mathbb{H} is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthonormal (dual) Riesz basis $\{x_n^*\}_{n=1}^\infty$, i.e., $\langle x_n, x_m^* \rangle_{\mathbb{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^\infty \langle x, x_n^* \rangle_{\mathbb{H}} x_n = \sum_{n=1}^\infty \langle x, x_n \rangle_{\mathbb{H}} x_n^*$$

hold for every $x \in \mathbb{H}$ (see [15] for more details and proofs).

In our case, assume that $K(z_n) = a_n x_n$, where $\{x_n\}_{n=1}^\infty$ is a Riesz basis for \mathbb{H} with dual basis $\{x_n^*\}_{n=1}^\infty$. Since T is an isometry, $\{S_n := T(x_n)\}_{n=1}^\infty$ and $\{S_n^* := T(x_n^*)\}_{n=1}^\infty$ will be dual Riesz bases in \mathcal{H} . Expanding any function f in \mathcal{H} in the basis $\{S_n^*\}_{n=1}^\infty$ one gets

$$f(z) = \sum_{n=1}^\infty \langle f, S_n \rangle_{\mathcal{H}} S_n^*(z) = \sum_{n=1}^\infty \frac{f(z_n)}{a_n} S_n^*(z).$$

Riesz bases are the only bases which ensure stable sampling (see [10, p. 103]).

Theorem 3.1 can also be restated as an abstract version of the analytic Kramer theorem in [5].

Theorem 3.2. *Let $K : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{H}$ be a kernel satisfying:*

- (1) Ω is an unbounded domain in \mathbb{C} ;
- (2) K is analytic in Ω ;
- (3) There exists a sequence $\{z_n\}_{n=1}^\infty$ in Ω satisfying $\lim_{n \rightarrow \infty} |z_n| = +\infty$ such that $\{K(z_n)\}_{n=1}^\infty$ is an orthogonal basis for \mathbb{H} .

Then, any function defined in Ω by $f(z) = \langle K(z), x \rangle_{\mathbb{H}}$, where $x \in \mathbb{H}$ can be expanded as

$$f(z) = \sum_{n=1}^\infty f(z_n) S_n(z),$$

where the sampling functions S_n are given by $S_n(z) := \|K(z_n)\|^{-2} \langle K(z), K(z_n) \rangle_{\mathbb{H}}$. The series converges absolutely and uniformly on compact subsets of Ω .

As in [9], we can similarly obtain sampling expansions which use samples of the derivative as well. To this end, we have also to impose some interpolation conditions to the coefficients of the expansions of $K(z)$ and $K'(z)$ with respect to a suitable orthonormal basis (or Riesz basis) $\{x_n\}_{n=1}^\infty \cup \{y_n\}_{n=1}^\infty$ for \mathbb{H} (this basis might be obtained by partitioning a given orthonormal basis into two arbitrary sequences). Namely, if $K(z)$ admits the expansion

$$K(z) = \sum_{n=1}^\infty S_n(z) x_n + \sum_{n=1}^\infty T_n(z) y_n,$$

then

$$K'(z) = \sum_{n=1}^\infty S'_n(z) x_n + \sum_{n=1}^\infty T'_n(z) y_n.$$

Assume that there exists a sequence $\{z_n\}_{n=1}^\infty$ in Ω such that the interpolation conditions

$$S_n(z_m) = a_n \delta_{n,m}; \quad T_n(z_m) = b_n \delta_{n,m}, \quad \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \subset \mathbb{C},$$

$$S'_n(z_m) = c_n \delta_{n,m}; \quad T'_n(z_m) = d_n \delta_{n,m}, \quad \{c_n\}_{n=1}^\infty, \{d_n\}_{n=1}^\infty \subset \mathbb{C}$$

hold with $\Delta_n := a_n d_n - b_n c_n \neq 0$ for all $n \in \mathbb{N}$. Under these hypotheses, the mapping T in (2.1) is one-to-one. Indeed, $f_x = f_y$ implies $f_{x-y} = 0$ and hence $f'_{x-y} = 0$. In particular,

$$f_{x-y}(z_n) = \langle K(z_n), x - y \rangle_{\mathbb{H}} = a_n \langle x_n, x - y \rangle_{\mathbb{H}} + b_n \langle y_n, x - y \rangle_{\mathbb{H}} = 0,$$

$$f'_{x-y}(z_n) = \langle K'(z_n), x - y \rangle_{\mathbb{H}} = c_n \langle x_n, x - y \rangle_{\mathbb{H}} + d_n \langle y_n, x - y \rangle_{\mathbb{H}} = 0$$

for all $n \in \mathbb{N}$. Since $\Delta_n := a_n d_n - b_n c_n \neq 0$ for all $n \in \mathbb{N}$, the completeness of the basis implies $x = y$. Consequently, the mapping T is a bijective isometry and $\{S_n = T(x_n)\}_{n=1}^\infty \cup \{T_n = T(y_n)\}_{n=1}^\infty$ is an orthonormal basis for \mathcal{H} . Expanding $f = T(x)$ in \mathcal{H} with respect this orthonormal basis we obtain

$$f = \sum_{n=1}^\infty [\langle f, S_n \rangle_{\mathcal{H}} S_n + \langle f, T_n \rangle_{\mathcal{H}} T_n]. \tag{3.3}$$

From the equalities

$$f(z_n) = \langle K(z_n), x \rangle_{\mathbb{H}} = a_n \langle x_n, x \rangle_{\mathbb{H}} + b_n \langle y_n, x \rangle_{\mathbb{H}},$$

$$f'(z_n) = \langle K'(z_n), x \rangle_{\mathcal{H}} = c_n \langle x_n, x \rangle_{\mathbb{H}} + d_n \langle y_n, x \rangle_{\mathbb{H}},$$

we get

$$\langle x_n, x \rangle_{\mathbb{H}} = \frac{d_n f(z_n) - b_n f'(z_n)}{\Delta_n} \quad \text{and} \quad \langle y_n, x \rangle_{\mathbb{H}} = \frac{a_n f'(z_n) - c_n f(z_n)}{\Delta_n}.$$

Since T is an anti-linear isometry, $\langle x_n, x \rangle_{\mathbb{H}} = \langle f, S_n \rangle_{\mathcal{H}}$ and $\langle y_n, x \rangle_{\mathbb{H}} = \langle f, T_n \rangle_{\mathcal{H}}$. Substituting in (3.3) and after some calculations we finally obtain the sampling formula

$$f(z) = \sum_{n=1}^\infty \left[f(z_n) \frac{d_n S_n(z) - c_n T_n(z)}{\Delta_n} + f'(z_n) \frac{a_n T_n(z) - b_n S_n(z)}{\Delta_n} \right].$$

4. Some illustrative examples

We put to use the sampling theory in Section 3 with some examples which use different inner products.

Example 4.1. The first example concerns the Whittaker–Shannon–Kotel’nikov sampling theorem in classical Paley–Wiener spaces $PW_{\pi\sigma}$:

$$PW_{\pi\sigma} := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \quad \text{supp } \hat{f} \subseteq [-\pi\sigma, \pi\sigma]\},$$

where \hat{f} stands for the Fourier transform. This space can also be expressed, by using the classical Paley–Wiener theorem [15, p. 100], as

$$PW_{\pi\sigma} = \{f \in \mathcal{H}(\mathbb{C}) : |f(z)| \leq A e^{\pi\sigma|z|}, \quad f|_{\mathbb{R}} \in L^2(\mathbb{R})\},$$

i.e., entire functions of exponential type at most $\pi\sigma$ such that their restriction to the real axis are square integrable. Note that any function f in $PW_{\pi\sigma}$ can be written as

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\sigma}^{\pi\sigma} \hat{f}(x)e^{izx} dx = \left\langle \frac{e^{izx}}{\sqrt{2\pi}}, \hat{f} \right\rangle_{L^2[-\pi\sigma, \pi\sigma]}$$

with $\hat{f} \in L^2[-\pi\sigma, \pi\sigma]$. Thus, in this case, the kernel K (the Fourier kernel) is given by

$$K : \mathbb{C} \rightarrow L^2[-\pi\sigma, \pi\sigma],$$

$$z \rightarrow K(z) = \frac{e^{izx}}{\sqrt{2\pi}}.$$

Consider the orthonormal basis $\{e^{inx/\sigma}/\sqrt{2\pi\sigma}\}_{n \in \mathbb{Z}}$ in $L^2[-\pi\sigma, \pi\sigma]$. Expanding $K(z) \in L^2[-\pi\sigma, \pi\sigma]$ in this basis, we obtain

$$K(z) = \frac{1}{2\pi\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \langle e^{izx}, e^{inx/\sigma} \rangle_{L^2[-\pi\sigma, \pi\sigma]} \frac{e^{inx/\sigma}}{\sqrt{2\pi\sigma}} = \sqrt{\sigma} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(\sigma z - n)}{\pi(\sigma z - n)} \frac{e^{inx/\sigma}}{\sqrt{2\pi\sigma}}.$$

Therefore, taking

$$S_n(z) = \sqrt{\sigma} \frac{\sin \pi(\sigma z - n)}{\pi(\sigma z - n)}, \quad z_n = \frac{n}{\sigma}, \quad n \in \mathbb{Z} \quad \text{and} \quad a_n = \sqrt{\sigma}$$

we obtain the Whittaker–Shannon–Kotel’nikov sampling theorem which reads:

- Any function f in the Paley–Wiener space $PW_{\pi\sigma}$ can be expanded as the cardinal series

$$f(z) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\sigma}\right) \frac{\sin \pi(\sigma z - n)}{\pi(\sigma z - n)},$$

where the convergence in the series is absolute and uniform on horizontal strips of \mathbb{C} since $\|K(z)\|_{L^2[-\pi\sigma, \pi\sigma]} \leq \sqrt{\sigma} e^{\pi\sigma|y|}$ for all $z = x + iy \in \mathbb{C}$.

Further examples of L^2 -spaces \mathbb{H} can be found in [6].

Example 4.2. In the second example $\mathbb{H} = \ell^2(\mathbb{N}_0)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the kernel $K : \mathbb{C} \rightarrow \ell^2(\mathbb{N}_0)$ is defined as $K(z) = \{P_n(z)\}_{n=0}^{\infty}$, where $\{P_n(z)\}_{n=0}^{\infty}$ denotes a sequence of orthonormal polynomials associated with an indeterminate Hamburger moment problem (see [7] for the details). Concretely, we consider the particular case of the so-called q^{-1} -Hermite polynomials ($0 < q < 1$). These polynomials have the explicit representation [11]

$$h_n(x | q) = \sum_{k=0}^n \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} (-1)^k q^{k(k-n)} (x + \sqrt{x^2 + 1})^{n-2k},$$

where the q -shifted factorial notation is used

$$(c_1, c_2, \dots, c_p; q)_n = \prod_{k=1}^n \prod_{j=1}^p (1 - c_j q^{k-1}),$$

where $n = 0, 1, \dots, \infty$. The moment problem associated with $\{h_n(x|q)\}_{n=0}^\infty$ is indeterminate (the moment problem does not have a unique solution), and the norms of the polynomials are given by $\|h_n\| = \sqrt{(q; q)_n / q^{n(n+1)/2}}$ [11]. In this case the kernel K_q is given by

$$\begin{aligned} K_q : \mathbb{C} &\rightarrow \ell^2(\mathbb{N}_0) \\ z &\rightarrow K_q(z) = \{P_n(z)\}_{n=0}^\infty, \end{aligned}$$

where $P_n = h_n / \|h_n\|$. The series $\sum_{n=0}^\infty |P_n(z)|^2$ is uniformly bounded on compact subsets of \mathbb{C} due to the indeterminacy of the moment problem [1]. Taking, for example, the points $\pm z_m = \pm \frac{1}{2}(q^{-m-1/2} - q^{m+1/2})$, $m \in \mathbb{N}_0$, we obtain [7] that

$$\{P_0(z_m), P_1(z_m), P_2(z_m), \dots\}_{m \in \mathbb{N}_0} \cup \{P_0(-z_m), P_1(-z_m), P_2(-z_m), \dots\}_{m \in \mathbb{N}_0}$$

is an orthogonal basis for $\ell^2(\mathbb{N}_0)$ (other choices for the sequence $\{z_m\}$ are possible, see [7]). In this case, the RKHS \mathcal{H} coincides with $L^2(\mu)$, where μ is any measure solution of the moment problem for which the polynomials are dense in $L^2(\mu)$ (the so-called N -extremal measures). Expanding $K_q(z) \in \ell^2(\mathbb{N}_0)$ following the above orthogonal basis in $\ell^2(\mathbb{N}_0)$, we obtain the sampling result (see [7] for the details):

- Any function f given by

$$f(z) = \sum_{n=0}^\infty c_n P_n(z) = \langle \{P_n(z)\}, \{c_n\} \rangle_{\ell^2(\mathbb{N}_0)}$$

is an entire function which can be expanded as

$$f(z) = \sum_{n=0}^\infty f(-z_n) \frac{D(z)}{(z + z_n)D'(-z_n)} + \sum_{n=0}^\infty f(z_n) \frac{D(z)}{(z - z_n)D'(z_n)},$$

where

$$D(z) = -\frac{(qe^{2\xi}, qe^{-2\xi}; q^2)_\infty}{(q; q^2)_\infty^2}, \quad z = \sinh \xi,$$

and the sampling points are $\pm z_n = \pm \frac{1}{2}(q^{-n-1/2} - q^{n+1/2})$, $n \in \mathbb{N}_0$.

Example 4.3. In our last example, we consider the Sobolev Hilbert space $H^1(-\pi, \pi)$ with its usual inner product

$$\langle f, g \rangle_1 = \int_{-\pi}^\pi f(x) \overline{g(x)} dx + \int_{-\pi}^\pi f'(x) \overline{g'(x)} dx, \quad f, g \in H^1(-\pi, \pi).$$

The sequence $\{e^{inx}\}_{n \in \mathbb{Z}} \cup \{\sinh x\}$ forms an orthogonal basis for $H^1(-\pi, \pi)$: It is easy to prove that the orthogonal complement of $\{e^{inx}\}_{n \in \mathbb{Z}}$ in $H^1(-\pi, \pi)$ is one-dimensional and $\sinh x$ belongs to it.

Given $a \in \mathbb{C} \setminus \mathbb{Z}$ we define a kernel

$$K_a : \mathbb{C} \rightarrow H^1(-\pi, \pi)$$

$$z \rightarrow K_a(z).$$

by setting

$$[K_a(z)](x) = (z - a)e^{izx} + \sin \pi z \sinh x \quad \text{for } x \in (-\pi, \pi).$$

Expanding $K_a(z) \in H^1(-\pi, \pi)$ in the former orthogonal basis we obtain

$$K_a(z) = [1 - i(z - a)] \sin \pi z \sinh x + (z - a) \sum_{n=-\infty}^{\infty} \frac{1 + zn}{1 + n^2} \operatorname{sinc}(z - n)e^{inx}.$$

As a consequence, we obtain the following sampling result:

- The entire function f given by

$$f(z) = \int_{-\pi}^{\pi} F(x)[K_a(z)](x) dx + \int_{-\pi}^{\pi} F'(x)[K_a(z)]'(x) dx = \langle K_a(z), \bar{F} \rangle_1, \quad z \in \mathbb{C},$$

where $F \in H^1(-\pi, \pi)$, can be recovered from its samples $\{f(n)\}_{n \in \mathbb{Z}} \cup \{f(a)\}$ by means of the sampling formula

$$f(z) = [1 - i(z - a)] \frac{\sin \pi z}{\sin \pi a} f(a) + \sum_{n=-\infty}^{\infty} f(n) \frac{z - a}{n - a} \frac{1 + zn}{1 + n^2} \operatorname{sinc}(z - n).$$

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