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On the aliasing error in wavelet subspaces

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Abstract

Shannon's sampling formula has been extended to subspaces $\{V_j\}_{j\in\mathbb{Z}}$ in multiresolution analysis. When we apply the corresponding sampling formula for V_0 to a function which does not belong to this space, the so-called aliasing error arises. This paper deals with the aliasing error, Ef, whenever f is in V_1. We obtain an expression for its Fourier transform, $\widehat{Ef}$, which allows us to give estimations of the aliasing error both in $L^2(\mathbb{R})$-norm and in $L^\infty(\mathbb{R})$-norm. Finally, we carry out a study of the aliasing error in the case when f belongs to V_2.

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1. Introduction

The Shannon sampling theorem states that any function f in the classical Paley–Wiener space $PW_\pi := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \operatorname{supp}\widehat{f} \subseteq [-\pi, \pi]\}$, where $\widehat{f}$ stands for the Fourier transform of f, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(t-n), \quad t \in \mathbb{R},$$

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where sinc denotes the cardinal sine function $\mathrm{sinc}(t) = \sin \pi t / \pi t$. The space PW_π can be seen as the subspace V_0 of the Shannon multiresolution analysis, whose scaling function is precisely $\phi = \mathrm{sinc}$.

In [11], Walter extended, under appropriate hypotheses, the Shannon sampling theorem to the subspace V_0 of a general multiresolution analysis $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$: For any $f \in V_0$ the sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) S(t-n), \quad t \in \mathbb{R} \tag{1}$$

holds, where $\widehat{S}(w) := \widehat{\phi}(w) / (\sum_{n\in\mathbb{Z}} \phi(n) \mathrm{e}^{-\mathrm{i}nw})$, and ϕ denotes the scaling function.

If the sampling formula (1) is applied to a function f which does not belong to V_0, the so-called aliasing error:

$$Ef(t) := f(t) - \sum_{n\in\mathbb{Z}} f(n) S(t-n), \quad t \in \mathbb{R}$$

arises. Concerning this error in Shannon's setting, a classic result by Brown [1] states that if $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$\left| f(t) - \sum_{n\in\mathbb{Z}} f(n)\, \mathrm{sinc}(t-n) \right| \leqslant \frac{2}{\sqrt{2\pi}} \int_{|w|>\pi} |\widehat{f}(w)|\, \mathrm{d}w, \quad t \in \mathbb{R}. \tag{2}$$

In addition, the function $f(t) = \mathrm{sinc}(2t-1)$ is an extremal solution for (2), i.e., there exists a value of t for which (2) holds with equality. Notice that if $f \in V_1 = \mathrm{PW}_{2\pi}$, then (2) can be written as

$$|Ef(t)| \leqslant \frac{2}{\sqrt{2\pi}} \| P_{W_0} f \|_{L^1(\mathbb{R})},$$

where P_{W_0} denotes the orthogonal projection onto W_0, the orthogonal complement of V_0 in V_1. Besides, Walter [11] has proved a similar result for functions in the subspace V_1 of a general multiresolution analysis. Specifically, for any $f \in V_1$, there exists a constant C such that

$$|Ef(t)| \leqslant C \| P_{W_0} f \|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}. \tag{3}$$

Notice that $\| P_{W_0} f \|_{L^2(\mathbb{R})}$ can be expressed in terms of the wavelet coefficients.

On the other hand, Janssen [6] generalized Walter's sampling formula by using shifted samples $\{f(n+a)\}_{n\in\mathbb{Z}}$, where $a \in [0, 1)$. As to the corresponding aliasing error $E_a(f)$, he proved the inequalities

$$K_0 \| P_{W_0} f \|_{L^2(\mathbb{R})} \leqslant \| E_a f \|_{L^2(\mathbb{R})} \leqslant K_\infty \| P_{W_0} f \|_{L^2(\mathbb{R})}, \quad f \in V_1.$$

In addition, he found the smallest possible value for the constant K_0 and the largest possible value for K_∞.

In this paper, we deal with the aliasing error function $E_a(f)$ for $f \in V_1$. We will calculate its Fourier transform in terms of the Fourier transform of $P_{W_0} f$. Besides recovering Janssen's inequalities, this will allow us to derive a precise bound like (3). Also, we provide extremal solutions in some cases. When ϕ is a cardinal scaling function, i.e., when it satisfies $\phi(n) = \delta_{n,0}$, the $L^2(\mathbb{R})$-norm of the aliasing error becomes especially simple: $\| Ef \|_{L^2(\mathbb{R})} = \sqrt{2} \| P_{W_0} f \|_{L^2(\mathbb{R})}$. Furthermore, we include some results concerning the aliasing error for functions $f \in V_2$. Finally, we illustrate the obtained results in some classical multiresolution analyses.

The aliasing error in Shannon's setting has been largely studied (see [4] and references therein). Besides the aforesaid Janssen's reference, see [10,14] for the general wavelet setting.

The paper is organized as follows: In Section 2 we include the needed preliminaries on multiresolution analysis and sampling. Section 3 is devoted to obtain the Fourier transform of the aliasing error function for $f \in V_1$. In Section 4 we obtain, using the main result in Section 3, estimations of the aliasing error function either in $L^2(\mathbb{R})$-norm or in $L^\infty(\mathbb{R})$-norm. In Section 5 we deal with the aliasing error for functions in V_2. Finally, in Section 6 we particularize our results in Shannon, Meyer and Spline multiresolution analyses.

2. Preliminaries on multiresolution analysis and sampling

Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis in $L^2(\mathbb{R})$ with an orthonormal scaling function ϕ, i.e., $\{V_j\}_{j\in\mathbb{Z}}$ is a sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying:

(i) $V_j = D^j V_0,\ j \in \mathbb{Z}$, where D is the dilation operator, defined by $Df(t) := \sqrt{2} f(2t)$.
(ii) $V_0 \subset V_1$.
(iii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(iv) $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0.

Let W_j be the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j$, and let $P_{W_j} f$ and $P_{V_j} f$ be the orthogonal projections of $f \in L^2(\mathbb{R})$ onto W_j and V_j, respectively.

It is well known (see for example [3] or [13]), that a function f belongs to V_1 if and only if there exists a unique 2π-periodic function in $L^2(0, 2\pi)$, denoted by m_f, such that

$$\widehat{f}(w) = m_f(w/2)\widehat{\phi}(w/2), \tag{4}$$

where the Fourier transform in $L^2(\mathbb{R})$ is defined as $\widehat{f}(w) := (1/\sqrt{2\pi}) \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-\mathrm{i}wt} \mathrm{d}t$. Equality (4) must be understood almost everywhere (a.e.), i.e., it holds except on a set of Lebesgue measure zero (this will be the case for other equalities throughout the paper). Besides, for each $f \in V_1$

$$\|f\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{\pi}} \|m_f\|_{L^2(0,2\pi)}. \tag{5}$$

Since $\phi \in V_0 \subset V_1$, we have $\widehat{\phi}(w) = m_\phi(w/2)\widehat{\phi}(w/2)$, which is called the scaling equation. Let ψ be the mother wavelet whose Fourier transform is

$$\widehat{\psi}(w) := \mathrm{e}^{\mathrm{i}w/2}\overline{m_\phi(w/2+\pi)}\widehat{\phi}(w/2)$$

and consequently, $m_\psi(w) = \mathrm{e}^{\mathrm{i}w}\overline{m_\phi(w+\pi)}$. The sequence $\{\psi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the subspace W_0. The functions m_ϕ and m_ψ satisfy the relationships:

$$|m_\phi(w)|^2 + |m_\phi(w+\pi)|^2 = 1, \quad |m_\psi(w)|^2 + |m_\psi(w+\pi)|^2 = 1,$$

$$m_\phi(w/2)\overline{m_\psi(w/2)} + m_\phi(w/2+\pi)\overline{m_\psi(w/2+\pi)} = 0. \tag{6}$$

For every $f \in V_1$, the functions given by

$$u_f(w) := m_f(w/2)\overline{m_\phi(w/2)} + m_f(w/2+\pi)\overline{m_\phi(w/2+\pi)},$$
$$v_f(w) := m_f(w/2)\overline{m_\psi(w/2)} + m_f(w/2+\pi)\overline{m_\psi(w/2+\pi)} \tag{7}$$

are the unique 2π-periodic functions in $L^2(0, 2\pi)$ verifying

$$\widehat{P_{V_0} f}(w) = u_f(w)\widehat{\phi}(w) \quad \text{and} \quad \widehat{P_{W_0} f}(w) = v_f(w)\widehat{\psi}(w). \tag{8}$$

Moreover, for $f \in V_1$ we have

$$\|P_{W_0} f\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|v_f\|^2_{L^2(0,2\pi)}. \tag{9}$$

See [7] for a reference on multiresolution analysis and applications.

The Zak transform of $f \in L^2(\mathbb{R})$, formally defined as

$$(Zf)(t, w) := \sum_{n\in\mathbb{Z}} f(n+t)\mathrm{e}^{-\mathrm{i}wn}, \quad t, w \in \mathbb{R},$$

will be an important tool in the sequel (see [5] for the properties on the Zak transform).

For sampling purposes, we assume that the scaling function ϕ is continuous in $\mathbb{R}$, and satisfies $\phi(t) = O(|t|^{-s})$), when $|t| \to \infty$, for some $s > 1/2$. In addition, we assume that the inequalities $0 < \|(Z\phi)(a, \cdot)\|_0 \leqslant \|(Z\phi)(a, \cdot)\|_\infty < \infty$ hold, where

$$\|(Z\phi)(a, \cdot)\|_\infty := \operatorname*{ess\,sup}_{w\in\mathbb{R}} |(Z\phi)(a, w)| \quad \text{and} \quad \|(Z\phi)(a, \cdot)\|_0 := \operatorname*{ess\,inf}_{w\in\mathbb{R}} |(Z\phi)(a, w)|.$$

(Essential supremum and essential infimum definitions can be found in [8]). Thus, for any $f \in V_0$ the following sampling formula holds

$$f(t) = \sum_{n\in\mathbb{Z}} f(n+a)\ S_a(t-n), \quad t \in \mathbb{R}, \tag{10}$$

where $\widehat{S_a}(w) := \widehat{\phi}(w)/(Z\phi)(a, w)$ (see [2] or [15]). The convergence of the series in (10) is in the $L^2(\mathbb{R})$-norm sense, absolute and uniform on $\mathbb{R}$.

In order to use the Poisson summation formula, we will assume throughout the paper that the series $\sum_{n\in\mathbb{Z}} |\widehat{\phi}(w + 2\pi n)|$ is uniformly bounded a.e. in $\mathbb{R}$. This condition is satisfied if, for instance, $\widehat{\phi}(w) = O((1 + |w|)^{-s})$, $w \in \mathbb{R}$, for some $s > 1$.

3. The Fourier transform of the aliasing error function for $f \in V_1$

For any $f \in V_j$, $j \geqslant 0$, the aliasing error function $E_a f$ is defined by

$$E_a f(t) := f(t) - \sum_{n\in\mathbb{Z}} f(n+a)S_a(t-n), \quad t \in \mathbb{R}.$$

Consider the subspace of V_1 defined by

$$M_a := \{f \in V_1 : f(n+a) = 0,\ n \in \mathbb{Z}\}.$$

In this section, for any $f \in V_1$, we will construct two functions, $Q_{V_0} f \in V_0$ and $Q_{M_a} f \in M_a$, such that $f = Q_{V_0} f + Q_{M_a} f$ holds. The function $Q_{M_a} f$ will be precisely the aliasing error function $E_a f$ because

$$E_a f = E_a Q_{V_0} f + E_a Q_{M_a} f = E_a Q_{M_a} f = Q_{M_a} f. \tag{11}$$

The two following lemmas are devoted to the Zak transform of functions in V_1.

Lemma 1. *Let f be in V_1. For a fixed $t \in \mathbb{R}$, we have*

$$(Zf)(t,w) = \sqrt{2\pi} \mathrm{e}^{\mathrm{i}wt} \sum_{n\in\mathbb{Z}} \widehat{f}(w + 2\pi n) \mathrm{e}^{\mathrm{i}2\pi nt}, \quad \text{a.e. } w \in \mathbb{R}. \tag{12}$$

Proof. If $f \in V_1$, using (4) and splitting the sum into odd and even terms we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{f}(w+2\pi n)| &= \sum_{n\in\mathbb{Z}} |m_f(w/2 + n\pi)\widehat{\phi}(w/2+\pi n)| \\ &= |m_f(w/2)| \sum_{n\in\mathbb{Z}} |\widehat{\phi}(w/2 + 2\pi n)| \\ &\quad + |m_f(w/2+\pi)| \sum_{n\in\mathbb{Z}} |\widehat{\phi}(w/2 + \pi + 2\pi n)|.\end{aligned}$$

Since we have supposed that $\sum_{n\in\mathbb{Z}} |\widehat{\phi}(w + 2\pi n)|$ is uniformly bounded, we obtain that the series $\sum_{n\in\mathbb{Z}} |\widehat{f}(w + 2\pi n)|$ converges a.e. to a function in $L^2(0, 2\pi)$. It can be easily checked that $\sqrt{2\pi}\mathrm{e}^{\mathrm{i}wt}\sum_{n\in\mathbb{Z}} \widehat{f}(w + 2\pi n)\ \mathrm{e}^{\mathrm{i}2\pi nt}$ belongs to $L^2(0, 2\pi)$, and its Fourier coefficients with respect to the orthonormal basis $\{\mathrm{e}^{-\mathrm{i}wn}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ are $\{\sqrt{2\pi}\ f(t+n)\}_{n\in\mathbb{Z}}$. Hence, the equality in (12) holds in the $L^2(0, 2\pi)$-norm sense. Since $\sum_{n\in\mathbb{Z}} |\widehat{f}(w + 2\pi n)|$ converges a.e., the series in (12) converges a.e. in $w \in (0, 2\pi)$. As the pointwise limit and the limit in the $L^2(0, 2\pi)$-norm coincide (see [8, Theorem 3.12]), the equality (12) follows. □

Lemma 2. *For each fixed $t \in \mathbb{R}$, the Zak transform of a function $f \in V_1$ can be expressed as*

$$(Zf)(t,w) = (Z\phi)(2t, w/2) m_f(w/2) + (Z\phi)(2t, w/2+\pi) m_f(w/2+\pi), \quad \text{a.e. } w \in \mathbb{R}.$$

Proof. Using Lemma 1 and splitting the sum into odd and even terms we obtain

$$\begin{aligned}(Zf)(t,w) &= \sqrt{2\pi}\mathrm{e}^{\mathrm{i}wt}\sum_{n\in\mathbb{Z}} \widehat{f}(w+2\pi n)\mathrm{e}^{\mathrm{i}\pi n2t} = \sqrt{2\pi}\mathrm{e}^{\mathrm{i}wt}\sum_{n\in\mathbb{Z}} m_f(w/2+\pi n)\widehat{\phi}(w/2+\pi n)\mathrm{e}^{\mathrm{i}\pi n2t} \\ &= \sqrt{2\pi}\mathrm{e}^{\mathrm{i}wt}\sum_{n\in\mathbb{Z}} [m_f(w/2)\widehat{\phi}(w/2+2\pi n)\mathrm{e}^{\mathrm{i}\pi 2n2t} \\ &\quad + m_f(w/2+\pi)\widehat{\phi}(w/2+\pi+2\pi n)\mathrm{e}^{\mathrm{i}\pi(2n+1)2t}].\end{aligned}$$

Using (12) again, the lemma follows. □

Eq. (12) gives $|(Z\phi)(t,w)| \leqslant \sqrt{2\pi}\sum_{n\in\mathbb{Z}} |\widehat{\phi}(w+2\pi n)|$ a.e. Since we have supposed that $\sum_{n\in\mathbb{Z}} |\widehat{\phi}(w+2\pi n)|$ is uniformly bounded, we have that $\|(Z\phi)(t,\cdot)\|_\infty < \infty$, for all $t \in \mathbb{R}$. Thus, the hypothesis $\|(Z\phi)(a,\cdot)\|_\infty < \infty$ is superfluous under the above hypothesis.

Theorem 1. *Let* $f \in V_1$ *and let* v_f *be the unique* 2π*-periodic function in* $L^2(0, 2\pi)$ *which satisfies* $\widehat{P_{W_0} f} = v_f(w)\widehat{\psi}(w)$. *Then*

$$\widehat{E_a f}(w) = \mathrm{e}^{\mathrm{i}w/2} \frac{(Z\phi)(2a, w/2+\pi)}{(Z\phi)(a, w)} v_f(w)\widehat{\phi}(w/2). \tag{13}$$

Proof. A function g in V_1 belongs to M_a if and only if $(Zg)(a, w) = \sum_{n\in\mathbb{Z}} g(n+a)\mathrm{e}^{-\mathrm{i}wn} = 0$. Then, by using Lemma 2 we obtain that a function g in V_1 belongs to M_a, if and only if

$$m_g(w/2)(Z\phi)(2a, w/2) + m_g(w/2+\pi)(Z\phi)(2a, w/2+\pi) = 0, \quad \text{a.e.} \tag{14}$$

This leads us to consider the function $\varphi_a \in M_a$, whose Fourier transform is

$$\widehat{\varphi}_a(w) := \mathrm{e}^{-\mathrm{i}w/2}(Z\phi)(2a, w/2+\pi)\widehat{\phi}(w/2). \tag{15}$$

Let m_f be the unique 2π-periodic function in $L^2(0, 2\pi)$ such that $\widehat{f}(w) = m_f(w/2)\widehat{\phi}(w/2)$. Next, we define

$$\widehat{Q_{V_0} f}(w) := \alpha_f(w)\widehat{\phi}(w), \quad \widehat{Q_{M_a} f}(w) := \beta_f(w)\widehat{\varphi}_a(w), \tag{16}$$

where

$$\alpha_f(w) := \frac{m_f(w/2)(Z\phi)(2a, w/2) + m_f(w/2+\pi)(Z\phi)(2a, w/2+\pi)}{(Z\phi)(a, w)},$$
$$\beta_f(w) := \mathrm{e}^{\mathrm{i}w} \frac{m_f(w/2)\overline{m_\psi(w/2)} + m_f(w/2+\pi)\overline{m_\psi(w/2+\pi)}}{(Z\phi)(a, w)}. \tag{17}$$

The function α_f is 2π-periodic, and also it belongs to $L^2(0, 2\pi)$ since $0 < \|(Z\phi)(a, \cdot)\|_0$ and $\|(Z\phi)(2a, \cdot)\|_\infty < \infty$. Therefore, taking into account the characterization of $\widehat{V_0}$ (see [13]), we get that $Q_{V_0} f \in V_0$. Similarly, we prove that $Q_{M_a} f \in V_1$. Since $g = Q_{M_a} f$ satisfies (14), it follows that it belongs to M_a. Using Lemma 2, we can check that

$$\widehat{f} = \widehat{Q_{V_0} f} + \widehat{Q_{M_a} f} \tag{18}$$

and, as a consequence, (11) implies $\widehat{E_a f} = \widehat{Q_{M_a} f}$. Comparing (7) with (17) we see that $\beta_f(w) = \mathrm{e}^{\mathrm{i}w} v_f(w)/(Z\phi)(a, w)$. Substituting β_f and $\widehat{\varphi}_a$ in (16), we obtain the desired result. □

It is worth noticing that the space V_1 can be expressed as the direct sum of the spaces V_0 and M_a, and Q_{V_0} and Q_{M_a} are the corresponding projections. Indeed, this is a consequence of (18) and of the sampling formula (10).

When the scaling function satisfies the interpolation condition $\phi(n) = \delta_{n,0},\ n \in \mathbb{Z}$, ϕ is called a cardinal scaling function (see [12] for properties and examples on cardinal scaling functions).

Corollary 1. *Let* ϕ *be a cardinal scaling function. Then,* $\|E_0 f\|_{L^2(\mathbb{R})} = \sqrt{2}\ \|P_{W_0} f\|_{L^2(\mathbb{R})}$ *for any* $f \in V_1$.

Proof. Since $(Z\phi)(0, w) = 1$, formula (13) becomes $\widehat{E_0 f}(w) = \mathrm{e}^{\mathrm{i}w/2} v_f(w)\widehat{\phi}(w/2)$ from which the result follows. □

4. Estimations of the aliasing error function for $f \in V_1$

Throughout this section we denote $H_\phi(w) := \sum_{n\in\mathbb{Z}} |\widehat{\phi}(w+2\pi n)|$, which we have supposed to be uniformly bounded. The main goal is to prove an estimation of the aliasing error function in the supremum norm.

Theorem 2. *For any $f \in V_1$ we have that $\widehat{E_a f}$ belongs to $L^1(\mathbb{R})$, and the following inequality holds*

$$|E_a f(t)| \leqslant 2 \left\| \frac{(Z\phi)(2a, w+\pi) H_\phi(w)}{(Z\phi)(a, 2w)} \right\|_{L^2(0,2\pi)} \|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}. \tag{19}$$

Proof. For $f \in V_1$, consider the function v_f such that $\widehat{P_{W_0} f} = v_f(w)\widehat{\psi}(w)$. We have that

$$\begin{aligned} &\int_{\mathbb{R}} \left| \frac{(Z\phi)(2a, w/2+\pi)}{(Z\phi)(a, w)} v_f(w) \widehat{\phi}(w/2) \right| \mathrm{d}w \\ &\quad = 2 \int_{\mathbb{R}} \left| \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} v_f(2w) \widehat{\phi}(w) \right| \mathrm{d}w \\ &\quad = 2 \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \left| \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} v_f(2w) \widehat{\phi}(w+2\pi n) \right| \mathrm{d}w \\ &\quad = 2 \int_0^{2\pi} \left| \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} v_f(2w) \right| H_\phi(w)\, \mathrm{d}w. \end{aligned} \tag{20}$$

Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \left| \frac{(Z\phi)(2a, w/2+\pi)}{(Z\phi)(a, w)} v_f(w) \widehat{\phi}(w/2) \right| \mathrm{d}w \\ &\quad \leqslant 2\sqrt{2\pi} \left\| \frac{(Z\phi)(2a, w+\pi) H_\phi(w)}{(Z\phi)(a, 2w)} \right\|_{L^2(0,2\pi)} \|P_{W_0} f\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence, $\widehat{E_a f}$ belongs to $L^1(\mathbb{R})$, and the inverse Fourier transform gives (19). □

Moreover, there exists an integral representation for $E_a f(t)$ for each $t \in \mathbb{R}$.

Corollary 2. *Let $f \in V_1$ and let v_f be the unique 2π-periodic function of $L^2(0, 2\pi)$ which satisfies $\widehat{P_{W_0} f} = v_f(w)\widehat{\psi}(w)$. Then, $E_a f \in C_0(\mathbb{R})$ and*

$$E_a f(t) = \frac{1}{\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}w} \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} (Z\phi)(2t, w) v_f(2w)\, \mathrm{d}w, \quad t \in \mathbb{R}. \tag{21}$$

Proof. From Theorem 2 we obtain that $\widehat{E_a f}$ belongs to $L^1(\mathbb{R})$ and hence, $E_a f \in C_0(\mathbb{R})$. Using the inverse Fourier transform, a calculation similar to the one in (20), gives

$$\begin{aligned} E_a f(t) &= \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{E_a f}(2w) \mathrm{e}^{\mathrm{i}2wt}\, \mathrm{d}w \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}w} \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} v_f(2w) \sum_{n\in\mathbb{Z}} \widehat{\phi}(w+2\pi n) \mathrm{e}^{\mathrm{i}2(w+2\pi n)t}\, \mathrm{d}w \end{aligned}$$

and (21) follows from (12). □

We denote by $\arg z$ the value of argument of $z \in \mathbb{C}$ which belongs to $[-\pi, \pi)$. In the following corollary, we get extremal solutions of the inequality (19) in the case when $\arg \phi(w)$ is a 2π-periodic function. Notice that this is the case whenever $\widehat{\phi}(w) \geqslant 0$ (e.g. in the Shannon and Meyer cases) or whenever ϕ has linear phase with phase $p \in \mathbb{Z}$ (see [3]).

Corollary 3. *If* $\arg \phi(w)$ *is* 2π*-periodic then, for any* $f \in V_1$*, we have the inequality*

$$|E_a f(t)| \leqslant \frac{2}{\sqrt{2\pi}} \left\| \frac{(Z\phi)(2a, w+\pi)(Z\phi)(0, w)}{(Z\phi)(a, 2w)} \right\|_{L^2(0,2\pi)} \|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}. \tag{22}$$

When $a = 0$ *or* $a = 1/2$ *the functions* $f \in V_1$ *such that*

$$\overline{v_f(w)} := \lambda \mathrm{e}^{\mathrm{i}w(1\pm 1)/2} (Z\phi)(2a, w/2+\pi)(Z\phi)(2a, w/2)/(Z\phi)(a, w), \quad \lambda \in \mathbb{C},$$

are extremal solutions for (22).

Proof. First, we have

$$|(Z\phi)(0, w)| = \sqrt{2\pi} \left| \sum_{n\in\mathbb{Z}} |\widehat{\phi}(w+2\pi n)| \mathrm{e}^{\mathrm{i}\ \arg\ \phi(w)} \right| = \sqrt{2\pi} H_\phi(w).$$

Therefore, the inequality (19) can be written in the form (22).

Let $f \in V_1$, which satisfies $\overline{v_f(w)} := \lambda \mathrm{e}^{\mathrm{i}w(1/2\pm 1/2)} (Z\phi)(2a, w/2+\pi)(Z\phi)(2a, w/2)/(Z\phi)(a, w)$ for some $\lambda \in \mathbb{C}$. There exists such a f in V_1 because $(Z\phi)(2a, w/2+\pi)(Z\phi)(2a, w/2)/(Z\phi)(a, w)$ is a 2π-periodic function in $L^2(0, 2\pi)$. Using (21) and the periodicity of the Zak transform, i.e., $(Z\phi)(\pm 1 + 2a, w) = \mathrm{e}^{\pm \mathrm{i}w}(Z\phi)(2a, w)$, we obtain

$$\begin{aligned} |E_a f(\pm 1/2 + a)| &= \frac{1}{\pi} \left| \int_0^{2\pi} v_f(2w) \mathrm{e}^{\mathrm{i}w(1\pm 1)} \frac{(Z\phi)(2a, w+\pi)(Z\phi)(2a, w)}{(Z\phi)(a, 2w)}\, \mathrm{d}w \right| \\ &= \frac{2}{\sqrt{2\pi}} \left\| \frac{(Z\phi)(2a, w+\pi)(Z\phi)(2a, w)}{(Z\phi)(a, 2w)} \right\|_{L^2(0,2\pi)} \|P_{W_0} f\|_{L^2(\mathbb{R})}. \end{aligned}$$

Finally, the result is obtained by noting that $|(Z\phi)(1, w)| = |(Z\phi)(0, w)|$. □

For each $f \in V_1$, Theorem 1 gives the $L^2(\mathbb{R})$-norm of the aliasing error

$$\|E_a f\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{\pi}} \left\| v_f(2w) \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} \right\|_{L^2(0,2\pi)} \tag{23}$$

which allows us to deduce the following result:

Theorem 3. *There exist two constants K_0 and K_∞ such that*

$$K_0 \|P_{W_0} f\|^2_{L^2(\mathbb{R})} \leqslant \|E_a f\|^2_{L^2(\mathbb{R})} \leqslant K_\infty \|P_{W_0} f\|^2_{L^2(\mathbb{R})}, \quad f \in V_1. \tag{24}$$

Furthermore, the largest value for the constant K_0 and the smallest value for the constant K_∞ are, respectively,

$$K_0 := \left\| \frac{|(Z\phi)(2a, w)|^2 + |(Z\phi)(2a, w+\pi)|^2}{|(Z\phi)(a, 2w)|^2} \right\|_0,$$
$$K_\infty := \left\| \frac{|(Z\phi)(2a, w)|^2 + |(Z\phi)(2a, w+\pi)|^2}{|(Z\phi)(a, 2w)|^2} \right\|_\infty. \tag{25}$$

Proof. From (23) we obtain

$$\begin{aligned}\|E_a f\|^2_{L^2(\mathbb{R})} &= \frac{1}{\pi} \int_0^\pi \left| v_f(2w) \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} \right|^2 \mathrm{d}w + \frac{1}{\pi} \int_\pi^{2\pi} \left| v_f(2w) \frac{(Z\phi)(2a, w+\pi)}{(Z\phi)(a, 2w)} \right|^2 \mathrm{d}w \\ &= \frac{1}{\pi} \int_0^\pi |v_f(2w)|^2 \frac{|(Z\phi)(2a, w)|^2 + |(Z\phi)(2a, w+\pi)|^2}{|(Z\phi)(a, 2w)|^2} \mathrm{d}w. \end{aligned} \tag{26}$$

Since $\|P_{W_0} f\|^2_{L^2(\mathbb{R})} = (1/\pi) \int_0^\pi |v_f(2w)|^2 \, \mathrm{d}w$, it follows (24) for the values of the constant K_0 and K_∞ given in (25).

For the optimality of the constant K_∞, set $\lambda < K_\infty$. There exists a set $D \subseteq (0, \pi)$ with positive Lebesgue measure such that $(|(Z\phi)(2a, w)|^2 + |(Z\phi)(2a, w+\pi)|^2)/|(Z\phi)(a, 2w)|^2 > \lambda$, for all $w \in D$. Consider $f \in V_1$ such that $v_f(w) = 0$ when $w/2 \in (0, \pi) - D$. Then, from (26), we obtain

$$\|E_a f\|^2_{L^2(\mathbb{R})} \geqslant \frac{\lambda}{\pi} \int_0^\pi |v_f(2w)|^2 \, \mathrm{d}w = \lambda \|P_{W_0} f\|^2_{L^2(\mathbb{R})}.$$

This proves that (25) gives the smallest value of K_∞ in (24). Similarly it can be proved that (25) gives the largest value of K_0. □

The optimal constants for (24) calculated by Janssen [6] are

$$K_0 = 1 + \left\| \frac{(Z\psi)(a, w)}{(Z\phi)(a, w)} \right\|_0^2, \quad K_\infty = 1 + \left\| \frac{(Z\psi)(a, w)}{(Z\phi)(a, w)} \right\|_\infty^2$$

which necessarily must coincide with those given in (25). A direct proof of this result is as follows: Using Lemma 2 for $f = \phi$ and $f = \psi$ and the relationships (6), we obtain that $\overline{m_\phi}(w)(Z\phi)(a, 2w) + \overline{m_\psi}(w)(Z\psi)(a, 2w) = (Z\phi)(2a, w)$. Hence, a straightforward calculation gives

$$|(Z\phi)(2a, w)|^2 + |(Z\phi)(2a, w+\pi)|^2 = |(Z\phi)(a, 2w)|^2 + |(Z\psi)(a, 2w)|^2,$$

from which the result comes out.

Closing the section, recall that the space V_1 is a reproducing kernel Hilbert space (see [12, p. 54]), with reproducing kernel $k_1(t,s) = 2\sum_{n\in\mathbb{Z}} \phi(2t-n)\overline{\phi(2s-n)}$. Thus, for each $t \in \mathbb{R}$, we have $|h(t)|^2 \leqslant k_1(t,t)\|h\|^2_{L^2(\mathbb{R})}$, $h \in V_1$. Therefore, from (24) we obtain a new bound for the aliasing error function of $f \in V_1$

$$|E_a f(t)|^2 \leqslant 2K_\infty \sum_{n\in\mathbb{Z}} |\phi(2t-n)|^2 \|P_{W_0} f\|^2_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}. \tag{27}$$

5. Aliasing error when $f \in V_2$

The aim in this section is to find the Fourier transform of the aliasing error for a function f in the subspace V_2 of the multiresolution analysis. For the sake of simplicity we assume that $a = 0$. For notational ease, we denote $G_f(w) := (Zf)(0, w)$ and $Ef := E_0 f$.

Any function h in V_2 can be expressed as $h = P_{V_1}h + P_{W_1}h$ and hence, $Eh = EP_{V_1}h + EP_{W_1}h$. Since Theorem 1 provides $\widehat{Ef}$ for $f \in V_1$, it is enough to get $\widehat{Ef}$ when $f \in W_1$.

Theorem 4. *Let f be a function in W_1, and let $v_{D^{-1}f}$ be the unique 2π-periodic function in $L^2(0, 2\pi)$ which satisfies $D^{-1}f = v_{D^{-1}f}(w)\widehat{\psi}(w)$. Then*

$$\widehat{Ef}(w) = \frac{1}{\sqrt{2}}[a_1(w/4)v_{D^{-1}f}(w/2) - a_2(w/4)v_{D^{-1}f}(w/2+\pi)]\widehat{\phi}(w/4), \tag{28}$$

where

$$\begin{aligned} a_1(w) &:= \frac{G_\phi(w+\pi)}{G_\phi(2w)}\mathrm{e}^{\mathrm{i}w} + \frac{G_\phi(2w+\pi)G_\psi(2w)}{G_\phi(4w)G_\phi(2w)} m_\phi(2w+\pi)m_\phi(w), \\ a_2(w) &:= \frac{G_\psi(2w+\pi)}{G_\phi(4w)} m_\phi(2w)m_\phi(w). \end{aligned} \tag{29}$$

Proof. For a given $f \in W_1$ we have $D^{-1}f \in V_1$. Thus $D^{-1}f = Q_{V_0}D^{-1}f + Q_{M_0}D^{-1}f$, and hence $f = DQ_{V_0}D^{-1}f + DQ_{M_0}D^{-1}f$. Since $DQ_{V_0}D^{-1}f \in V_1$ we can split f as

$$f = Q_{V_0}DQ_{V_0}D^{-1}f + Q_{M_0}DQ_{V_0}D^{-1}f + DQ_{M_0}D^{-1}f\,.$$

Since $Q_{V_0}DQ_{V_0}D^{-1}f \in V_0$ and $Q_{M_0}DQ_{V_0}D^{-1}f(n) = DQ_{M_0}D^{-1}f(n) = 0$ for $n \in \mathbb{Z}$, the aliasing error for f is given by

$$Ef = Q_{M_0}DQ_{V_0}D^{-1}f + DQ_{M_0}D^{-1}f. \tag{30}$$

Taking the Fourier transform, and having in mind that $Q_{M_0}D^{-1}f = ED^{-1}f$, we have

$$\begin{aligned} \mathscr{F}(DQ_{M_0}D^{-1}f)(w) &= \frac{1}{\sqrt{2}}\mathscr{F}(Q_{M_0}D^{-1}f)(w/2) = \frac{G_\phi(w/4+\pi)}{\sqrt{2}G_\phi(w/2)}\mathrm{e}^{\mathrm{i}w/4}v_{D^{-1}f}(w/2)\widehat{\phi}(w/4), \\ \mathscr{F}(Q_{M_0}DQ_{V_0}D^{-1}f)(w) &= \frac{G_\phi(w/2+\pi)}{G_\phi(w)}\mathrm{e}^{\mathrm{i}w/2}v_{DQ_{V_0}D^{-1}f}(w)\widehat{\phi}(w/2). \end{aligned} \tag{31}$$

Next we derive $v_{DQ_{V_0}D^{-1}f}$ in terms of $v_{D^{-1}f}$. For,

$$\mathscr{F}(DQ_{V_0}D^{-1}f)(w) = (1/\sqrt{2})\mathscr{F}(Q_{V_0}D^{-1}f)(w/2) = (1/\sqrt{2})\alpha_{D^{-1}f}(w/2)\widehat{\phi}(w/2)$$

(see (16)). As a consequence, $m_{DQ_{V_0}D^{-1}f} = (1/\sqrt{2})\alpha_{D^{-1}f}$. Therefore, from (7)

$$v_{DQ_{V_0}D^{-1}f}(w) = (1/\sqrt{2})[\alpha_{D^{-1}f}(w/2)\overline{m_\psi(w/2)} + \alpha_{D^{-1}f}(w/2+\pi)\overline{m_\psi(w/2+\pi)}].$$

Now we find the relationship between $\alpha_{D^{-1}f}$ and $v_{D^{-1}f}$. From (16) and (15) we have

$$m_{D^{-1}f}(w/2) = \alpha_{D^{-1}f}(w)m_\phi(w/2) + \beta_{D^{-1}f}(w)\mathrm{e}^{-\mathrm{i}w/2}G_\phi(w/2+\pi).$$

As $D^{-1}f \in W_0$, we have $m_{D^{-1}f}(w/2) = v_{D^{-1}f}(w)m_\psi(w/2)$. Therefore,

$$v_{D^{-1}f}(w)m_\psi(w/2) = \alpha_{D^{-1}f}(w)m_\phi(w/2) + \beta_{D^{-1}f}(w)G_\phi(w/2+\pi)\mathrm{e}^{-\mathrm{i}w/2}.$$

Multiplying this equality by $G_\phi(w/2)$ and adding the obtained equality to its own 2π-shifted version, we obtain

$$\alpha_{D^{-1}f}(w) = \frac{G_\psi(w)}{G_\phi(w)}v_{D^{-1}f}(w),$$

where we have used Lemma 2. Substituting in (31) and taking into account (30), we finally obtain (28). □

By using (28), for any $f \in W_1$ we get

$$\begin{aligned}\|Ef\|_{L^2(\mathbb{R})} &= \|D\widehat{Ef}\|_{L^2(\mathbb{R})} = \|[a_1(w/2)v_{D^{-1}f}(w) - a_2(w/2)v_{D^{-1}f}(w+\pi)]\widehat{\phi}(w/2)\|_{L^2(\mathbb{R})}\\ &= \frac{1}{\sqrt{\pi}}\|a_1(w)v_{D^{-1}f}(2w) - a_2(w)v_{D^{-1}f}(2w+\pi)\|_{L^2(0,2\pi)}. \qquad (32)\end{aligned}$$

Finally, taking into account that $\|P_{W_0}D^{-1}f\|_{L^2(\mathbb{R})} = \|P_{W_1}f\|_{L^2(\mathbb{R})}$, the second term in (24) allows us to derive the inequality

$$\|Ef\|_{L^2(\mathbb{R})} \leqslant \sqrt{K_\infty}\|P_{W_0}f\|_{L^2(\mathbb{R})} + \sqrt{2}(\|a_1\|_\infty + \|a_2\|_\infty)\|P_{W_1}f\|_{L^2(\mathbb{R})}$$

for any $f \in V_2$.

Proceeding as in Theorem 2, from (28) we obtain that

$$|Ef(t)| \leqslant 2\sqrt{2}(\|a_1(w)H_\phi(w)\|_{L^2(0,2\pi)} + \|a_2(w)H_\phi(w)\|_{L^2(0,2\pi)})\|f\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}$$

for $f \in W_1$. Now, for $f \in V_2$ by using (19), we obtain that,

$$|Ef(t)| \leqslant L_1\|P_{W_0}f\|_{L^2(\mathbb{R})} + L_2\|P_{W_1}f\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R},$$

where

$$L_1 := 2\left\|\frac{G_\phi(w+\pi)H_\phi(w)}{G_\phi(2w)}\right\|_{L^2(0,2\pi)}$$

$$L_2 := 2\sqrt{2}(\|a_1(w)H_\phi(w)\|_{L^2(0,2\pi)} + \|a_2(w)H_\phi(w)\|_{L^2(0,2\pi)}).$$

6. Examples

Ending the paper, we illustrate the obtained results with some classical examples.

6.1. Shannon wavelet subspaces

Taking the cardinal sine function as the scaling function $\phi = \mathrm{sinc}$, the corresponding multiresolution analysis contains the Paley–Wiener classes [12]

$$V_j = PW_{2^j\pi} = \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \operatorname{supp}\widehat{f} \subseteq [-2^j\pi, 2^j\pi]\}.$$

Hence, for $f \in L^2(\mathbb{R})$,

$$\widehat{P_{W_0}f}(w) = \mathscr{X}_{\{\pi<|w|<2\pi\}}(w)\widehat{f}(w), \quad \widehat{P_{W_1}f}(w) = \mathscr{X}_{\{2\pi<|w|<4\pi\}}(w)\widehat{f}(w),$$

where $\mathscr{X}_A$ denotes the characteristic function of A. For $f \in PW_{2\pi}$, Corollary 1 gives,

$$\|Ef(t)\|^2_{L^2(\mathbb{R})} = 2\int_{|w|>\pi} |\widehat{f}(w)|^2\,\mathrm{d}w. \tag{33}$$

Hence, after a change of variable, the reconstruction of a bandlimited signal to the interval $[-2\pi\sigma, 2\pi\sigma]$ by means of its samples $\{f(n/\sigma)\}_{n\in\mathbb{Z}}$ leads to the following expression for the energy of the aliasing error

$$\left\| f(t) - \sum_{n\in\mathbb{Z}} f(n/\sigma)\,\mathrm{sinc}(\sigma t - n)\right\|^2_{L^2(\mathbb{R})} = 2\int_{|w|>\sigma\pi} |\widehat{f}(w)|^2\,\mathrm{d}w.$$

Since $\widehat{\phi}(w) = 1/\sqrt{2\pi}\mathscr{X}_{\{|w|<\pi\}}(w) \geqslant 0$ and $G_\phi(w) := (Z\phi)(0, w) = 1$, inequality (22) leads to

$$|Ef(t)|^2 \leqslant 4\int_{|w|>\pi} |\widehat{f}(w)|^2\,\mathrm{d}w, \quad t \in \mathbb{R} \tag{34}$$

for $f \in PW_{2\pi}$. Notice that the above inequality follows also from (27) since $\sum_{n\in\mathbb{Z}} |\mathrm{sinc}(t-n)|^2 = 1$ and $K_\infty = 1$. Extremal solutions for (34), having in mind Corollary 3, are, for instance, functions f such that $v_f(w) = \lambda$ or $v_f(w) = \lambda \mathrm{e}^{-\mathrm{i}w}$, for some $\lambda \in \mathbb{C}$. In particular, the mother wavelet ψ, and also the extremal solution $f(t) = \mathrm{sinc}(2t-1)$ given by Brown [1] for inequality (2), because $v_{\mathrm{sinc}(2t-1)}(w) = 1/2\ \mathrm{e}^{-\mathrm{i}w}$.

Now, we derive a sharp bound of $\|Ef\|_{L^2(\mathbb{R})}$ for $f \in V_2$. Using that

$$\widehat{\psi}(w) = (1/\sqrt{2\pi})\mathrm{e}^{\mathrm{i}w/2}\mathscr{X}_{\{\pi<|w|<2\pi\}}(w) \quad \text{and} \quad G_\psi(w) = -\mathrm{e}^{-\mathrm{i}w/2}, \ \ |w| < \pi$$

we obtain $a_1(w) = \mathrm{e}^{\mathrm{i}w}[\mathscr{X}_{\{|w|<\pi/4\}}(w) + \mathscr{X}_{\{\pi/2<|w|\}}(w)]$, $a_2(w) = \mathrm{sgn}(w)\mathrm{i}\mathrm{e}^{\mathrm{i}w}\mathscr{X}_{\{|w|<\pi/4\}}(w)$, when $|w| < \pi$. Thus, from (32), we obtain that, when $h \in W_1$,

$$\begin{aligned}\|Eh\|^2 = {} & \frac{1}{\pi}\int_{|w|<\pi/4} |v_{D^{-1}h}(2w) - \mathrm{i}\,\mathrm{sgn}(w)v_{D^{-1}h}(2w+\pi)|^2\,\mathrm{d}w \\ & + \frac{1}{\pi}\int_{\pi/2<|w|<\pi} |v_{D^{-1}h}(2w)|^2\,\mathrm{d}w \\ \leqslant {} & \frac{2}{\pi}\int_{|w|<\pi/4} (|v_{D^{-1}h}(2w)|^2 + |v_{D^{-1}h}(2w+\pi)|^2)\,\mathrm{d}w + \|h\|^2 \leqslant 3\|h\|^2.\end{aligned}$$

Now, for $f \in PW_{4\pi}$, since $Ef = EP_{V_1}f + EP_{W_1}f$, we have

$$\|Ef\|_{L^2(\mathbb{R})} \leqslant \sqrt{2}\|P_{W_0}f\|_{L^2(\mathbb{R})} + \sqrt{3}\,\|P_{W_1}f\|_{L^2(\mathbb{R})}.$$

The constants $\sqrt{2}$ and $\sqrt{3}$ are optimal because $\|Ef\|_{L^2(\mathbb{R})} = \sqrt{2}\,\|P_{W_0}f\|_{L^2(\mathbb{R})}$ when $f \in V_1$, and $\|Ef\|_{L^2(\mathbb{R})} = \sqrt{3}\|P_{W_1}f\|_{L^2(\mathbb{R})}$ when $f \in W_1$ with $v_{D^{-1}f}(w+\pi) = \mathrm{i}\,\mathrm{sgn}(w)v_{D^{-1}f}(w),\ |w| < \pi/2$.

6.2. Spline wavelet subspaces of odd degree

The cardinal B-spline of degree $2r-1$, N_{2r}, is a Riesz scaling function of the spline multiresolution analysis of degree $2r-1$ (see [3]). Let ϕ_{2r} be the corresponding orthogonal scaling function, whose Fourier transform is $\widehat{\phi}_{2r}(w) := \widehat{N}_{2r}(w)/(2\pi\sum_{n\in\mathbb{Z}}|\widehat{N}_{2r}(w+2\pi n)|^2)^{1/2}$ (see [3, p. 216]).

Using the equalities (see [3, pp. 88–89]),

$$2\pi\sum_{n\in\mathbb{Z}}|\widehat{N}_k(2w+2\pi n)|^2 = \mathrm{e}^{\mathrm{i}2kw}\sum_{n\in\mathbb{Z}}N_{2k}(n)\mathrm{e}^{-\mathrm{i}n2w} = \frac{-\sin^{2k}w}{(2k-1)!}\frac{\mathrm{d}^{2k-1}}{\mathrm{d}w^{2k-1}}\cot\,w, \quad k \in \mathbb{N} \tag{35}$$

and the Poisson summation formula, one obtains

$$|G_{\phi_{2r}}(2w)|^2 = \frac{-(4r-1)!}{[(2r-1)!]^2}\left(\frac{\mathrm{d}^{2r-1}}{\mathrm{d}w^{2r-1}}\cot\,w\right)^2 \Big/ \frac{\mathrm{d}^{4r-1}}{\mathrm{d}w^{4r-1}}\cot\,w. \tag{36}$$

The function ϕ_{2r} has linear phase with phase r and, as a consequence, inequality (22) holds with extremal solutions. Numerical approximations to the involved constant $(2/\sqrt{2\pi})\|G_{\phi_{2r}}(w+\pi)G_{\phi_{2r}}(w)/G_{\phi_{2r}}(2w)\|_{L^2(0,2\pi)}$ can be obtained using (36). For instance, for $r = 1, 2, 3, 4$ we obtain the values 2.678, 2.253, 2.169 and 2.128, respectively.

Next, we estimate the constants in inequality (24). Using (25) and (36), after some calculations we can check that, for $r = 1, 2, 3, 4$, the optimal constants are

$$K_0 = 1, \quad K_\infty = 1 + G^2_{\phi_{2r}}(\pi).$$

The values of K_∞ are 4, $52/17 \approx 3.0588$, $2077/691 \approx 3.0057$, and $2,789,284/929,569 \approx 3.0006$ for $r = 1, 2, 3, 4$, respectively.

Finally, we apply the bound of the aliasing error (22) to the choice of the sampling points $\{n+a\}_{n\in\mathbb{Z}}$ in the case $r = 2$ (cubic splines). Using (12) we obtain that for $a \in [0, 1)$,

$$\begin{aligned}(Z\phi_4)(a,w) &= \frac{(ZN_4)(a,w)}{(2\pi\sum_{n\in\mathbb{Z}}|\widehat{N}_4(w+2\pi n)|^2)^{1/2}} \\ &= \frac{a^3 + (1+3a+3a^2-3a^3)\mathrm{e}^{-\mathrm{i}w} + (4-6a^2+3a^3)\mathrm{e}^{-2\mathrm{i}w} + (1-a)^3\mathrm{e}^{-3\mathrm{i}w}}{6(2\pi\sum_{n\in\mathbb{Z}}|\widehat{N}_4(w+2\pi n)|^2)^{1/2}}\end{aligned}$$

from which we can easily check that the condition $0 < \|(Z\phi_4)(a,\cdot)\|_0 \leqslant \|(Z\phi_4)(a,\cdot)\|_\infty < \infty$ is satisfied when $a \in [0, 1)$ and $a \neq 1/2$. Thus, for this case, the sampling formula (10) holds. Since ϕ_4 has linear phase the inequality (22) holds with extremal solutions (see Corollary 3). A possible choice for the sampling points $\{n+a\}_{n\in\mathbb{Z}}$ consists in taking a in such a way that $\sup\{|E_a f(t)| : t \in \mathbb{R}, f \in V_1, \|f\|_{L^2(\mathbb{R})} = 1\}$, or equivalently, the constant in (22)

$$\frac{2}{\sqrt{2\pi}}\left\| \frac{(Z\phi_4)(2a, w+\pi)(Z\phi_4)(0,w)}{(Z\phi_4)(a,2w)} \right\|_{L^2(\mathbb{R})}$$

is minimized. A numerical calculation using (35) gives $a \approx 0.21$.

See [9] and references therein for Splines' uses in signal processing and, in particular, in image processing.

6.3. Meyer wavelet subspaces

Following Wojtaszczyk [13, p. 49], let ϕ be a function of $L^2(\mathbb{R})$ such that its Fourier transform $\widehat{\phi}$ satisfies

$$\begin{aligned}
&\widehat{\phi} \in C^2(\mathbb{R}), \quad \widehat{\phi}(w) = \widehat{\phi}(-w), \quad 0 \leqslant \widehat{\phi}(w) \leqslant \frac{1}{\sqrt{2\pi}}, \quad w \in \mathbb{R},\\
&\widehat{\phi}(w) = \frac{1}{\sqrt{2\pi}}, \quad |w| < \frac{2}{3}\pi, \quad \widehat{\phi}(w) = 0, \quad |w| > \frac{4}{3}\pi,\\
&\widehat{\phi}^2(w) + \widehat{\phi}^2(w-2\pi) = \frac{1}{2\pi}, \quad 0 \leqslant w \leqslant 2\pi.
\end{aligned}$$

Notice that $1 \leqslant G_\phi(w) \leqslant \sqrt{2}$. Indeed, $G_\phi(w) = 1$, when $w \in [0, 2\pi/3] \cup [4\pi/3, 2\pi]$, and $G_\phi(w) = \sqrt{2\pi}[\widehat{\phi}(w) + \widehat{\phi}(w-2\pi)]$, when $w \in [2\pi/3, 4\pi/3)$. Using that $\widehat{\phi}^2(w) + \widehat{\phi}^2(w-2\pi) = 1/(2\pi)$, we obtain the conclusion. Denoting $h(w) := [G_\phi^2(w) + G_\phi^2(w+\pi)]/G_\phi^2(2w)$, it can be proved that $1 \leqslant h(w) \leqslant 3$, for $w \in \mathbb{R}$. Namely, since $\widehat{\phi}$ is continuous, $\widehat{\phi}(2\pi/3) = 1/\sqrt{2\pi}$, and $\widehat{\phi}(4\pi/3) = 0$, there exists $w_0 \in (2\pi/3, 4\pi/3)$ such that $\widehat{\phi}(w_0) = 1/\sqrt{4\pi}$, and finally, $h(w_0/2) = 1$ and $h(w_0) = 3$. Hence $K_0 = 1$ and $K_\infty = 3$. Therefore, (24) reads as

$$\|P_{W_0} f\|^2_{L^2(\mathbb{R})} \leqslant \|Ef\|^2_{L^2(\mathbb{R})} \leqslant 3\|P_{W_0} f\|^2_{L^2(\mathbb{R})}, \quad f \in V_1 .$$

Moreover, the constants 1 and 3 are optimal in these inequalities.

Since $\widehat{\phi}$ is positive, the inequality (22) holds, and there exist extremal solutions. Concerning the constant in (22), notice that

$$\begin{aligned}
&\int_0^{2\pi} \frac{G_\phi^2(w+\pi) G_\phi^2(w)}{G_\phi^2(2w)}\,\mathrm{d}w\\
&\quad = 2\int_0^{\pi/3} G_\phi^2(w+\pi)\,\mathrm{d}w + 2\int_{\pi/3}^{2\pi/3} \frac{1}{G_\phi^2(2w)}\,\mathrm{d}w + 2\int_{2\pi/3}^{\pi} G_\phi^2(w)\,\mathrm{d}w\\
&\quad = \int_{2\pi/3}^{4\pi/3} \left[2G_\phi^2(w) + \frac{1}{G_\phi^2(w)} \right] \mathrm{d}w.
\end{aligned}$$

Using that $1 \leqslant G_\phi(w) \leqslant \sqrt{2}$ and $3 \leqslant 2x^2 + 1/x^2 \leqslant 9/2$, when $x \in [1, \sqrt{2}]$, we finally obtain that

$$2 \leqslant \frac{2}{\sqrt{2\pi}} \left\| \frac{G_\phi(w+\pi)G_\phi(w)}{G_\phi(2w)} \right\|_{L^2(0,2\pi)} \leqslant \sqrt{6}.$$

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