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Letter to the Editor

Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces

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Abstract

The aim of this article is to derive stable generalized sampling in a shift-invariant space by using some special dual frames in $L^2(0,1)$. These sampling formulas involve samples of filtered versions of the functions in the shift-invariant space. The involved samples are expressed as the frame coefficients of an appropriate function in $L^2(0,1)$ with respect to some particular frame in $L^2(0,1)$. Since any shift-invariant space with stable generator is the image of $L^2(0,1)$ by means of a bounded invertible operator, our generalized sampling is derived from some dual frame expansions in $L^2(0,1)$.

Keywords: Shift-invariant spaces; Dual frames; Generalized sampling

1. Statement of the problem

Suppose that s linear-time invariant systems (filters) $\mathcal{L}_j$, $j=1,2,\dots,s$, are defined on a shift-invariant space V_φ of $L^2(\mathbb{R})$

$$V_\varphi := \left\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n):\ \{a_n\} \in \ell^2(\mathbb{Z}) \right\},$$

where the function $\varphi \in L^2(\mathbb{R})$ is a stable generator for V_φ. The main aim in this work is to recover any function $f \in V_\varphi$ by means of a stable sampling formula. More precisely, by using a frame expansion which involves the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$, where the sampling period $r \in \mathbb{N}$ necessarily satisfies $r \leqslant s$. Whenever $s > r$ we are in the oversampling case. The advantages of the oversampling technique in practical applications are well-known (see, for instance, Refs. [10], [18] or [23]).

This problem goes back to a paper of Papoulis [19] where a sampling formula is given, which allows to recover a bandlimited function f by using the sequence of samples $\{(\mathcal{L}_j f)(sn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$, which involves s filtered versions of f. Note that, according to the Whittaker–Shannon–Kotel'nikov sampling theorem, the space of functions bandlim-

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ited to an interval $[-\sigma,\sigma]$, i.e., the classical Paley–Wiener space $PW_\sigma := \{f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})\colon\ \sup \hat{f} \subseteq [-\sigma,\sigma]\}$, where $\hat{f}$ stands for the Fourier transform $\hat{f}(w) := \int_{-\infty}^{\infty} f(t)e^{-2\pi i t w}\,\mathrm{d}t$, is an example of a shift-invariant space where the generator is a scaled version of the cardinal sine function $\operatorname{sinc} t := \sin \pi t/\pi t$. Wavelet subspaces are also important examples of shift-invariant spaces.

Papoulis' result has been extended to a general shift-invariant space by using the filter banks technique. Concretely, Djokovic and Vaidyanathan [11] extended Papoulis' result for some important particular filters, and Unser and Zerubia [25,26] extended it in the general case.

The case where the number of channels s is larger than the sampling period r, i.e., the oversampling case, has been also considered in [11] by means of an example. Later, Venkataramani and Bresler [27] studied this general setting for classical bandlimited functions.

In this paper we propose a new approach involving the theory of frames in a separable Hilbert space $\mathcal{H}$. Recall that a sequence $\{f_k\}$ is a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|f\|^2 \leqslant \sum_k \big|\langle f, f_k\rangle\big|^2 \leqslant B\|f\|^2 \quad \text{for all } f \in \mathcal{H}.$$

Given a frame $\{f_k\}$ for $\mathcal{H}$ the representation property of any vector $f \in \mathcal{H}$ as a series $f = \sum_k c_k f_k$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients c_k which depend continuously and linearly on f are obtained by using the dual frames $\{g_k\}$ of $\{f_k\}$, i.e., $\{g_k\}$ is another frame for $\mathcal{H}$ such that $f = \sum_k \langle f, g_k\rangle f_k = \sum_k \langle f, f_k\rangle g_k$ for each $f \in \mathcal{H}$. For more details on the frame theory see the superb monograph [8] and references therein.

The shift-invariant space V_φ is the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T} : L^2(0,1) \to V_\varphi$, which maps the orthonormal basis $\{e^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ.

Our starting point is to write the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},\ j=1,2,\ldots,s}$ as the frame coefficients with respect to a particular frame in $L^2(0,1)$ of the function $F = \mathcal{T}^{-1} f \in L^2(0,1)$. Searching for its dual frames we obtain those expansions for F in $L^2(0,1)$ having the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},\ j=1,2,\ldots,s}$ as frame coefficients. Thus, applying the isomorphism $\mathcal{T}$ to the above frame expansions of F we will obtain sampling expansions for $f = \mathcal{T}F$ in V_φ involving its samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},\ j=1,2,\ldots,s}$.

The use of several different dual frames allow us to obtain a variety of reconstruction functions. Thus we can try to find some reconstruction functions with "good properties." For instance, following an idea in [11], those with compact support. All these steps will be carried out throughout the remaining sections.

2. Preliminaries on shift-invariant spaces

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space

$$V_\varphi := \left\{ \sum_{n\in\mathbb{Z}} a_n \varphi(\cdot - n)\colon\ \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}),$$

i.e., the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ if and only if

$$0 < \|\Phi\|_0 \leqslant \|\Phi\|_\infty < \infty,$$

where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\hat{\varphi}(w+k)|^2$ in $(0,1)$, and $\|\Phi\|_\infty$ its essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [8, p. 143].

We assume throughout the paper that the functions in the shift-invariant space V_φ are continuous on $\mathbb{R}$. Equivalently, that the generator φ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [29]). Thus, any $f \in V_\varphi$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$.

Besides, V_φ is a reproducing kernel Hilbert space (RKHS) since the evaluation functionals are bounded in V_φ. Indeed, for each fixed $t \in \mathbb{R}$ we have

$$\big|f(t)\big|^2 \leqslant \frac{\|f\|^2}{\|\Phi\|_0} \sum_{n\in\mathbb{Z}} \big|\varphi(t-n)\big|^2, \quad f \in V_\varphi, \tag{1}$$

where we have used Cauchy–Schwartz's inequality in $f(t)=\sum_{n\in\mathbb{Z}} a_n\varphi(t-n)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{n\in\mathbb{Z}} |a_n|^2 \leqslant \|f\|^2 \leqslant \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} |a_n|^2, \quad f\in V_\varphi.$$

Inequality (1) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$.

The reproducing kernel of V_φ is given by $k(t,s)=\sum_{n\in\mathbb{Z}} \varphi(t-n)\overline{\varphi^*(s-n)}$ where the sequence $\{\varphi^*(\cdot-n)\}_{n\in\mathbb{Z}}$ denotes the dual Riesz basis of $\{\varphi(\cdot-n)\}_{n\in\mathbb{Z}}$. Recall that the function φ^* has Fourier transform $\widehat{\varphi^*}=\hat{\varphi}/\Phi$ [4].

On the other hand, the space V_φ is the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T}: L^2(0,1)\to V_\varphi$ which maps the orthonormal basis $\{e^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ (see [12]), i.e.,

$$(\mathcal{T}F)(t) := \sum_{n\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi i n\cdot}\big\rangle_{L^2(0,1)} \varphi(t-n), \quad F\in L^2(0,1).$$

Notice that for each $F\in L^2(0,1)$ the function $f=\mathcal{T}F$ is given by

$$f(t)=\langle F, K_t\rangle_{L^2(0,1)}, \quad t\in\mathbb{R}.$$

The kernel transform $t\in\mathbb{R}\to K_t\in L^2(0,1)$ is defined as $K_t(x):=\overline{Z\varphi}(t,x)$, where $Z\varphi$ denotes the Zak transform of φ. Recall that the Zak transform of $f\in L^2(\mathbb{R})$ is formally defined in $\mathbb{R}^2$ as $(Zf)(t,w):=\sum_{n\in\mathbb{Z}} f(t+n)e^{-2\pi i n w}$. See [14] for properties and uses of the Zak transform.

The following shifting property of $\mathcal{T}$ will be used later: For $F\in L^2(0,1)$, $r\in\mathbb{N}$ and $n\in\mathbb{Z}$ we have

$$\mathcal{T}\big[F(\cdot)e^{-2\pi i r n\cdot}\big](t)=\mathcal{T}[F](t-rn), \quad t\in\mathbb{R}. \tag{2}$$

We close this section citing Refs. [5,6,20] for the general theory of shift-invariant spaces and their applications. Whenever the generator φ is a B-spline, the corresponding shift-invariant space has been proved to be very fruitful in signal processing applications [24]. Besides, sampling in shift-invariant spaces has been a topic largely studied in recent years, see, for instance, the papers by Aldroubi and Gröchenig [3], Aldroubi and Unser [4], Chen et al. [7], Janssen [17], Sun and Zhou [21], or Walter [28]. Average sampling in shift-invariant spaces is also an important topic related to generalized sampling (see, for instance, [2] or [22] and references therein).

3. An expression for the samples

Throughout the paper we distinguish two types of linear-time invariant system $\mathcal{L}$:

(a) The impulse response l of $\mathcal{L}$ belongs to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Thus, for any $f\in V_\varphi$ we have

$$(\mathcal{L}f)(t) := [f*l](t) = \int_{-\infty}^{\infty} f(x)l(t-x)\,\mathrm{d}x = \big\langle f(\cdot), \phi(\cdot-t)\big\rangle_{L^2(\mathbb{R})}, \quad t\in\mathbb{R},$$

where $\phi(t):=\overline{l(-t)}$. Notice that $\mathcal{L}f$ is a continuous and bounded function in $L^2(\mathbb{R})$.

(b) The impulse response l has the form $l=\sum_{k=0}^{N} c_k\delta^{(k)}(t+d_k)$, where $\delta^{(k)}$ denotes the kth derivative of the Dirac delta and c_k, d_k are constants for $k=0,1,\dots,N$. For each $f\in V_\varphi$ we have

$$(\mathcal{L}f)(t) := \sum_{k=0}^{N} c_k f^{(k)}(t+d_k), \quad t\in\mathbb{R}.$$

In this case we also assume that $\varphi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\varphi^{(k)}(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ for each $k=0,1,\dots,N$.

Given a linear-time invariant system $\mathcal{L}$ of the type (a) or (b), next lemma assures that, for each fixed $t\in\mathbb{R}$, the Zak transform $(Z\mathcal{L}\varphi)(t,w)=\sum_{n\in\mathbb{Z}} \mathcal{L}\varphi(t+n)e^{-2\pi i n w}$ defines a function in $L^2(0,1)$.

Lemma 1. *Let $\mathcal{L}$ be a linear-time invariant system of the type* (a) *or* (b) *above. For any $t\in\mathbb{R}$ the sequence $\{(\mathcal{L}\varphi)(t+n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$.*

Proof. Whenever $\mathcal{L}$ is of the type (b), the result trivially holds. Assume that $\mathcal{L}$ is a system of the type (a) with impulse response l. Then, for any $t \in \mathbb{R}$ we have

$$\sum_{n\in\mathbb{Z}}\big|(\mathcal{L}\varphi)(t+n)\big|^2 = \sum_{n\in\mathbb{Z}}\left|\int_{-\infty}^{\infty} l(x)\varphi(t+n-x)\,\mathrm{d}x\right|^2 \leqslant \left(\int_{-\infty}^{\infty}\left(\sum_{n\in\mathbb{Z}}\big|l(x)\varphi(t+n-x)\big|^2\right)^{1/2}\mathrm{d}x\right)^2$$
$$\leqslant \left(\int_{-\infty}^{\infty}\big|l(x)\big|\left(\sum_{n\in\mathbb{Z}}\big|\varphi(t+n-x)\big|^2\right)^{1/2}\mathrm{d}x\right)^2 \leqslant M\|l\|_1^2,$$

where $M := \sup_{x\in\mathbb{R}}\sum_{n\in\mathbb{Z}}|\varphi(x-n)|^2$, and we have used a version of the Minkowski inequality for integrals [15, p. 148]. □

Now, consider s linear-time invariant systems $\mathcal{L}_j$, $j = 1, 2, \dots, s$, of the type (a), (b), or both. For notational ease we choose $t = 0$ without loss of generality. The apparently more general set of samples $\{\mathcal{L}_j f(rn + e_j)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$, where $e_j \in \mathbb{R}$ for $j = 1, 2, \dots, s$, is reduced to the case considered here by taking the appropriate shifted systems.

For $j = 1, 2, \dots, s$, the function g_j in $L^2(0,1)$ defined by

$$g_j(w) := \sum_{n\in\mathbb{Z}}\mathcal{L}_j\varphi(n)e^{-2\pi i n w} = (Z\mathcal{L}_j\varphi)(0,w), \tag{3}$$

plays an important role throughout this paper. Indeed, next lemma gives an expression for the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$, which involves the functions g_j, $j = 1, 2, \dots, s$ and the function $F = \mathcal{T}^{-1} f$ in $L^2(0,1)$:

Lemma 2. *Let f be a function in V_φ such that $f = \mathcal{T}F$ where $F \in L^2(0,1)$. For every $j = 1, 2, \dots, s$, we have*

$$(\mathcal{L}_j f)(rn) = \big\langle F(\cdot), \bar{g}_j(\cdot)\,e^{-2\pi i rn\cdot}\big\rangle_{L^2(0,1)}, \quad n \in \mathbb{Z}. \tag{4}$$

Proof. Assume that $\mathcal{L}_j$ is a filter of the type (a). For each $n \in \mathbb{Z}$ we have

$$(\mathcal{L}_j f)(rn) = \big\langle f(\cdot), \phi_j(\cdot - rn)\big\rangle_{L^2(\mathbb{R})} = \left\langle \sum_{k\in\mathbb{Z}}\big\langle F(\cdot), e^{-2\pi i k\cdot}\big\rangle_{L^2(0,1)}\varphi(\cdot - k), \phi_j(\cdot - rn)\right\rangle_{L^2(\mathbb{R})}$$
$$= \sum_{k\in\mathbb{Z}}\big\langle F(\cdot), e^{-2\pi i k\cdot}\big\rangle_{L^2(0,1)}\mathcal{L}_j\varphi(rn-k).$$

Parseval's equality and a change in the summation index gives

$$(\mathcal{L}_j f)(rn) = \left\langle F(\cdot), \sum_{k\in\mathbb{Z}}\overline{\mathcal{L}_j\varphi}(rn-k)\,e^{-2\pi i k\cdot}\right\rangle_{L^2(0,1)} = \big\langle F(\cdot), \bar{g}_j(\cdot)e^{-2\pi i rn\cdot}\big\rangle_{L^2(0,1)}.$$

Assume now that $\mathcal{L}_j$ is a filter of the type (b). Under our hypotheses on $\mathcal{L}_j$ we have that $f^{(k)}(t) = \sum_{l\in\mathbb{Z}}\langle F(\cdot), e^{-2\pi i l\cdot}\rangle_{L^2(0,1)}\varphi^{(k)}(t-l)$. Hence, for each $n \in \mathbb{Z}$, one gets

$$(\mathcal{L}_j f)(rn) = \sum_{k=0}^{N} c_k f^{(k)}(rn + d_k) = \sum_{k=0}^{N} c_k \sum_{l\in\mathbb{Z}}\big\langle F(\cdot), e^{-2\pi i l\cdot}\big\rangle_{L^2(0,1)}\varphi^{(k)}(rn + d_k - l)$$
$$= \left\langle F(\cdot), \sum_{k=0}^{N}\bar{c}_k\sum_{l\in\mathbb{Z}}\overline{\varphi}^{(k)}(rn + d_k - l)e^{-2\pi i l\cdot}\right\rangle_{L^2(0,1)} = \big\langle F(\cdot), \bar{g}_j(\cdot)e^{-2\pi i rn\cdot}\big\rangle_{L^2(0,1)}. \qquad \square$$

Observe that, under appropriate hypotheses, the Poisson summation formula gives a different expression for the functions g_j. For instance, assuming that $\sum_{n\in\mathbb{Z}}|\widehat{\mathcal{L}_j\varphi}(w+n)| \in L^2(0,1)$, one has

$$g_j(w) = \sum_{n\in\mathbb{Z}}\widehat{\mathcal{L}_j\varphi}(w+n) = \sum_{n\in\mathbb{Z}}\hat{l}_j(w+n)\hat{\varphi}(w+n) \quad \text{in } L^2(0,1), \tag{5}$$

where we have used that $l_j \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ when $\mathcal{L}_j$ is a system of the type (a).

Lemma 2 leads us to study when the sequence $\{a_j(\cdot)\, e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ (or equivalently the sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$), where $a_j \in L^2(0,1)$ for each $j = 1,2,\dots,s$, is a Bessel sequence or a frame for $L^2(0,1)$. To this end, associated with the functions a_j, $j = 1,2,\dots,s$, we introduce the $s \times r$ matrix function defined for $w \in (0,1)$ as

$$\mathbf{A}(w) := \begin{pmatrix} a_1(w) & a_1(w+\frac{1}{r}) & \cdots & a_1(w+\frac{r-1}{r}) \\ a_2(w) & a_2(w+\frac{1}{r}) & \cdots & a_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ a_s(w) & a_s(w+\frac{1}{r}) & \cdots & a_s(w+\frac{r-1}{r}) \end{pmatrix} = \left[a_j\left(w+\frac{k-1}{r}\right)\right]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,r}},$$

and its related constants

$$\alpha_{\mathbf{A}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}\left[\mathbf{A}^*(w)\mathbf{A}(w)\right], \qquad \beta_{\mathbf{A}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}\left[\mathbf{A}^*(w)\mathbf{A}(w)\right],$$

where $\mathbf{A}^*(w)$ denotes the transpose conjugate of the matrix $\mathbf{A}(w)$, and $\lambda_{\min}$ (respectively, $\lambda_{\max}$) the smallest (respectively, the largest) eigenvalue of the positive semidefinite matrix $\mathbf{A}^*(w)\mathbf{A}(w)$. Observe that $0 \leqslant \alpha_{\mathbf{A}} \leqslant \beta_{\mathbf{A}} \leqslant \infty$. Notice that in the definition of the matrix $\mathbf{A}(w)$ we are considering the 1-periodic extensions of the involved functions a_j, $j = 1,2,\dots,s$.

Lemma 3. *Let a_j be in $L^2(0,1)$ for $j = 1,2,\dots,s$ and let $\mathbf{A}(w)$ be its associated matrix. Then*:

(i) *The sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$ if and only if $a_j \in L^\infty(0,1)$ for $j = 1,\dots,s$. In this case, the optimal Bessel bound is $\beta_{\mathbf{A}}/r$.*

(ii) *The sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbf{A}} \leqslant \beta_{\mathbf{A}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbf{A}}/r$ and $\beta_{\mathbf{A}}/r$.*

Proof. Notice that the equivalence between the spectral and the Frobenius norms (see [16]) gives $a_j \in L^\infty(0,1)$ for $j = 1,2,\dots,s$ if and only if $\beta_{\mathbf{A}} < \infty$.

For $p = r$ or s, $L^2_p(0,1/r)$ denotes the space of the functions $\mathbf{H} = [h_1,\dots,h_p]^\top$ such that

$$\|\mathbf{H}\|^2_{L^2_p(0,1/r)} := \int_0^{1/r} \left|\mathbf{H}(w)\right|^2 \mathrm{d}w = \sum_{j=1}^{p} \|h_j\|^2_{L^2(0,1/r)} < \infty,$$

where $|\mathbf{H}(w)|$ is the Euclidean norm of $\mathbf{H}(w)$ in $\mathbb{C}^p$.

For any $F \in L^2(0,1)$ we have

$$\left\langle F(\cdot), \bar{a}_j(\cdot)e^{2\pi irn\cdot}\right\rangle_{L^2(0,1)} = \int_0^{1/r} \sum_{k=1}^{r} a_j\left(w+\frac{k-1}{r}\right) F\left(w+\frac{k-1}{r}\right) e^{-2\pi irnw}\, \mathrm{d}w.$$

Denote $\mathbf{F}(w) := [F(w), F(w+\frac{1}{r}),\dots,F(w+\frac{r-1}{r})]^\top$; whenever $\mathbf{A}(w)\mathbf{F}(w) \in L^2_s(0,1/r)$, we obtain

$$\begin{aligned}\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left|\left\langle F(\cdot), \bar{a}_j(\cdot)e^{2\pi irn\cdot}\right\rangle_{L^2(0,1)}\right|^2 &= \frac{1}{r}\sum_{j=1}^{s}\left\| \sum_{k=1}^{r} a_j\left(\cdot+\frac{k-1}{r}\right)F\left(\cdot+\frac{k-1}{r}\right)\right\|^2_{L^2(0,1/r)} \\ &= \frac{1}{r}\left\|\mathbf{A}(\cdot)\mathbf{F}(\cdot)\right\|^2_{L^2_s(0,1/r)} = \frac{1}{r}\int_0^{1/r} \mathbf{F}^*(w)\mathbf{A}^*(w)\mathbf{A}(w)\mathbf{F}(w)\,\mathrm{d}w. \end{aligned} \tag{6}$$

If $\beta_{\mathbf{A}} < \infty$ then, for each $F \in L^2(0,1)$, we have

$$\int_0^{1/r} \mathbf{F}^*(w)\mathbf{A}^*(w)\mathbf{A}(w)\mathbf{F}(w)\,\mathrm{d}w \leqslant \beta_{\mathbf{A}}\|\mathbf{F}\|^2_{L^2_r(0,1/r)} = \beta_{\mathbf{A}}\|F\|^2_{L^2(0,1)},$$

from which $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,\dots,s}$ is a Bessel sequence of $L^2(0,1)$ and the optimal Bessel bound is less than or equal to $\beta_{\mathbf{A}}/r$.

Let $K < \beta_{\mathbf{A}}$. Then, there exists a set $\Omega_K \subset (0,1/r)$ of positive measure such that $\lambda_{\max}[\mathbf{A}^*(w)\mathbf{A}(w)] \geqslant K$ for $w \in \Omega_K$. Let $F \in L^2(0,1)$ such that its associated vector function $\mathbf{F}(w)$ is 0 if $w \in (0,1/r) \setminus \Omega_K$ and $\mathbf{F}(w)$ is an eigenvector of norm 1 associated with the largest eigenvalue of $\mathbf{A}^*(w)\mathbf{A}(w)$ if $w \in \Omega_K$. We have that $\mathbf{A}(w)\mathbf{F}(w) \in L_s^2(0,1/r)$ and, using (6), we obtain

$$\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\bigl|\bigl\langle F(\cdot), \bar{a}_j(\cdot)e^{2\pi irn\cdot}\bigr\rangle_{L^2(0,1)}\bigr|^2 \geqslant \frac{1}{r}\int_0^{1/r} K\bigl|\mathbf{F}(w)\bigr|^2\,\mathrm{d}w = \frac{K}{r}\|F\|^2_{L^2(0,1)}.$$

Therefore if $\beta_{\mathbf{A}} = \infty$ then $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,\dots,s}$ is not a Bessel sequence in $L^2(0,1)$, and if $\beta_{\mathbf{A}} < \infty$ then the optimal Bessel bound is $\beta_{\mathbf{A}}/r$. This completes the proof of (i).

To prove part (ii) of the Lemma, assume first that $0 < \alpha_{\mathbf{A}} \leqslant \beta_{\mathbf{A}} < \infty$. By using part (i), the sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$. Moreover, using (6) and the Rayleigh–Ritz theorem (see [16, p. 176]), for each $F \in L^2(0,1)$ we obtain

$$\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\bigl|\bigl\langle F(\cdot), \bar{a}_j(\cdot)e^{2\pi irn\cdot}\bigr\rangle_{L^2(0,1)}\bigr|^2 \geqslant \frac{\alpha_{\mathbf{A}}}{r}\|\mathbf{F}\|^2_{L_r^2(0,1/r)} = \frac{\alpha_{\mathbf{A}}}{r}\|F\|^2_{L^2(0,1)}.$$

Hence, the sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ with optimal lower frame bound bigger or equal that $\alpha_{\mathbf{A}}/r$.

Conversely, if $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ we know by part (i) that $\beta_{\mathbf{A}} < \infty$. In order to prove that $\alpha_{\mathbf{A}} > 0$, consider any constant $K > \alpha_{\mathbf{A}}$. Then, there exists a set $\Omega_K \subset (0,1/r)$ with positive measure such that $\lambda_{\min}[\mathbf{A}^*(w)\mathbf{A}(w)] \leqslant K$ for $w \in \Omega_K$. Let $F \in L^2(0,1)$ such that its associated vector function $\mathbf{F}(w)$ is 0 if $w \in (0,1/r) \setminus \Omega_K$ and $\mathbf{F}(w)$ is an eigenvector of norm 1 associated with the smallest eigenvalue of $\mathbf{A}^*(w)\mathbf{A}(w)$ if $w \in \Omega_K$. Since F is bounded, we have that $\mathbf{A}(w)\mathbf{F}(w) \in L_s^2(0,1/r)$. From (6) we get

$$\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\bigl|\bigl\langle F(\cdot), \bar{a}_j(\cdot)e^{2\pi irn\cdot}\bigr\rangle_{L^2(0,1)}\bigr|^2 \leqslant \frac{K}{r}\int_0^{1/r} \bigl|\mathbf{F}(w)\bigr|^2\,\mathrm{d}w = \frac{K}{r}\|F\|^2_{L^2(0,1)}.$$

Denoting by A the optimal lower frame bound of $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$, we have obtained that $K/r \geqslant A$ for each $K > \alpha_{\mathbf{A}}$. Thus $\alpha_{\mathbf{A}}/r \geqslant A$ and consequently, $\alpha_{\mathbf{A}} > 0$. Moreover, under the hypotheses of part (ii) we deduce that $\alpha_{\mathbf{A}}/r$ and $\beta_{\mathbf{A}}/r$ are the optimal frame bounds. □

In order to complete the statement of Lemma 3, it is worth mentioning that one can also prove that the sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a Riesz basis for $L^2(0,1)$ if and only if it is a frame for $L^2(0,1)$ and $r = s$.

Consider the functions g_j, $j = 1,2,\dots,s$, given in (3), and its related matrix $\mathbf{G}$. It is worth to point out that, Lemmas 2, 3, and the isomorphism $\mathcal{T}$ gives the following result: There exist two constants $0 < A \leqslant B$ such that

$$A\|f\|^2 \leqslant \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\bigl|\mathcal{L}_j f(rn)\bigr|^2 \leqslant B\|f\|^2 \quad \text{for all } f \in V_\varphi \tag{7}$$

if and only if $0 < \alpha_{\mathbf{G}} \leqslant \beta_{\mathbf{G}} < \infty$.

Equation (7) coincides with the definition of *stable uniform averaging sampler* given by Aldroubi and co-workers in [1] (when the sampling period is 1). In [1] a necessary and sufficient condition for (7) is given for a shift-invariant space with several generators. That condition is equivalent to this given above as one can easily check.

4. The resulting sampling theory

The main aim in this section is to recover any function f in the shift-invariant space V_φ from its samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ by means of a stable sampling formula, i.e., the sampling formula will be an expansion with respect to an appropriate frame for V_φ.

Having in mind Lemma 2, for each $j = 1, 2, \dots, s$ we have

$$(\mathcal{L}_j f)(rn) = \left\langle \sum_{k=0}^{r-1} F\left(\cdot + \frac{k}{r}\right) g_j\left(\cdot + \frac{k}{r}\right), e^{-2\pi i n r \cdot} \right\rangle_{L^2(0,1/r)}, \quad n \in \mathbb{Z},$$

where $f = \mathcal{T}F$. Assuming that $g_j \in L^\infty(0,1)$, for each $j = 1, 2, \dots, s$, we obtain that

$$r \sum_{n\in\mathbb{Z}} (\mathcal{L}_j f)(rn) e^{-2\pi i n r w} = \sum_{k=0}^{r-1} F\left(w + \frac{k}{r}\right) g_j\left(w + \frac{k}{r}\right) \quad \text{in } L^2(0, 1/r).$$

The above expansions also hold in $L^2(0,1)$ by considering the 1-periodic extensions of F and g_j, $j = 1, 2, \dots, s$. Thus we have the matrix expression

$$\mathbf{G}(w)\mathbf{F}(w) = r\left[\sum_{n\in\mathbb{Z}} (\mathcal{L}_1 f)(rn) e^{-2\pi i n r w}, \dots, \sum_{n\in\mathbb{Z}} (\mathcal{L}_s f)(rn) e^{-2\pi i n r w}\right]^\top \quad \text{in } L^2(0,1), \tag{8}$$

where $\mathbf{G}(w) := [g_j(w + \frac{k-1}{r})]_{j=1,2,\dots,s,\, k=1,2,\dots,r}$ and $\mathbf{F}(w) := [F(w), F(w + \frac{1}{r}), \dots, F(w + \frac{r-1}{r})]^\top$. In order to recover F, assume that there exists a vector $[a_1(w), \dots, a_s(w)]$ with entries in $L^\infty(0,1)$ such that

$$\left[a_1(w), \dots, a_s(w)\right]\mathbf{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0,1).$$

As it will be proved later (see Theorem 2 below), a necessary and sufficient condition for the existence of such a vector (not necessarily unique) is that $\alpha_{\mathbf{G}} > 0$.

If we left multiply (8) by $[a_1(w), \dots, a_s(w)]$ we get

$$\begin{aligned} F(w) &= r\left[a_1(w), \dots, a_s(w)\right]\left[\sum_{n\in\mathbb{Z}} (\mathcal{L}_1 f)(rn) e^{-2\pi i n r w}, \dots, \sum_{n\in\mathbb{Z}} (\mathcal{L}_s f)(rn) e^{-2\pi i n r w}\right]^\top \\ &= r \sum_{j=1}^{s} a_j(w) \sum_{n\in\mathbb{Z}} (\mathcal{L}_j f)(rn) e^{-2\pi i n r w} = r \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn) a_j(w) e^{-2\pi i n r w}, \end{aligned}$$

in the $L^2(0,1)$-sense. Finally, the isomorphism $\mathcal{T}$ gives

$$f(t) = r \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn)(\mathcal{T}a_j)(t - rn) \quad \text{in } V_\varphi,$$

where we have used (2). In addition, much more can be said about the above sampling expansion. In fact, the following result holds:

Theorem 1. *Assume that $g_j \in L^\infty(0,1)$ for $j = 1, 2, \dots, s$. If there exists a vector $[a_1(w), \dots, a_s(w)]$ with entries in $L^\infty(0,1)$ such that*

$$\left[a_1(w), \dots, a_s(w)\right]\mathbf{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0,1) \tag{9}$$

then, for each $f \in V_\varphi$, we have

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j f)(rn) S_j(t - rn), \quad t \in \mathbb{R}, \tag{10}$$

where $S_j = r\mathcal{T}a_j$, $j = 1, \dots, s$. Moreover, the sequence $\{S_j(t - rn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a frame for V_φ with frame bounds $r\alpha_{\mathbf{A}}\|\Phi\|_0$ and $r\beta_{\mathbf{A}}\|\Phi\|_\infty$. The convergence of the series in (10) is in the $L^2(\mathbb{R})$-sense, absolute and uniform on $\mathbb{R}$.

Proof. Given $f \in V_\varphi$, consider $F = \mathcal{T}^{-1} f$ in $L^2(0,1)$. Above we have proved that

$$F(w) = r \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \left\langle F(\cdot), \bar{g}_j(\cdot)\, e^{-2\pi i r n \cdot} \right\rangle_{L^2(0,1)} a_j(w) e^{-2\pi i n r w} \quad \text{in } L^2(0,1). \tag{11}$$

Thus, the sequences $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ and $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ are Bessel sequences for $L^2(0,1)$ satisfying the representation property (11). According to Lemma 5.6.2 in [8] we obtain that they are dual frames for $L^2(0,1)$.

Next, applying the isomorphism $\mathcal{T}$ to (11) one gets the sampling expansion (10) in V_φ, where $S_j = r\mathcal{T}a_j$, $j=1,2,\ldots,s$. Moreover, the sequence $\{S_j(t-rn)\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ is a frame for V_φ. From Lemma 3 the optimal frame bounds for $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ are $r\alpha_{\mathbf{A}}$ and $r\beta_{\mathbf{A}}$. Hence, $r\alpha_{\mathbf{A}}\|\mathcal{T}^{-1}\|^{-2} = r\alpha_{\mathbf{A}}\|\Phi\|_0$ and $r\beta_{\mathbf{A}}\|\mathcal{T}\|^2 = r\beta_{\mathbf{A}}\|\Phi\|_\infty$ are frame bounds for $\{S_j(\cdot - nr)\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s} = \{\mathcal{T}[ra_j(\cdot)e^{-2\pi irn\cdot}]\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ (see Corollary 5.3.2 and Proposition 3.6.8 in [8]).

Pointwise convergence in the sampling series is absolute due to the unconditional convergence of a frame expansion. The uniform convergence on $\mathbb{R}$ is a consequence of (1). □

Notice that the frame bounds in Theorem 1 are optimal whenever $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_φ because, in this case, $\mathcal{T}$ is an unitary operator. In the general case, the optimal frame bounds could be computed orthonormalizing the Riesz basis $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ as $\{\tilde{\varphi}(\cdot - n)\}_{n\in\mathbb{Z}}$, where the orthonormal generator $\tilde{\varphi}$ has Fourier transform $\hat{\tilde{\varphi}} := \hat{\varphi}/\sqrt{\Phi}$ (see [8, Proposition 7.3.9]), and using (5).

The functions S_j for $j=1,2,\ldots,s$ are determined from the Fourier coefficients of a_j with respect to the orthonormal basis $\{e^{2\pi in\cdot}\}_{n\in\mathbb{Z}}$. Indeed,

$$S_j(t) = r(\mathcal{T}a_j)(t) = r\sum_{n\in\mathbb{Z}}\langle a_j(\cdot), e^{-2\pi in\cdot}\rangle_{L^2(0,1)}\varphi(t-n). \tag{12}$$

The Fourier transform in (12) gives $\hat{S}_j(w) = ra_j(w)\hat{\varphi}(w)$, $j=1,2,\ldots,s$, where we have used Lemma 7.2.1 in [8].

Observe that condition (9) is equivalent to $\mathbf{A}^\top(w)\mathbf{G}(w) = \mathbf{I}_r$ a.e. in $(0,1)$. In particular, this matrix equality implies that $\operatorname{rank}[\mathbf{G}(w)] = r$ a.e. in $(0,1)$ and, as a consequence, necessarily $s \geqslant r$.

In the next result we give a characterization of the existence of a sampling formula like (10). It is also proved that Theorem 1 provides all these formulas.

Theorem 2. *Assume that $g_j \in L^\infty(0,1)$ for $j=1,2,\ldots,s$. Then the following statements are equivalent*:

(i) *There exists a frame for V_φ having the form $\{S_j(t-rn)\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ such that for each $f\in V_\varphi$,*

$$f = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)S_j(\cdot - rn) \quad \text{in } L^2(\mathbb{R}), \tag{13}$$

(ii) $\alpha_{\mathbf{G}} > 0$.

If these equivalent conditions hold, the reconstruction functions are given by $S_j = r\mathcal{T}a_j$, where the functions $a_j \in L^\infty(0,1)$, $j=1,2,\ldots,s$, satisfy $\mathbf{A}^\top(w)\mathbf{G}(w) = \mathbf{I}_r$ a.e. in $(0,1)$.

Proof. First, assume that $\{S_j(t-rn)\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ is a frame for V_φ for which formula (13) holds. Applying the isomorphism $\mathcal{T}^{-1}$ to (13) we get

$$F(w) = r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)a_j(w)e^{-2\pi irnw} \quad \text{in } L^2(0,1),$$

where $ra_j = \mathcal{T}^{-1}S_j$, $j=1,2,\ldots,s$. The sequence $\{ra_j(w)e^{-2\pi irnw}\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$ and therefore, the functions $a_j \in L^\infty(0,1)$. Since

$$F(w) = r\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\langle F(\cdot), \bar{g}_j(\cdot)\,e^{-2\pi irn\cdot}\rangle_{L^2(0,1)}a_j(w)e^{-2\pi inrw},$$

and $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ is a Bessel sequence for $L^2(0,1)$, we obtain that the sequences $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ and $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$ are dual frames for $L^2(0,1)$ (see [8, Lemma 5.6.2]).

In particular, according to Lemma 3, we deduce that $\alpha_{\mathbf{G}} > 0$. This proves (ii). Besides, for each $F_1, F_2 \in L^2(0,1)$ we have [8, Lemma 5.6.2]:

$$\langle F_1, F_2\rangle_{L^2(0,1)} = r\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\langle F_1(\cdot), a_j(\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)}\langle \bar{g}_j(\cdot)e^{-2\pi irn\cdot}, F_2(\cdot)\rangle_{L^2(0,1)}. \tag{14}$$

Having in mind that

$$\langle F_1(\cdot), a_j(\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)} = \left\langle \sum_{k=0}^{r-1} F_1\left(\cdot+\frac{k}{r}\right)\bar{a}_j\left(\cdot+\frac{k}{r}\right), e^{-2\pi inr\cdot}\right\rangle_{L^2(0,1/r)},$$

$$\langle \bar{g}_j(\cdot)e^{-2\pi irn\cdot}, F_2(\cdot)\rangle_{L^2(0,1)} = \left\langle e^{-2\pi inr\cdot}, \sum_{k=0}^{r-1} F_2\left(\cdot+\frac{k}{r}\right)g_j\left(\cdot+\frac{k}{r}\right)\right\rangle_{L^2(0,1/r)},$$

Parseval's equality allows us to write the right-hand side in (14) as

$$\sum_{j=1}^{s}\left\langle \sum_{k=0}^{r-1} F_1\left(\cdot+\frac{k}{r}\right)\bar{a}_j\left(\cdot+\frac{k}{r}\right), \sum_{k=0}^{r-1} F_2\left(\cdot+\frac{k}{r}\right)g_j\left(\cdot+\frac{k}{r}\right)\right\rangle_{L^2(0,1/r)}$$
$$= \int_0^{1/r}\sum_{j=1}^{s}\mathbf{F_1}^\top(w)\bar{\mathbf{a}}_{\mathbf{j}}(w)\mathbf{g_j}^*(w)\bar{\mathbf{F}}_{\mathbf{2}}(w)\,\mathrm{d}w = \int_0^{1/r}\mathbf{F_1}^\top(w)\mathbf{A}^*(w)\overline{\mathbf{G}}(w)\bar{\mathbf{F}}_{\mathbf{2}}(w)\,\mathrm{d}w.$$

Since the left-hand side in (14) equals $\int_0^{1/r}\mathbf{F_1}^\top(w)\bar{\mathbf{F}}_{\mathbf{2}}(w)\,\mathrm{d}w$, we obtain that $\mathbf{A}^\top(w)\mathbf{G}(w) = \mathbf{I}_r$ a.e. in $(0,1)$.

Conversely, assume that $\alpha_{\mathbf{G}} > 0$. Hence, the inverse matrix $[\mathbf{G}^*(w)\ \mathbf{G}(w)]^{-1}$ exists a.e. in $(0,1)$. Consider the first row $[a_1(w),\dots,a_s(w)]$ of the pseudo-inverse matrix $\mathbf{G}^\dagger(w) = [\mathbf{G}^*(w)\ \mathbf{G}(w)]^{-1}\mathbf{G}^*(w)$ of $\mathbf{G}(w)$. Its entries a_j are essentially bounded in $(0,1)$ since the functions g_j and $\det^{-1}[\mathbf{G}^*(w)\ \mathbf{G}(w)]$ are essentially bounded in $(0,1)$. From $\mathbf{G}^\dagger(w)\mathbf{G}(w) = \mathbf{I}_r$ we obtain that $[a_1(w),\dots,a_s(w)]\ \mathbf{G}(w) = [1,0,\dots,0]$ a.e. in $(0,1)$. Thus, (i) comes out by using Theorem 1. □

Whenever the functions g_j are continuous on $\mathbb{R}$, the condition $\alpha_{\mathbf{G}} > 0$ is equivalent to $\det[\mathbf{G}^*(w)\ \mathbf{G}(w)] \neq 0$ on $\mathbb{R}$.

It can be proved that the first row $[a_1(w),\dots,a_s(w)]$ of the pseudo-inverse matrix $\mathbf{G}^\dagger(w)$ gives precisely the canonical dual frame $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ of the frame $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$.

Other suitable solutions for (9) are given by the first row of the matrix $\mathbf{G}^\dagger(w) + \mathbf{U}(w)[\mathbf{I}_s - \mathbf{G}(w)\mathbf{G}^\dagger(w)]$, where $\mathbf{U}(w)$ is any $r\times s$ matrix function with entries in $L^\infty(0,1)$.

Whenever $r = s$ we are in the Riesz bases setting, and the following result holds:

Corollary 1. *Assume that $r = s$ and $\alpha_{\mathbf{G}} > 0$. Then, there exists a unique frame $\{S_j(t-sn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ for V_φ for which the sampling formula* (13) *holds. In this case, this frame is a Riesz basis for V_φ with Riesz bounds $s\|\Phi\|_0/\beta_{\mathbf{G}}$ and $s\|\Phi\|_\infty/\alpha_{\mathbf{G}}$. Moreover, the functions a_j, $j = 1,2,\dots,s$, form the first row of the matrix $\mathbf{G}^{-1}$. The functions S_j, $j = 1,2,\dots,s$, satisfy the interpolation property $(\mathcal{L}_l S_j)(sn) = \delta_{j,l}\delta_{n,0}$, where $j,l = 1,2,\dots,s$ and $n\in\mathbb{Z}$.*

Proof. In this case, the unique solution of $[a_1(w),\dots,a_s(w)]\mathbf{G}(w) = [1,0,\dots,0]$ is given by the first row of $\mathbf{G}^\dagger = \mathbf{G}^{-1}$. By using that $\mathbf{G}(w)\mathbf{G}^{-1}(w) = \mathbf{I}_s$ we obtain

$$\langle sa_j(\cdot)e^{-2\pi isn\cdot}, \bar{g}_l(\cdot)e^{-2\pi ims\cdot}\rangle_{L^2(0,1)} = s\int_0^1 a_j(w)g_l(w)e^{2\pi i(m-n)sw}\,\mathrm{d}w$$
$$= s\int_0^{1/s}\sum_{k=0}^{s-1} a_j\left(w+\frac{k}{s}\right)g_l\left(w+\frac{k}{s}\right)e^{2\pi i(m-n)sw}\,\mathrm{d}w = \delta_{l,j}\delta_{n,m}.$$

Therefore, the dual frames $\{sa_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ and $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ are biorthogonal. Hence [8, Theorem 6.1.1], they form a pair of biorthogonal Riesz bases. The Riesz bounds for $\{S_j(t-rn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ follow from Theorem 1 having in mind that, in this case, $\alpha_{\mathbf{A}}\beta_{\mathbf{G}} = \alpha_{\mathbf{G}}\beta_{\mathbf{A}} = 1$.

Finally, sampling formula (13) for S_j gives $S_j(t) = \sum_{n\in\mathbb{Z}}\sum_{l=1}^{s}(\mathcal{L}_l S_j)(sn)S_l(t-sn)$. The uniqueness of the coefficients of an expansion with respect to a Riesz basis implies $(\mathcal{L}_l S_j)(sn) = \delta_{j,l}\delta_{n,0}$. $\square$

We finish this section pointing out two generalizations related to the theory exhibited in this article.

4.1. *The case where $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an overcompleted frame for V_φ*

First, recall that $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a frame sequence with bounds $0 < A \leqslant B < \infty$, i.e., a frame for its closed linear span, if and only if $A \leqslant \Phi(w) \leqslant B$ a.e. in $(0,1)\setminus N$, where $N := \{w \in (0,1)\colon \Phi(w) = 0\}$ [8, Theorem 7.2.3]. In this case, $\mathcal{T}$ is a bounded surjective operator from $L^2(0,1)$ onto V_φ. For any $f = \mathcal{T}F \in V_\varphi$ we have $\hat{f} = F\hat{\varphi}$. Since $\hat{\varphi}(w+n) = 0$ a.e. in N and $n \in \mathbb{Z}$, we deduce that $\mathcal{T}F_1 = \mathcal{T}F_2$, where $F_1, F_2 \in L^2(0,1)$, if and only if $F_1 = F_2$ in $L^2\big((0,1)\setminus N\big)$. Under the new hypothesis

$$\big[a_1(w), \dots, a_s(w)\big]\mathbf{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0,1)\setminus N,$$

the sampling result (10) in Theorem 1 also holds. One can check that the proof in Theorem 1 applies, having in mind that the operator $\tilde{\mathcal{T}} : L^2((0,1)\setminus N) \to V_\varphi$ defined for $F \in L^2((0,1)\setminus N)$ as

$$\tilde{\mathcal{T}}F := \mathcal{T}\tilde{F}, \quad \text{where } \tilde{F}(w) = \begin{cases} F(w) & \text{if } w \in (0,1)\setminus N, \\ 0 & \text{if } w \in N, \end{cases}$$

is an isomorphism satisfying the shifting property (2). In this case, the sequence $\{S_j(t-rn)\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$, where $S_j = r\tilde{\mathcal{T}}a_j$, $j = 1,2,\dots,s$, is also a frame for V_φ.

4.2. *Generalized irregular sampling*

Notice that the function g_j defined in (3) is nothing but the Zak transform $(Z\mathcal{L}_j\varphi)(0,\cdot)$. As in Lemma 2, one can prove for any $f \in V_\varphi$ that

$$(\mathcal{L}_j f)(rn + \varepsilon_n) = \big\langle F(\cdot), \overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_n, \cdot)e^{-2\pi irn\cdot}\big\rangle_{L^2(0,1)},$$

where $F = \mathcal{T}^{-1}f$, and $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$. As a consequence, stable generalized irregular sampling in V_φ depends on whether the sequence $\{\overline{(Z\mathcal{L}_j\varphi)}(\varepsilon_n, w)e^{-2\pi irnw}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$. This sequence can be seen as a perturbation of the frame $\{\overline{(Z\mathcal{L}_j\varphi)}(0, w)e^{-2\pi irnw}\}_{n\in\mathbb{Z},\, j=1,2,\dots,s}$ appearing in Theorem 1. Hence, by using similar techniques as those in [12], the theory on perturbation of frames (see [8, Chapter 15]) yields generalized irregular sampling in the shift-invariant space V_φ, for suitable error sequences $\{\varepsilon_n\}_{n\in\mathbb{Z}}$. This is work in progress and will appear elsewhere [13].

5. An illustrative example: Cubic splines

In the oversampling setting, i.e., $s > r$, Theorem 1 allows us different choices for the vector $\mathbf{a}(w) := [a_1(w), \dots, a_s(w)]$ and consequently, different reconstruction functions S_j. One may use this flexibility in order to obtain appropriate sampling functions S_j. For instance, if the generator φ and the impulse responses of the linear-time invariant systems $\mathcal{L}_j$ have compact support, the functions g_j are trigonometric polynomials and we can choose $\mathbf{a}(w)$ in order to obtain sampling functions S_j with compact support (which involves low computational complexities and avoids truncation errors). We illustrate this assertion in the case of cubic splines:

The cubic B-spline is defined as $N_4 := N_1 * N_1 * N_1 * N_1$, where N_1 denotes the characteristic function of the interval $(0,1)$. It is known that N_4 is a stable generator for the cubic splines in $L^2(\mathbb{R})$ with nodes at the integers (see [9]). Consider the $s = 3$ linear-time invariant systems defined as

$$\mathcal{L}_1 f(x) := \int_{x}^{x+1/3} f(t)\,\mathrm{d}t, \qquad \mathcal{L}_2 f(x) := \int_{x+1/3}^{x+2/3} f(t)\,\mathrm{d}t, \qquad \mathcal{L}_3 f(x) := \int_{x+2/3}^{x+1} f(t)\,\mathrm{d}t,$$

and the sampling period $r = 2$. Denoting by

$$\mathsf{g}_j(z) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j\varphi(n)z^n, \qquad \mathsf{G}(z) := \left[\mathsf{g}_j\left(ze^{-2\pi i(k-1)/r}\right)\right]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,r}}$$

if there exists a vector $\mathbf{b}(z) := [b_1(z), b_2(z), b_3(z)]$ whose entries are polynomials, and such that $\mathbf{b}(z)\mathsf{G}(z) = [z^l, 0]$ for some non-negative integer l, then the vector $\mathbf{a}(w) := e^{2\pi ikw}\mathbf{b}(e^{-2\pi iw})$, whose entries are trigonometric polynomials, satisfies $\mathbf{a}(w)\mathbf{G}(w) = [1, 0]$. Thus we have obtained reconstruction functions S_j with compact support (see Eq. (12)).

In particular, solving a linear system of 12 equations with 12 unknowns we find a vector $\mathbf{b}(z)$ whose entries are polynomials of degree 3, satisfying $\mathbf{b}(z)\mathsf{G}(z) = [z, 0]$. The corresponding sampling functions S_j, $j = 1, 2, 3$, are

$$\begin{aligned}
S_1(t) &= 418^{-1}\left[-5395N_4(t+1) + 22687N_4(t) + 188N_4(t-1) - 705N_4(t-2)\right],\\
S_2(t) &= 418^{-1}\left[7943N_4(t+1) - 41438N_4(t) - 892N_4(t-1) + 3345N_4(t-2)\right],\\
S_3(t) &= 418^{-1}\left[-1750N_4(t+1) + 21715N_4(t) + 1160N_4(t-1) - 4350N_4(t-2)\right].
\end{aligned}$$

The associated sampling formula for $f \in V_{N_4}$ reads:

$$f(t) = \sum_{n\in\mathbb{Z}}\left[\mathcal{L}_1 f(2n)S_1(t-2n) + \mathcal{L}_2 f(2n)S_2(t-2n) + \mathcal{L}_3 f(2n)S_3(t-2n)\right], \quad t \in \mathbb{R},$$

uniformly on $\mathbb{R}$.

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