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Regular sampling in wavelet subspaces involving two sequences of sampling points

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Abstract

Shannon's sampling formula has been extended for subspaces of a multiresolution analysis in $L^2(\mathbb{R})$. Thus, any function in the subspace V_0 of a multiresolution analysis can be recovered from its samples at the shifted integers $\{a+n\}_{n\in\mathbb{Z}}$ by means of a sampling formula, whenever a certain condition on the Zak transform of the scaling function is satisfied. In this paper it is proved that a natural condition, which involves again the Zak transform of the scaling function, allow us to recover any function in V_0 from its samples at the sequences $\{a+2n\}_{n\in\mathbb{Z}}$ and $\{b+2n\}_{n\in\mathbb{Z}}$ by using an appropriate sampling expansion.

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1 Introduction

The Whittaker-Shannon-Kotel'nikov sampling theorem states that any function f in the classical Paley-Wiener space $PW_{1/2} := \big\{f \in L^2(\mathbb{R})\cap C(\mathbb{R}) \;:\; \operatorname{supp}\widehat{f} \subset [-1/2,1/2]\big\}$, where $\widehat{f}$ stands for the Fourier transform $\widehat{f}(w) := \int_{\mathbb{R}} f(t)\, e^{-2\pi i wt}dt$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ at the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\operatorname{sinc}(t-n)\,, \quad t\in\mathbb{R}\,,$$

where sinc denotes the cardinal sine function, $\operatorname{sinc}(t) = \sin \pi t/\pi t$. Actually, the sampling points need not be taken at the integers to recover functions in $PW_{1/2}$. Indeed, any function f in $PW_{1/2}$ can be recovered from its samples at the integers shifted by a real constant a by means of the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(a+n)\,\operatorname{sinc}(t-a-n)\,, \quad t\in\mathbb{R}\,.$$

See, for instance, references [5, 13] on general sampling theory. Notice that the space $PW_{1/2}$ corresponds to the subspace V_0 in Shannon's multiresolution analysis.

In a general multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, the above sampling results have been extended to the subspace V_0, provided that a certain condition on the Zak transform of the scaling function is satisfied [1, 7, 11] (see infra Theorem 1).

On the other hand, it is also known that we can recover any function $f \in PW_{1/2}$ from its samples $\{f(a+2n)\}_{n\in\mathbb{Z}}$ and $\{f(b+2n)\}_{n\in\mathbb{Z}}$ whenever $a \neq b$ in $[0,2)$. This result goes back to a paper by Kohlenberg [8] (see also [5]). In engineering literature this sampling is known as interlaced sampling or periodically nonuniform sampling [3, 10]. In the present paper we show that, under a natural condition which involves the Zak transform of the scaling function and the points $a, b \in [0,2)$, the same result also holds in a general wavelet setting. Furthermore, the sampling functions in the corresponding sampling formula are explicitly given by their Fourier transforms.

2 Preliminaries

Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis in $L^2(\mathbb{R})$ with a Riesz scaling function φ, i.e., $\{V_j\}_{j\in\mathbb{Z}}$ is a increasing sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying:

(i) $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}\,, \quad j \in \mathbb{Z}$

(ii) $f \in V_0 \Rightarrow f(\cdot - n) \in V_0\,, \quad n \in \mathbb{Z}$

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(iv) $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_0

Recall that a function f belongs to V_1 if and only if there exists a unique 1-periodic function in $L^2(0,1)$, denoted by m_f, such that $\widehat{f}(w) = m_f(w/2)\widehat{\varphi}(w/2)$.

In order to use the Poisson summation formula we assume, throughout this paper, the following hypothesis on $\widehat{\varphi}$:

$$\operatorname*{ess\,sup}_{w\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(w+n)| < \infty\,. \tag{1}$$

This condition is satisfied if, for example, $\widehat{\varphi}(w) = O\big((1+|w|)^{-s}\big)$, $w \in \mathbb{R}$, for some $s > 1$.

The Zak transform of $f \in L^2(\mathbb{R})$, formally defined as

$$(Zf)(t,w) := \sum_{n\in\mathbb{Z}} f(t+n)e^{-2\pi iwn}\,, \quad t, w \in \mathbb{R}\,,$$

will be an important tool in the sequel. The Zak transform is an unitary map of $L^2(\mathbb{R})$ onto $L^2([0,1)\times[0,1))$, and it satisfies the quasi-periodicity properties: $(Zf)(t+1,w) = e^{2\pi iw}(Zf)(t,w)$ and $(Zf)(t,w+1) = (Zf)(t,w)$. See, for instance, [4, 6] for the properties and uses of the Zak transform. The following two lemmas, concerning the Zak transform, will be needed later.

Lemma 1 *Any function f in V_1 is continuous on $\mathbb{R}$. Moreover, for a fixed $t \in \mathbb{R}$, its Zak transform satisfies*

$$(Zf)(t,w) = \sum_{n\in\mathbb{Z}} \widehat{f}(w+n)e^{2\pi i(w+n)t}\,, \quad a.e. \ in\ \mathbb{R}\,. \tag{2}$$

Equality (2) *also holds in the $L^2(0,1)$-norm sense.*

Proof. For $f \in V_1$ we have

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)| = \sum_{n\in\mathbb{Z}} \Big| m_f\Big(\frac{w}{2}+\frac{n}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}+\frac{n}{2}\Big)\Big|$$
$$= \Big|m_f\Big(\frac{w}{2}\Big)\Big| \sum_{n\in\mathbb{Z}} \Big|\widehat{\varphi}\Big(\frac{w}{2}+n\Big)\Big| + \Big|m_f\Big(\frac{w}{2}+\frac{1}{2}\Big)\Big| \sum_{n\in\mathbb{Z}} \Big|\widehat{\varphi}\Big(\frac{w}{2}+\frac{1}{2}+n\Big)\Big|, \quad \text{a.e.}$$

From hypothesis (1) we have that $\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)| \in L^2(0,1)$. Taking in to account that $L^2(0,1) \subset L^1(0,1)$ and

$$\int_{\mathbb{R}} |\widehat{f}(w)|dw = \sum_{n\in\mathbb{Z}} \int_n^{n+1} |\widehat{f}(w)|dw = \sum_{n\in\mathbb{Z}} \int_0^1 |\widehat{f}(w+n)|dw = \int_0^1 \sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)|dw,$$

we have that $\widehat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and so that f is continuous. Since $\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)| \in L^2(0,1)$ we deduce that, for a fixed $t \in \mathbb{R}$, the function

$$g_t(w) := \sum_{n\in\mathbb{Z}} \widehat{f}(w+n)e^{2\pi i(w+n)t},$$

belongs to $L^2(0,1)$. Using the inverse Fourier transform, it can be easily checked that the Fourier coefficients of g_t with respect to the orthonormal basis $\{e^{-2\pi i wn}\}_{n\in\mathbb{Z}}$ are $\{f(t+n)\}_{n\in\mathbb{Z}}$. Hence, the equality in (2) holds in the $L^2(0,1)$-norm sense. Since $\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)|$ converges a.e., the series in (2) converges a.e. As the pointwise limit and the limit in the $L^2(0,1)$-norm coincide (see [9, Th 3.12]), the equality holds also a.e.

■

Applying the Parseval equality to (2) we obtain

$$\sum_{n\in\mathbb{Z}} |f(t+n)|^2 = \| \sum_{n\in\mathbb{Z}} \widehat{f}(w+n)e^{2\pi i(w+n)t}\|^2_{L^2(0,1)} \le \| \sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)|\|^2_{L^2(0,1)} < \infty\,.$$

Therefore, for each $f \in V_1$,

$$\sup_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |f(t+n)|^2 < \infty\,. \tag{3}$$

Notice that, taking $f = \varphi$, Lemma 1 gives $(Z\varphi)(t,w) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(w+n)e^{2\pi i(w+n)t}$ a.e., and since we have supposed (1), we obtain that $\operatorname{ess\,sup}_{(t,w)\in\mathbb{R}^2} |Z\varphi(t,w)| < \infty$.

Lemma 2 *Fixed $t \in \mathbb{R}$, for any $f \in V_1$ we have*

$$(Zf)\Big(\frac{t}{2},w\Big) = m_f\Big(\frac{w}{2}\Big)(Z\varphi)\Big(t,\frac{w}{2}\Big) + m_f\Big(\frac{w}{2}+\frac{1}{2}\Big)(Z\varphi)\Big(t,\frac{w}{2}+\frac{1}{2}\Big), \quad a.e.\ in\ \mathbb{R}\,.$$

Proof: Using Lemma 1 and splitting the sum into odd and even terms we obtain

$$(Zf)\Big(\frac{t}{2},w\Big)=\sum_{n\in\mathbb{Z}}\widehat{f}(w+n)e^{2\pi i(w+n)t/2}=\sum_{n\in\mathbb{Z}}m_f\Big(\frac{w}{2}+\frac{n}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}+\frac{n}{2}\Big)e^{2\pi i(w+n)t/2}$$

$$=m_f\Big(\frac{w}{2}\Big)\sum_{n\in\mathbb{Z}}\widehat{\varphi}\Big(\frac{w}{2}+n\Big)e^{2\pi i(w/2+n)t}+m_f\Big(\frac{w}{2}+\frac{1}{2}\Big)\sum_{n\in\mathbb{Z}}\widehat{\varphi}\Big(\frac{w}{2}+\frac{1}{2}+n\Big)e^{2\pi i(w/2+1/2+n)t}.$$

Applying again Lemma 1 for $f=\varphi$, the result follows.

■

Next, we characterize the subspace of V_1 containing the functions vanishing at the sequence $\{a/2+n\}_{n\in\mathbb{Z}}$, for a fixed $a\in\mathbb{R}$.

Lemma 3 *Let f be a function in V_1. Then $f(a/2+n)=0$ for all $n\in\mathbb{Z}$ if and only if*

$$m_f(w)(Z\varphi)(a,w)+m_f\Big(w+\frac{1}{2}\Big)(Z\varphi)\Big(a,w+\frac{1}{2}\Big)=0,\quad a.e.\ in\ \mathbb{R}\,.$$

Proof: Since $(Zf)(a/2,w)=0$ a.e. if and only if $f(a/2+n)=0$ for all $n\in\mathbb{Z}$, the result follows from Lemma 2.

■

At this point we remind some necessary concepts on shift-invariant spaces generated by a single function ϕ. The function $\phi\in L^2(\mathbb{R})$ such that $\{\phi(\cdot-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for its closed linear span, is said to be a stable generator for

$$V_\phi:=\Big\{\sum_{n\in\mathbb{Z}}a_n\ \phi(\cdot-n)\ :\ \{a_n\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})\Big\}\subset L^2(\mathbb{R})\,.$$

Equivalently, the function defined as $\Phi_\phi(w):=\sum_{k\in\mathbb{Z}}|\widehat{\phi}(w+k)|^2$ must satisfy the condition $0<\|\Phi_\phi\|_0\le\|\Phi_\phi\|_\infty<\infty$, where $\|\Phi_\phi\|_0$ denotes the essential infimum of the function in $(0,1)$, and $\|\Phi_\phi\|_\infty$ its essential supremum [2]. Recall that a Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator.

Closing the section we state a sampling theorem for shift-invariant spaces which will be used later. It can be found in [14, Th. 1].

Theorem 1 *Let ϕ be in $L^2(\mathbb{R})$ a continuous stable generator for V_ϕ, satisfying that $\sup_{t\in\mathbb{R}}\sum_{n\in\mathbb{Z}}|\phi(t+n)|^2<\infty$ and let $a\in\mathbb{R}$ such that $0<\|(Z\phi)(a,\cdot)\|_0\le\|(Z\phi)(a,\cdot)\|_\infty<\infty$. Then, for any $f\in V_\phi$, the sampling expansion*

$$f(t)=\sum_{n=-\infty}^{\infty}f(a+n)T_a(t-n),\quad t\in\mathbb{R}\,,$$

holds, where $\widehat{T_a}(w):=\widehat{\phi}(w)/(Z\phi)(a,w)$. The convergence of the series is absolute and uniform on $\mathbb{R}$. It also converges in the $L^2(\mathbb{R})$-norm sense.

3 The sampling result

The aim in this Section is to prove a sampling formula for V_0 which involves the samples at $\{a+2n\}_{n\in\mathbb{Z}}$ and $\{b+2n\}_{n\in\mathbb{Z}}$. This sampling result relies on a condition about a function $\Gamma_{a,b}$ which includes the parameters $a, b \in [0,2)$. It is defined by

$$\Gamma_{a,b}(w) := (Z\varphi)(b,w)(Z\varphi)\Big(a, w+\frac{1}{2}\Big) - (Z\varphi)\Big(b, w+\frac{1}{2}\Big)(Z\varphi)(a,w)\,, \qquad \text{a.e. in } \mathbb{R}\,.$$

Theorem 2 *Let $a, b \in [0,2)$ such that $\|\Gamma_{a,b}\|_0 > 0$. Then, any function $f \in V_0$ can be recovered from its samples $\{f(a+2n)\}_{n\in\mathbb{Z}}$ and $\{f(b+2n)\}_{n\in\mathbb{Z}}$ by means of the sampling formula*

$$f(t) = \sum_{n=-\infty}^{\infty} \big[f(a+2n)S_1(t-2n) + f(b+2n)S_2(t-2n)\big]\,, \quad t \in \mathbb{R}\,, \tag{4}$$

where the functions S_a and S_b in V_0 are given by their Fourier transforms

$$\widehat{S}_1(w) := \frac{-2(Z\varphi)(b,w+1/2)}{\Gamma_{a,b}(w)}\widehat{\varphi}(w), \quad \widehat{S}_2(w) := \frac{2(Z\varphi)(a,w+1/2)}{\Gamma_{a,b}(w)}\widehat{\varphi}(w).$$

The convergence of the series in (4) is absolute and uniform on $\mathbb{R}$. It also converges in the $L^2(\mathbb{R})$-norm sense.

Proof: In order to prove the sampling formula (4) we proceed as follows: We write any function $f \in V_1$ as $f = f_a + f_b$ where f_a (respectively f_b) belongs to a suitable shift-invariant space V_{φ_a} (respectively V_{φ_b}) whose functions vanish at the sequence $\{a/2+n\}_{n\in\mathbb{Z}}$ (respectively $\{b/2+n\}_{n\in\mathbb{Z}}$). Then, applying Theorem 1 in V_{φ_a} and V_{φ_b} we will obtain a sampling formula in V_1 which, restated by dilation for V_0, gives (4).

In so doing, Lemma 3 leads us to consider the functions φ_a and φ_b in V_1 whose Fourier transforms are given by

$$\widehat{\varphi}_a(w) := e^{-i\pi w}(Z\varphi)\Big(a, \frac{w}{2}+\frac{1}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}\Big) \quad \text{and} \quad \widehat{\varphi}_b(w) := e^{-i\pi w}(Z\varphi)\Big(b, \frac{w}{2}+\frac{1}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}\Big)\,.$$

We prove that φ_a (respectively φ_b) is a stable generator for V_{φ_a} (respectively V_{φ_b}). To this end,

$$\begin{aligned}\Phi_{\varphi_a}(w) &= \sum_{n\in\mathbb{Z}} |\widehat{\varphi}_a(w+n)|^2 = \sum_{n\in\mathbb{Z}} \Big|(Z\varphi)\Big(a, \frac{w}{2}+\frac{1}{2}+\frac{n}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}+\frac{n}{2}\Big)\Big|^2 \\ &= \Big|(Z\varphi)\Big(a, \frac{w}{2}+\frac{1}{2}\Big)\Big|^2 \Phi_\varphi\Big(\frac{w}{2}\Big) + \Big|(Z\varphi)\Big(a, \frac{w}{2}\Big)\Big|^2 \Phi_\varphi\Big(\frac{w}{2}+\frac{1}{2}\Big)\,, \qquad \text{a.e.}\end{aligned}$$

Since $\|(Z\varphi)(a,\cdot)\|_\infty < \infty$ and $\|\Phi_\varphi\|_\infty < \infty$ we have that $\|\Phi_{\varphi_a}\|_\infty < \infty$. On the other hand, using $\|\Phi_\varphi\|_0 > 0$, and

$$\begin{aligned}&\big|(Z\varphi)\big(a, w+\frac{1}{2}\big)\big| + \big|(Z\varphi)(a,w)\big| \geq \\ &\geq \frac{|(Z\varphi)(b,w)(Z\varphi)(a,w+1/2)| + |(Z\varphi)(b,w+1/2)(Z\varphi)(a,w)|}{\|(Z\varphi)(b,\cdot)\|_\infty} \geq \frac{\|\Gamma_{a,b}\|_0}{\|(Z\varphi)(b,\cdot)\|_\infty}\,, \ \text{a.e.}\ ,\end{aligned}$$

we obtain that $\|\Phi_{\varphi_a}\|_0 > 0$. Notice that $\|Z\varphi)(b,\cdot)\|_\infty > 0$ since $\|\Gamma_{a,b}\|_0 > 0$. Therefore, φ_a is a stable generator for V_{φ_a}. Similarly, it is proved that φ_b is a stable generator for V_{φ_b}

Next, for a given $f \in V_1$, consider the functions $f_a \in V_{\varphi_a}$ and $f_b \in V_{\varphi_b}$, whose Fourier transform are $\widehat{f_a}(w) := \alpha_f(w)\widehat{\varphi}_a(w)$ and $\widehat{f_b}(w) := \beta_f(w)\widehat{\varphi}_b(w)$, where α_f y β_f are the 1-periodic functions in $L^2(0,1)$ defined respectively by

$$\begin{aligned}\alpha_f(w) &:= e^{i\pi w}\frac{m_f(w/2)(Z\varphi)(b,w/2) + m_f(w/2+1/2)(Z\varphi)(b,w/2+1/2)}{\Gamma_{a,b}(w/2)},\\ \beta_f(w) &:= -e^{i\pi w}\frac{m_f(w/2)(Z\varphi)(a,w/2) + m_f(w/2+1/2)(Z\varphi)(a,w/2+1/2)}{\Gamma_{a,b}(w/2)}.\end{aligned}$$

We can easily check that $\widehat{f} = \widehat{f_a} + \widehat{f_b}$ and, as a consequence, $f = f_a + f_b$.

Lemma 2 gives the relationship $(Z\varphi_a)(b/2,w) = e^{-i\pi w}\Gamma_{a,b}(w/2)$. Since $\|\Gamma_{a,b}\|_0 > 0$ we have that $\|(Z\varphi_a)(b/2,\cdot)\|_0 > 0$ and, since $Z\varphi$ is uniformly bounded a.e., we have that $\|(Z\varphi_a)(b/2,\cdot)\|_\infty < \infty$ as well. Moreover, as $\varphi_b \in V_1$ then φ_b is continuous and from (3) we have that $\sup_{t\in\mathbb{R}}\sum_{n\in\mathbb{Z}}|\varphi_b(t+n)|^2 < \infty$. Thus, the hypotheses in Theorem 1 for the stable generator φ_a of V_{φ_a} and the point $b/2$ are satisfied. Therefore, as $f_a \in V_{\varphi_a}$,

$$f_a(t) = \sum_{n=-\infty}^{\infty} f_a\Big(\frac{b}{2}+n\Big)T_{a,b/2}(t-n)\,,$$

where

$$\widehat{T}_{a,b/2}(w) = \frac{\widehat{\varphi}_a(w)}{(Z\varphi_a)(b/2,w)} = \frac{(Z\varphi)(a,w/2+1/2)}{\Gamma_{a,b}(w/2)}\widehat{\varphi}\Big(\frac{w}{2}\Big)\,.$$

The convergence of the series is in the $L^2(\mathbb{R})$-norm sense, absolute and uniform on $\mathbb{R}$. Similarly, we obtain that $f_b(t) = \sum_{n\in\mathbb{Z}} f_b(a/2+n)T_{b,a/2}(t-n)$, $t\in\mathbb{R}$, where $\widehat{T}_{b,a/2}(w) = -(Z\varphi)(b,w/2+1/2)\widehat{\varphi}(w/2)/\Gamma_{a,b}(w/2)$.

By using Lemma 3, f_a vanish at the sequence $\{a/2+n\}_{n\in\mathbb{Z}}$. Hence, $f(a/2+n) = f_b(a/2+n)$ for all $n\in\mathbb{Z}$. Similarly, $f(b/2+n) = f_a(b/2+n)$ for all $n\in\mathbb{Z}$.

Therefore, for each $f \in V_1$, we have the sampling formula

$$f(t) = f_a(t)+f_b(t) = \sum_{n=-\infty}^{\infty}\Big[f\big(\frac{b}{2}+n\big)T_{a,b/2}(t-n) + f\big(\frac{a}{2}+n\big)T_{b,a/2}(t-n)\Big]\,,\quad t\in\mathbb{R}\,. \quad (5)$$

Finally, this sampling formula for V_1 yields, by dilation, the sampling formula (4) for V_0.

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Some comments about Theorem 2 are in order:

• First, notice that the characterization for the subspace $M_{a/2} := \big\{f\in V_1 : f(a/2+n) = 0,\, n\in\mathbb{Z}\big\}$ given in Lemma 3, along with a similar technique that those used to derive a mother wavelet in a multiresolution analyis [12, p. 35], proves that $M_{a/2} = V_{\varphi_a}$ provided $\|\Gamma_{a,b}\|_0 > 0$. In addition, the sampling formula (5) gives $M_{a/2}\cap M_{b/2} = \{0\}$. Therefore, the condition $\|\Gamma_{a,b}\|_0 > 0$ implies that the subspace V_1 can be written as the direct sum $V_1 = M_{a/2}\oplus M_{b/2}$ of its closed subspaces $M_{a/2}$ and $M_{b/2}$.

• The sequence $\{S_1(\cdot - 2n)\}_{n\in\mathbb{Z}} \cup \{S_2(\cdot - 2n)\}_{n\in\mathbb{Z}}$ forms a Riesz basis for V_0. It is a straightforward consequence of Theorem 2 and [2, Lemma 3.6.2] since the sequence

$\{T_a(\cdot - n)\}_{n\in\mathbb{Z}}$ in Theorem 1 is a Riesz basis for V_ϕ. As a consequence, the interpolation property $S_1(a+2n) = S_2(b+2n) = \delta_{n,0}$, $n \in \mathbb{Z}$, holds.

• Sampling formula (4) also holds for any f in a shift-invariant space V_φ, where φ is a stable generator for V_φ, provided the condition $\|\Gamma_{a,b}\|_0 > 0$. Indeed, it is enough to consider the dilated space $V_1 := \{f(2t) : f \in V_\varphi\}$, and to proceed as in the proof of Theorem 2.

• The sampling formula (4) for $a \in [0,1)$ and $b = a+1$, reduces to Theorem 1 as we can easily check by using that $(Z\varphi)(a+1,w) = e^{2\pi i w}(Z\varphi)(a,w)$.

Closing the paper, we illustrate the sampling result (4) with two examples:

Example 1: The Paley-Wiener space $PW_{1/2}$ corresponds to the subspace V_0 in the Shannon multiresolution analysis. As $\varphi = \text{sinc}$, we have that $(Z\,\text{sinc})(t,w) = e^{2\pi i w t}$ when $|w| < 1/2$ and, as a consequence,

$$\Gamma_{a,b}(w) = e^{2\pi i w(a+b)}\left[e^{-i\pi a} - e^{-i\pi b}\right], \quad w \in (0, 1/2).$$

Since $\Gamma_{a,b}(w \pm 1/2) = -\Gamma_{a,b}(w)$, we have $\|\Gamma_{a,b}\|_0 > 0$ if and only if $a \neq b$, with $a, b \in [0,2)$. For any $f \in PW_{1/2}$ sampling formula (4) reads

$$f(t) = \sum_{n=-\infty}^{\infty} \big[f(a+2n)S(t-2n-a) + f(b+2n)S(b+2n-t)\big], \quad t \in \mathbb{R},$$

where

$$S(t) := \frac{\sin \pi t - \sin \pi(t+a-b) + \sin \pi(a-b)}{\pi t[1 - \cos \pi(a-b)]}.$$

Example 2: Let φ be the scaling function of the Meyer multiresolution analysis given in [12, p. 49]. Namely, let φ be a function in $L^2(\mathbb{R})$ such that its Fourier transform satisfies the following conditions:

$$0 \le \widehat{\varphi}(w) \le 1,\ \ w \in \mathbb{R}, \qquad \widehat{\varphi}(w) = 1,\ \ |w| < \frac{1}{3}, \qquad \widehat{\varphi}(-w) = \widehat{\varphi}(w),\ \ w \in \mathbb{R},$$
$$\widehat{\varphi}(w) = 0,\ \ |w| > \frac{2}{3}, \qquad \widehat{\varphi}^2(w) + \widehat{\varphi}^2(w-1) = 1,\ \ 0 \le w \le 1.$$

One can easily check that

$$(Z\varphi)(t,w) = e^{2\pi i w t}\begin{cases} 1 & w \in (0,1/3) \\ \widehat{\varphi}(w) + \widehat{\varphi}(w-1)e^{-2\pi i t} & w \in (1/3, 2/3) \\ e^{-2\pi i t} & w \in (2/3, 1). \end{cases}$$

After some calculations, one get

$$\Gamma_{a,b}(w) = e^{i2\pi w(a+b)}\begin{cases} \widehat{\varphi}(w+1/2)\overline{C} + \widehat{\varphi}(w-1/2)C & w \in (0,1/6) \\ C & w \in (1/6, 1/3) \\ C\widehat{\varphi}(w) - C\widehat{\varphi}(w-1)e^{-\pi i(a+b)} & w \in (1/3, 1/2), \end{cases}$$

where $C := e^{-\pi i a} - e^{-i\pi b}$. Provided $a \neq b$ and $a + b \neq 2$ $(a, b \in [0,2)\,)$ it can be checked that $\|\Gamma_{a,b}\|_0 > 0$.

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