SAMPLING THEORY IN SIGNAL AND IMAGE PROCESSING Vol. 6, No. 1, Jan. 2007, pp. 53-69

ISSN: 1530-6429

On the Truncation Error of Generalized Sampling Expansions in Shift-Invariant Spaces

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Abstract

Any function in a shift-invariant space with stable generator φ can be recovered, under suitable hypotheses, from regular samples of some filtered versions of the function itself. This work concerns with the truncation error for the corresponding sampling expansion. In particular, whenever the generator φ and the impulse responses of the involved filters have compact support we obtain the asymptotic behaviour of the truncation error without a previous knowledge of the sampling functions. We illustrate the obtained results with some examples in the cubic spline space.

Key words and phrases: Shift-invariant spaces, Generalized sampling, Truncation error.

2000 AMS Mathematics Subject Classification: 42C40, 94A20

1 Introduction

Sampling in shift-invariant spaces has been a topic largely studied in recent years. (See, for instance, the papers by Aldroubi and Gröchenig [2], Aldroubi and Unser [3, 20], Sun and Zhou [18], Unser [22] or Walter [25], and references therein.) Average sampling in shift-invariant spaces is also an important topic (see [1], [19]). Classical and average sampling may be seen as particular cases of the so-called generalized sampling.

Concretely, suppose that s linear-time invariant systems (filters) $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, are defined on a shift-invariant space V_φ of $L^2(\mathbb{R})$,

$$V_\varphi := \Big\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\ \varphi(t-n)\ :\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\},$$

where the function $\varphi \in L^2(\mathbb{R})$ is a stable generator for V_φ. Unser and Zerubia obtained in [23, 24] a generalized sampling formula that allows for the recovery of any function $f \in V_\varphi$ from the samples $\{(\mathcal{L}_j f)(sn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$. This generalized sampling formula for $f \in V_\varphi$ reads

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(sn)\ S_j(t-sn)\,, \quad t \in \mathbb{R}\,, \tag{1}$$

where the sequence $\{S_j(t-sn)\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ is a Riesz basis for V_φ. In [10] a necessary and sufficient condition for the existence of a Riesz basis $\{S_j(t-sn)\}$ satisfying (1) is given. The case where the number of channels is bigger than the sampling period is also treated there. The sampling formula (1) is a very general sampling formula since it includes as particular cases the Shannon sampling formula ($\varphi = \operatorname{sinc}$, $s = 1$ and $\mathcal{L}_1 f = f$), the uniform sampling formula for shift-invariant spaces ($s = 1$ and $\mathcal{L}_1 f = f$), and sampling formulas involving local averages or derivatives.

An appropriate scaling in formula (1) gives the corresponding sampling formula for the wavelet subspaces in a multiresolution analysis: The generator to be considered is the scaling function of the multiresolution analysis. Shannon and Meyer wavelet subspaces are examples where the Fourier transform of the scaling function has compact support. Examples with better time localization are Spline and Daubechies wavelet subspaces whose scaling function has compact support (see [9, 17]).

Observe that formula (1) allows for the recovery of any function $f \in V_\varphi$ from the samples $\{\widetilde{\mathcal{L}}_j f(sn+a)\}$, instead of $\{\widetilde{\mathcal{L}}_j f(sn)\}$, by using the shifted systems $(\mathcal{L}_j f)(t) := (\widetilde{\mathcal{L}}_j f)(t+a)$.

In practice, only finitely many samples are available and so one would like to study the truncation error for this formula, i.e.,

$$E_N f(t) := f(t) - \sum_{|n|<N} \sum_{j=1}^{s} (\mathcal{L}_j f)(sn) S_j(t-sn)\,, \quad t \in \mathbb{R}\,.$$

For the Shannon sampling formula this error has been rather extensively studied in the engineering and mathematical literature. See [14] for an account of results in this direction, and [15, 16] for more recent results. For instance, Beutler [6] obtained, under some weak conditions, a bound for the truncation error which shows that $E_N f(t) = O(1/N)$ uniformly on $\mathbb{R}$.

Atreas and Karanikas [4, 5] have studied the truncation error for the regular sampling formula $f(t) = \sum_{n\in\mathbb{Z}} f(n)\; S(t-n)$ in a shift-invariant space whose stable generator φ has polynomial decay, i.e., $\varphi = O(1/|t|^p)$ for some $p \geq 1$. They have obtained that $E_N f(t) = O(1/N^{p-1/2})$ as $N \to \infty$ uniformly on compact subsets.

In this paper we study the truncation error for the generalized sampling formula (1) whenever the generator, φ, and the impulse response of the filter, $\mathcal{L}_j$ $j = 1, 2, \ldots, s$, have compact support. In this case it is proved that $E_N f(t) = o(N^p \rho^{sN})$ as $N \to \infty$ uniformly on compact subsets of $\mathbb{R}$, where the rate $\rho \in (0,1)$ and $p \in \mathbb{N} \cup \{0\}$ depends only on the generator and on the systems $\mathcal{L}_j$ through the zeros of an involved Laurent polynomial. For the most common situation, $p = 0$, which corresponds with the case when the aforesaid Laurent polynomial has only simple zeros on the boundary of an annulus containing the unit circle, an explicit estimation for the truncation error is given.

Mostly often, the systems $\mathcal{L}_j$ depend on some parameters. For example, they could depend on the shift parameter a of the samples $\{\widetilde{\mathcal{L}}_j f(sn + a)\}$. A criterion for the optimal choice of these parameters might be to choose those for which the rate ρ is minimum. We illustrate this optimization procedure with some examples in the cubic spline space.

2 Preliminaries on generalized sampling in shift-invariant spaces

In this Section we introduce the preliminaries on generalized sampling in shift-invariant spaces needed in the sequel. Let $\varphi \in L^2(\mathbb{R})$ be a stable generator of

$$V_\varphi := \Big\{ f(t) = \sum_{n\in\mathbb{Z}} a_n \;\varphi(t-n) \;:\; \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})\,,$$

i.e., the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ. We assume that the functions in V_φ are continuous on $\mathbb{R}$. Equivalently (see [26]), the generator φ is continuous on $\mathbb{R}$ and $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$. Thus, any $f \in V_\varphi$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$.

The shift-invariant space V_φ can be seen as the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T}_\varphi \;:\; L^2(0,1) \to V_\varphi$ which maps the orthonormal basis $\{e^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ, i.e.,

$$(\mathcal{T}_\varphi F)(t) := \sum_{n\in\mathbb{Z}} \big\langle F(\cdot), e^{-2\pi i n \cdot} \big\rangle_{L^2(0,1)} \varphi(t-n), \qquad F \in L^2(0,1). \tag{2}$$

Let $\mathcal{L}_j$, $j = 1, 2, \ldots, s$ be s linear-time invariant systems defined on V_φ. We assume that for $j = 1, 2, \ldots, s$, the impulse response l_j of the system $\mathcal{L}_j$ belongs to the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, or it has the form $l_j = \sum_{k=0}^{N} c_k \delta^{(k)}(t + d_k)$ where $\delta^{(k)}$ denotes the kth derivative of the Dirac delta and c_k, d_k are constants for $k = 0, 1, \dots, N$. In this last case we also assume that $\varphi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\varphi^{(k)}(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ for each $k = 0, 1, 2, \dots, N$.

Whenever $\mathcal{L}_j$ is a filter as above, the sequence $\{(\mathcal{L}_j\varphi)(n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ (see [10, Lemma 1]). Thus, for $j = 1, 2, \dots, s$ we consider the function g_j in $L^2(0,1)$ defined by

$$g_j(w) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j\varphi(n) e^{-2\pi i n w}\,, \tag{3}$$

and the $s \times s$ matrix

$$\mathbf{G}(w) := \Big[g_j\Big(w + \frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,s}}.$$

In [10] it has been proven that in the case where $g_j \in L^\infty(0,1)$ for $j = 1, 2, \dots, s$, the condition $\alpha_{\mathbf{G}} := \operatorname{ess\,inf}_{w\in(0,1/s)} \lambda_{\min}[\mathbf{G}^*(w)\mathbf{G}(w)] > 0$ is a necessary and sufficient condition for the existence of a Riesz basis $\{S_j(t-sn)\}_{n\in\mathbb{Z}, j=1,2,\cdots,s}$ for V_φ such that the sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{s} \big(\mathcal{L}_j f\big)(sn)\ S_j(t-sn)\,, \quad t \in \mathbb{R} \tag{4}$$

holds in V_φ. The convergence is in the $L^2(\mathbb{R})$-sense, absolute and uniform on $\mathbb{R}$. As usual, $\mathbf{G}^*(w)$ denotes the transpose conjugate of the matrix $\mathbf{G}(w)$ and $\lambda_{\min}$ the smallest eigenvalue of the matrix $\mathbf{G}^*(w)\mathbf{G}(w)$. The reconstruction functions, S_j, are given by

$$S_j(t) = s\mathcal{T}_\varphi\big[a_j(\cdot)\big](t)\,, \quad t \in \mathbb{R}\,,$$

where the functions a_j, $j = 1, 2, \dots, s$, form the first row of the matrix $\mathbf{G}^{-1}(w)$ and $\mathcal{T}_\varphi$ is the isomorphism defined in (2).

3 The truncation error

For any $f \in V_\varphi$ and $N \in \mathbb{N}$ we define the truncation error corresponding to the sampling formula (4) evaluated at $t \in \mathbb{R}$ as

$$E_N f(t) := f(t) - \sum_{|n|<N} \sum_{j=1}^{s} (\mathcal{L}_j f)(sn) S_j(t-sn) = \sum_{|n|\geq N} \sum_{j=1}^{s} (\mathcal{L}_j f)(sn) S_j(t-sn)\,.$$

The Cauchy-Schwarz inequality gives a first estimation for this error:

$$|E_N f(t)|^2 \leq \Big(\sum_{|n|\geq N} \sum_{j=1}^{s} |(\mathcal{L}_j f)(sn)|^2 \Big) \Big(\sum_{|n|\geq N} \sum_{j=1}^{s} |S_j(t-sn)|^2 \Big). \tag{5}$$

Since $\{(\mathcal{L}_j f)(sn)\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ are the coefficients of f with respect to the Riesz basis $\{S_j(t-sn)\}_{n\in\mathbb{Z},j=1,2,\cdots,s}$, the factor $\sum_{|n|\geq N}\sum_{j=1}^{s}|(\mathcal{L}_j f)(sn)|^2$ in (5) goes to 0 when $N\to\infty$. Without a prior knowledge on the function $f\in V_\varphi$, nothing can be said about its convergence speed. However, an estimation for $\|f\|^2_{L^2(\mathbb{R})}$ gives a bound for this factor. In fact, the Riesz basis condition for $\{S_j(t-sn)\}_{n\in\mathbb{Z},j=1,2,\cdots,s}$ gives

$$\sum_{|n|\geq N}\sum_{j=1}^{s}|(\mathcal{L}_j f)(sn)|^2 \leq \frac{1}{A}\|f\|^2_{L^2(\mathbb{R})} - \sum_{|n|<N}\sum_{j=1}^{s}|(\mathcal{L}_j f)(sn)|^2\,, \tag{6}$$

where A is a lower Riesz bound for $\{S_j(t-sn)\}_{n\in\mathbb{Z},j=1,2,\cdots,s}$. For the calculation of the optimal Riesz bounds of $\{S_j(t-sn)\}_{n\in\mathbb{Z},j=1,2,\cdots,s}$ (see references [10, 11, 12, 23]). Extra assumptions on the function f allow interesting results. Indeed, see references [13, 15, 16] for the Shannon case, or [19] for the average sampling case.

The factor $\sum_{|n|\geq N}\sum_{j=1}^{s}|S_j(t-sn)|^2$ in (5) is independent of the function $f\in V_\varphi$, and it plays an important role in the study of the convergence speed. The following lemma allows us, whenever the reconstruction functions, S_j, have polynomial or exponential decay, to obtain a bound for this factor.

Lemma 1 *Let S be a complex function of real variable. The following statements are satisfied:*

(i) If $|S(t)|\leq L\,\mu^{|t|}$ *for* $|t|\geq b$*, where* $\mu\in(0,1)$ *and* $L>0$*, then*

$$\sum_{|n|\geq N}|S(t-sn)|^2 \leq \frac{2L^2\mu^{-2|t|}}{1-\mu^{2s}}\mu^{2sN}\,, \qquad |t|\leq sN-b\,.$$

(ii) If $|S(t)|\leq L\,|t|^{-\mu}$ *for* $|t|\geq b$*, where* $\mu>1/2$*,* $b>s$ *and* $L>0$*, then*

$$\sum_{|n|\geq N}|S(t-sn)|^2 \leq \frac{2L^2}{s(2\mu-1)(sN-s-|t|)^{2\mu-1}}\,, \quad |t|\leq sN-b\,.$$

Proof: Observe that if $|t|\leq sN-b$ and $|n|\geq N$ then $|t-sn|\geq b$. Hence, for $|t|\leq sN-b$ we have

$$\sum_{|n|\geq N}|S(t-sn)|^2 \leq L^2\sum_{|n|\geq N}\mu^{2|t-sn|} \leq 2L^2\mu^{-2|t|}\sum_{n=N}^{\infty}\mu^{2sn} = \frac{2L^2\mu^{-2|t|}}{1-\mu^{2s}}\mu^{2sN}\,,$$

by using the hypothesis in (i), or

$$\sum_{|n|\geq N}|S(t-sn)|^2 \leq L^2\sum_{|n|\geq N}\frac{1}{(|sn|-|t|)^{2\mu}} \leq \frac{2L^2}{s}\int_{sN-s-|t|}^{\infty}\frac{dx}{x^{2\mu}}$$
$$= \frac{2L^2}{s(2\mu-1)(sN-s-|t|)^{2\mu-1}}\,,$$

by using the hypothesis in (ii). □

The use of inequality (6) and Lemma 1 in (5) gives an explicit estimation for the truncation error. A couple of examples are shown in Section 4 (see examples 1 and 2).

When the generator φ and the impulse responses of the systems $\mathcal{L}_j$ have compact support, one can study the convergence speed of $E_N f(t)$ as $N \to \infty$ without any previous knowledge of the reconstruction functions, S_j. This is the subject of the next section.

3.1 The case of a stable generator φ with compact support

Assume that the generator φ and the impulse response of the systems $\mathcal{L}_j$ have compact support. For $j = 1, 2, \dots, s$, consider

$$\mathsf{g}_j(z) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j\varphi(n) z^n \qquad \text{and} \qquad \mathsf{G}(z) := \left[\mathsf{g}_j(e^{-2\pi i(k-1)/s}\, z)\right]_{j,k=1,2,\dots,s}.$$

Thus, $g_j(w) = \mathsf{g}_j(e^{-2\pi i w})$ and $\mathbf{G}(w) = \mathsf{G}(e^{-2\pi i w})$ for any $w \in \mathbb{R}$. As a consequence, $a_j(w) = \mathsf{a}_j(e^{-2\pi i w})$, where $a_j(w)$ (respectively $\mathsf{a}_j(z)$) is the jth entry in the first row of $\mathbf{G}^{-1}(w)$ (respectively $\mathsf{G}^{-1}(z)$).

In this case, only a finite number of terms in $\{(\mathcal{L}_j\varphi)(n)\}_{n\in\mathbb{Z}}$ are different from zero. Hence, the functions

$$\mathsf{g}_j(e^{-2\pi i(k-1)/s}\, z) = \sum_{n\in\mathbb{Z}} (\mathcal{L}_j\varphi)(n) e^{-2\pi i n(k-1)/s} z^n\,, \quad j,k = 1,2,\dots,s$$

and $\det \mathsf{G}(z)$ are Laurent polynomials. The condition $\alpha_{\mathbf{G}} > 0$ implies, by continuity, that $\det \mathbf{G}(w) \neq 0$ for all $w \in \mathbb{R}$ or, equivalently, $\det \mathsf{G}(z) \neq 0$ on the unit circle $|z| = 1$.

As we will see, the convergence speed of the sampling expansion (4) depends on the zeros of the Laurent polynomial $\det \mathsf{G}(z)$. As it is proved in the next result, the best situation occurs when $\det \mathsf{G}(z)$ has no zeros in $\mathbb{C} \setminus \{0\}$. In this case the sampling functions S_j, $j = 1, 2, \dots, s$, have compact support and, consequently, for N large enough there is no truncation error in (4).

Theorem 1 *Assume that φ and the impulse response of $\mathcal{L}_j$, $j = 1, 2, \cdots, s$ have compact support. If $\det \mathsf{G}(z)$ has no zeros in $\mathbb{C} \setminus \{0\}$, then the reconstruction functions S_j, $j = 1, 2, \dots, s$, have compact support.*

Proof: The function $S_j(t)$ can be obtained from the Laurent expansion of $\mathsf{a}_j(z)$ in an annulus containing the unit circle, i.e., $\mathsf{a}_j(z) := \sum_{n\in\mathbb{Z}} c_{j,n} z^n$.

Indeed, for $j = 1, 2, \dots, s$,

$$\begin{aligned} S_j(t) &= s[\mathcal{T}_\varphi a_j](t) = s[\mathcal{T}_\varphi \mathsf{a}_j(e^{-2\pi i \cdot})](t) = s\Big[\mathcal{T}_\varphi \sum_{n\in\mathbb{Z}} c_{j,n} e^{-2\pi n i \cdot}\Big](t) \\ &= s\sum_{n\in\mathbb{Z}} c_{j,n} (\mathcal{T}_\varphi e^{-2\pi i n \cdot})(t) = s\sum_{n\in\mathbb{Z}} c_{j,n}\varphi(t-n)\,, \quad t \in \mathbb{R}, \end{aligned}$$

whenever the coefficients $\{c_{j,n}\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. As the Laurent polynomial $\det \mathsf{G}(z)$ has no zeros in $\mathbb{C}\setminus\{0\}$, then necessarily it has the form $\det \mathsf{G}(z) = Kz^n$ for some constant $K\in\mathbb{C}$ and $n\in\mathbb{Z}$. Hence, the entries of the matrix $\mathsf{G}^{-1}(z)$ and, in particular, $\mathsf{a}_j(z)$, $j=1,2,\dots,s$, are Laurent polynomials. As a consequence, only a finite number of coefficients $\left\{c_{j,n}\right\}_{n\in\mathbb{Z}}$ are different from zero. As φ has compact support, the functions S_j, $j=1,2\dots,s$, also have compact support. □

In general, the Laurent polynomial $\det \mathsf{G}(z)$ has zeros in $\mathbb{C}\setminus\{0\}$. Let ρ be the smallest number in $(0,1)$ such that $\det \mathsf{G}(z)$ is not zero at the annulus

$$C_\rho := \{z\in\mathbb{C} : \rho < |z| < 1/\rho\}\,.$$

The next theorem gives an explicit bound for the truncation error whenever $\det \mathsf{G}(z)$ has simple zeros on ∂C_ρ, the boundary of C_ρ.

Theorem 2 *Assume that φ and the impulse response of $\mathcal{L}_j$, $j=1,2,\cdots,s$ have compact support, and that $\operatorname{supp}\varphi\subseteq[c,d]$. If $\det \mathsf{G}(z)$ is not zero on C_ρ and has only simple zeros on ∂C_ρ, then for each $f\in V_\varphi$*

$$|E_N f(t)| \le K\, h_f(N)\, \rho^{-|t|}\ \rho^{sN}\,, \qquad |t|\le sN - \max\{|c|,|d|\}\,,$$

where $h_f^2(N) := \sum_{|n|\ge N}\sum_{j=1}^s |(\mathcal{L}_j f)(sn)|^2$ and the constant K is given by

$$K := \frac{s\|\varphi\|_\infty(d-c)}{\rho^{\max\{|c|,|d|\}}}\sqrt{\frac{2\sum_{j=1}^s \Upsilon_j^2}{1-\rho^{2s}}}\,,$$

with

$$\begin{aligned}\Upsilon_j := \max\Big\{ &\max_{|z|=\rho}|f_j(z)| + \sum_{k=1}^{m_j^{(\rho)}} |\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(\rho)})|\rho^{-1},\\ &\max_{|z|=1/\rho}|f_j(z)| + \sum_{k=1}^{m_j^{(1/\rho)}} |\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(1/\rho)})|\rho\Big\}\,,\quad j=1,2,\dots,s\end{aligned}$$

and

$$f_j(z) := \mathsf{a}_j(z) - \sum_{k=1}^{m_j^{(\rho)}} \frac{\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(\rho)})}{z - z_{j,k}^{(\rho)}} - \sum_{k=1}^{m_j^{(1/\rho)}} \frac{\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(1/\rho)})}{z - z_{j,k}^{(1/\rho)}}\,,$$

where for $\mu=\rho,1/\rho$ and $j=1,2,\dots,s$, the points $z_{j,1}^{(\mu)},\dots,z_{j,m_j^{(\mu)}}^{(\mu)}$ are the poles of $\mathsf{a}_j(z)$ on the circle $|z|=\mu$.

Proof: For the sake of clarity the proof is divided into three steps:

Step 1: First we prove that for each $j = 1, 2, \dots, s$, the coefficients $c_{j,n}$ in the Laurent expansion $\mathsf{a}_j(z) := \sum_{n\in\mathbb{Z}} c_{j,n} z^n$ in the annulus C_ρ satisfy

$$|c_{j,n}| \le \Upsilon_j \rho^{|n|}, \quad n \in \mathbb{Z}. \tag{7}$$

Notice that $\mathsf{a}_j(z)$ is a rational function with simple poles on ∂C_ρ. Thus,

$$\mathsf{a}_j(z) = f_j(z) + \sum_{k=1}^{m_j^{(\rho)}} \frac{\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(\rho)})}{z - z_{j,k}^{(\rho)}} + \sum_{k=1}^{m_j^{(1/\rho)}} \frac{\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(1/\rho)})}{z - z_{j,k}^{(1/\rho)}},$$

where the function $f_j(z)$ is analytic in an annulus containing C_ρ. The coefficients of its Laurent expansion,

$$f_j(z) = \sum_{n\in\mathbb{Z}} d_{j,n} z^n, \quad z \in C_\rho,$$

can be expressed as

$$d_{j,n} := \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f_j(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_{|z|=1/\rho} \frac{f_j(z)}{z^{n+1}}\,dz.$$

Consequently,

$$|d_{j,n}| \le \max_{|z|=1/\rho} |f_j(z)|\rho^{|n|}, \quad n \ge 0, \qquad |d_{j,n}| \le \max_{|z|=\rho} |f_j(z)|\rho^{|n|}, \quad n < 0.$$

Now, bearing in mind the expansions

$$\frac{1}{z - z_{j,k}^{(\rho)}} = \sum_{n=1}^{\infty} \frac{\big[z_{j,k}^{(\rho)}\big]^{n-1}}{z^n}, \quad |z| > \rho, \qquad \frac{1}{z_{j,k}^{(1/\rho)} - z} = \sum_{n=0}^{\infty} \frac{z^n}{\big[z_{j,k}^{(1/\rho)}\big]^{n+1}}, \quad |z| < 1/\rho$$

we obtain that the Laurent coefficients of $\mathsf{a}_j(z)$ satisfy the inequalities

$$|c_{j,n}| \le \Big[\max_{|z|=1/\rho} |f_j(z)| + \sum_{k=1}^{m_j^{(1/\rho)}} |\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(1/\rho)})|\rho\Big]\rho^{|n|}, \quad n \ge 0,$$

$$|c_{j,n}| \le \Big[\max_{|z|=\rho} |f_j(z)| + \sum_{k=1}^{m_j^{(\rho)}} |\operatorname{Res}(\mathsf{a}_j, z_{j,k}^{(\rho)})|\rho^{-1}\Big]\rho^{|n|}, \quad n < 0,$$

which proves inequality (7).

Step 2: For $j = 1, 2, \dots, s$ the reconstruction function, S_j, satisfies

$$|S_j(t)| \le s\|\varphi\|_\infty \rho^{-\max\{|c|,|d|\}}(d-c)\Upsilon_j \rho^{|t|}, \quad |t| \ge \max\{|c|,|d|\}. \tag{8}$$

As $S_j(t) = s\sum_{n\in\mathbb{Z}} c_{j,n}\varphi(t-n)$ and $\operatorname{supp}\varphi \subseteq [c,d]$, we have

$$|S_j(t)| \le s\sum_{n\in\mathbb{Z}} |c_{j,n}\varphi(t-n)| = s\sum_{n=[t-d]+1}^{[t-c]} |c_{j,n}\varphi(t-n)|$$
$$\le s\|\varphi\|_\infty \Upsilon_j \sum_{n=[t-d]+1}^{[t-c]} \rho^{|n|}\,,$$

where $[t-d]$ (respectively $[t-c]$) denotes the integer part of $t-d$ (respectively $t-c$). Whenever $t \ge |d|$ and $n \in [t-d, t-c]$ we have $n \ge t-d = |t|-d \ge |t|-|d| \ge 0$ and, hence, $|n| = n \ge |t|-|d|$. Whenever $t \le -|c|$ and $n \in [t-d, t-c]$ we have $n \le t-c \le t+|c| \le 0$ and, hence, $|n| = -n \ge -t-|c| = |t|-|c|$. Therefore, for $|t| > \max\{|c|,|d|\}$ we obtain

$$|S_j(t)| \le s\|\varphi\|_\infty \Upsilon_j \sum_{n=[t-d]+1}^{[t-c]} \rho^{|t|-\max\{|c|,|d|\}} \le s\|\varphi\|_\infty (d-c)\Upsilon_j \rho^{-\max\{|c|,|d|\}}\rho^{|t|}.$$

Step 3: From (8) and Lemma 1 we prove the theorem. Indeed, taking $\mu = \rho$ and $L = s\|\varphi\|_\infty (d-c)\Upsilon_j \rho^{-\max\{|c|,|d|\}}$ in Lemma 1, the inequality (8) gives

$$\sum_{j=1}^{s}\sum_{|n|\ge N} |S_j(t-sn)|^2 \le \sum_{j=1}^{s} \frac{2\big(s\|\varphi\|_\infty (d-c)\Upsilon_j \rho^{-\max\{|c|,|d|\}}\big)^2 \rho^{-2|t|}}{1-\rho^{2s}} \rho^{2sN}\,,$$

for $|t| \le sN - \max\{|c|,|d|\}$. Finally, the inequality (5) concludes the proof. □

Notice that the sequence $h_f(N)$ goes to 0 when $N \to \infty$. As a consequence we deduce the following corollary:

Corollary 1 *As in Theorem 2, for any $f \in V_\varphi$ we have $E_N f(t) = o\,(\rho^{sN})$ as $N \to \infty$ uniformly in compact subsets of $\mathbb{R}$.*

Proof: For $t \in [-C, C]$ Theorem 2 gives

$$|E_N f(t)| \le K\, h_f(N)\, \rho^{-C}\ \rho^{sN}\,, \qquad \text{for any } N \ge \frac{C + \max\{|c|,|d|\}}{s}\,.$$

□

Corollary 1 admits the following generalization in the general case.

Theorem 3 *Assume that φ and the impulse response of $\mathcal{L}_j$, $j = 1, 2, \cdots, s$ have compact support. If $\det \mathsf{G}(z)$ has zeros in $\mathbb{C}\setminus\{0\}$, then*

$$E_N f(t) = o\big(N^{v-1}\rho^{sN}\big) \quad \text{as } N \to \infty$$

uniformly in compact subsets of $\mathbb{R}$, where ρ is the smallest number in $(0,1)$ such that $\det \mathsf{G}(z)$ is not zero on the annulus $C_\rho := \{z \in \mathbb{C} : \rho < |z| < 1/\rho\}$ and v is the greatest order among the zeros of $\det \mathsf{G}(z)$ on ∂C_ρ.

Proof: As in Theorem 2 we divide the proof into three steps:

Step 1: First, for $j=1,2,\dots,s$, there exists a constant $\Theta_j>0$ such that the Laurent coefficients in the expansion $\mathsf{a}_j(z):=\sum_{n\in\mathbb{Z}} c_{j,n}z^n$ on the annulus C_ρ satisfy the inequality

$$|c_{j,n}|\le \Theta_j |n|^{\upsilon-1}\rho^{|n|}\,,\quad \forall n\in\mathbb{Z}\,. \tag{9}$$

Notice that the function $\mathsf{a}_j(z)$, $j=1,2\dots,s$ has no poles of order greater than υ on ∂C_ρ. As a consequence,

$$\mathsf{a}_j(z)=f_j(z)+\frac{K_1}{(z-z_1)^{\sigma_1}}+\ldots+\frac{K_{m_j}}{(z-z_{m_j})^{\sigma_{m_j}}}$$

where the function $f_j(z)$ is analytic on an annulus containing C_ρ, $K_l\in\mathbb{C}$, $\sigma_l\in\mathbb{N}$, $\sigma_l\le\upsilon$ and the points $z_1,\dots,z_{m_j}$ (not necessarily different) are located on ∂C_ρ. The Laurent coefficients $d_{j,n}$ of $f_j(z)$ on the annulus C_ρ verify

$$|d_{j,n}|\le \max_{\rho\le|z|\le 1/\rho}|f_j(z)|\,\rho^{|n|}\,,\quad n\in\mathbb{Z}\,.$$

Bearing in mind the expansions

$$\frac{1}{(z_l-z)^{\sigma_l}}=\sum_{n=\sigma_l}^{\infty}\frac{(n-1)(n-2)\dots(n-\sigma_l+1)\,z^{n-\sigma_l}}{(\sigma_l-1)!\,z_l^n}\,,\quad |z|<1/\rho,$$

valid when $|z_l|=1/\rho,$ and

$$\frac{1}{(z-z_l)^{\sigma_l}}=\sum_{n=\sigma_l}^{\infty}\frac{(n-1)(n-2)\dots(n-\sigma_l+1)\,z_l^{n-\sigma_l}}{(\sigma_l-1)!\,z^n}\,,\quad |z|>\rho,$$

valid when $|z_l|=\rho,$ we obtain (9).

Step 2: Let $b>0$ be such that $\operatorname{supp}\varphi\subseteq[-b,b]$. Next we prove that for each $j=1,2,\dots,s$ there exists a constant Ψ_j such that

$$|S_j(t)|\le \Psi_j|t|^{\upsilon-1}\rho^{|t|}\,,\quad |t|\ge b\,. \tag{10}$$

As $S_j(t)=s\sum_{n\in\mathbb{Z}}c_{j,n}\varphi(t-n)$, we have

$$\begin{aligned}|S_j(t)|&\le s\sum_{n\in\mathbb{Z}}|c_{j,n}\varphi(t-n)|=s\sum_{n=[t-b]+1}^{[t+b]}|c_{j,n}\varphi(t-n)|\\&\le s\Theta_j\|\varphi\|_\infty\sum_{n=[t-b]+1}^{[t+b]}|n|^{\upsilon-1}\rho^{|n|}\,,\end{aligned}$$

where $[t \pm b]$ denotes the integer part of $t \pm b$ and $\|\varphi\|_\infty = \max_{t\in\mathbb{R}} |\varphi(t)|$. If $t \geq b$, then $|n| = n \geq t - b = |t| - b$ for $n \in [t-b, t+b]$. If $t \leq -b$, then $|n| = -n \geq -t - b = |t| - b$ for $n \in [t-b, t+b]$. As a consequence, for $|t| \geq b$ we have

$$\begin{aligned} |S_j(t)| &\leq s\Theta_j \rho^{|t|-b} \|\varphi\|_\infty \sum_{n=[t-b]+1}^{[t+b]} (|t|+b)^{v-1} \\ &\leq s\Theta_j \rho^{|t|-b} (|t|+b)^{v-1} \|\varphi\|_\infty 2b \,, \end{aligned}$$

which gives (10).

Step 3: For $|t| \leq sN - b$ and $|n| \geq N$ we have that $|t - sn| \geq b$. By using inequality (10), for $|t| \leq sN - b$ we obtain

$$\begin{aligned} \sum_{|n|\geq N} |S_j(t-sn)|^2 &\leq \Psi_j^2 \sum_{|n|\geq N} |t-sn|^{2v-2} \rho^{2|t-sn|} \\ &\leq \Psi_j^2 \sum_{|n|\geq N} (|t| + |sn|)^{2v-2} \rho^{2|sn|-2|t|} \\ &\leq 2\Psi_j^2 \rho^{-2|t|} \sum_{n=N}^{\infty} (|t| + sn)^{2v-2} \rho^{2sn} \,. \end{aligned}$$

For $t \in [-C, C]$ we have

$$\sum_{|n|\geq N} |S_j(t-sn)|^2 \leq 2\Psi_j^2 \rho^{-2C} \sum_{n=N}^{\infty} (C + sn)^{2v-2} \rho^{2sn}$$

whenever $N > (C+b)/s$.

Successive derivation with respect to x in $\sum_{n=N}^{\infty} x^n = x^N/(1-x)$, where $|x| < 1$, shows that for $l \in \mathbb{N} \cup \{0\}$ $\sum_{n=N}^{\infty} n^l x^n = p_l(N) x^N$ where p_l is a polynomial of degree l whose coefficients depend on x. As a consequence, for fixed x, l, the relationship $\sum_{n=N}^{\infty} n^l x^n = O(N^l x^N)$ as $N \to \infty$ holds. In our case, $\sum_{n=N}^{\infty} (C+sn)^{2v-2} \rho^{2sn} = O\big(N^{2v-2} \rho^{2sN}\big)$. Therefore,

$$\sum_{j=1}^{s} \sum_{|n|\geq N} |S(t-sn)|^2 = O(N^{2v-2} \rho^{2sN})$$

uniformly in $[-C, C]$.

Bearing in mind that $\sum_{|n|\geq N} \sum_{j=1}^{s} |(\mathcal{L}_j f)(sn)|^2$ goes to 0 as $N \to \infty$, the desired result follows from inequality (5). □

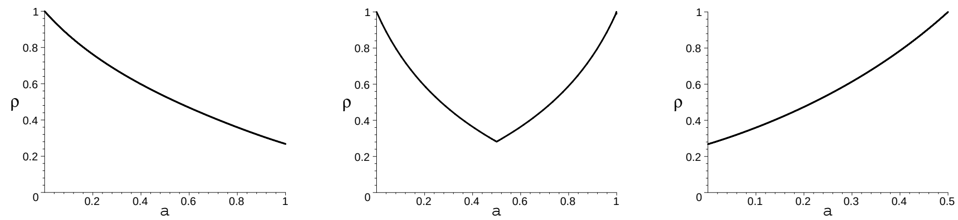

Figure 1: The rate, ρ, as a function of a in examples 1, 2, and 3, respectively

4 Examples for the cubic spline space

The B-splines are important examples of stable generators with compact support. For fixed $m \in \mathbb{N}$ the B-spline N_m of degree $m-1$ is defined as the convolution $N_m := N_1 * N_1 * \ldots * N_1$ (m times) where N_1 denotes the characteristic function of the interval $(0,1)$. The corresponding shift-invariant space V_{N_m} is the space of splines of degree $m-1$ in $L^2(\mathbb{R})$ with nodes at the integers (see [8]). As pointed out in [21], these spaces have been proven to be very fruitful in signal processing applications.

We illustrate the obtained results with three examples in the space of cubic splines, V_{N_4}. The considered systems in these examples depend on a parameter a: we calculate the convergence rate ρ in Theorem 2 in terms of it. For the value of the parameter a that makes the rate ρ minimum, we get an explicit bound for the truncation error $E_N f(t)$.

The cubic B-spline N_4 is

$$N_4(t) := (N_1 * N_1 * N_1 * N_1)(t) = \frac{t^3}{6}\mathcal{X}_{[0,1)}(t) + \Big(\frac{2}{3} - 2t + 2t^2 - \frac{t^3}{2}\Big)\mathcal{X}_{[1,2)}(t) + \Big(-\frac{22}{3} + 10t - 4t^2 + \frac{t^3}{2}\Big)\mathcal{X}_{[2,3)}(t) + \Big(\frac{32}{3} - 8t + 2t^2 - \frac{t^3}{6}\Big)\mathcal{X}_{[3,4)}(t)\,.$$

1. Interlaced sampling. We consider the reconstruction of a cubic spline from its samples at $2n$ and $2n+a$, $n \in \mathbb{Z}$. Without loss of generality we can assume that $a \in (0,1]$. This is a particular case of generalized sampling theory which corresponds to the systems $\mathcal{L}_1 f = f$ and $\mathcal{L}_2 f(t) = f(t+a)$. We have that

$$\begin{aligned}\det \mathbf{G}(z) =& \sum_{n=0}^{3} N_4(n) z^n \sum_{n=0}^{3} (-1)^n N_4(n+a) z^n \\ &- \sum_{n=0}^{3} N_4(n+a) z^n \sum_{n=0}^{3} (-1)^n N_4(n) z^n \\ =& \frac{a^3}{18} z + \Big(\frac{8a^3}{9} - a^2 - \frac{2a}{3}\Big) z^3 + \Big(\frac{7a^3}{18} - a^2 + \frac{2a}{3}\Big) z^5\,.\end{aligned}$$

The zeros of $\det \mathbf{G}(z)$ are $-\sqrt{\lambda}$, $-\sqrt{\mu}$, 0, $\sqrt{\mu}$, $\sqrt{\lambda}$, where $\mu < 1 < \lambda$ are the zeros of $a^3/18 + (8a^3/9 - a^2 - 2a/3)x + (7a^3/18 - a^2 + 2a/3)x^2$. Since $1/\lambda > \mu$, we have that the convergence rate ρ in Theorem 2 is given by

$$\rho^2 = \frac{1}{\lambda} = \frac{12 - 18a + 7a^2}{6 + 9a - 8a^2 + \sqrt{36 + 108a - 27a^2 - 126a^3 + 57a^4}}\,.$$

The rate ρ is a decreasing function of the interlaced parameter $a \in (0,1]$, and so the minimum rate ρ occurs when $a = 1$ (see Figure 1). Thus the best situation occurs in uniform sampling ($a = 1$) for which $\rho = 2 - \sqrt{3} \approx 0.26$. It is worth noting that Unser and Zerubia [23] obtained the same value $a = 1$ for stability purposes. For any $f \in V_{N_4}$ the corresponding sampling formula reads

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) L_4(t-n)\,, \quad t \in \mathbb{R}\,, \tag{11}$$

where $L_4(t) = \sqrt{3}\sum_{n\in\mathbb{Z}}(-2+\sqrt{3})^{|n|} N_4(t-n+2)$. We have that

$$E_N f(t) = \sum_{|n|\geq N} f(n) L_4(t-n) = o\big[(2-\sqrt{3}\,)^{2N}\big] \quad \text{when}\, N \to \infty\,,$$

uniformly on compact subsets of $\mathbb{R}$.

Now we give an explicit bound for this error. Remembering that the support of N_4 is $[0,4]$ and that $\sum_{n\in\mathbb{Z}} N_4(t+n) = 1$, $t \in \mathbb{R}$, we obtain

$$|L_4(t)| \leq \sqrt{3}\sum_{n\in\mathbb{Z}}(2-\sqrt{3}\,)^{|t|-2} N_4(t-n+2) \leq \sqrt{3}\,(2-\sqrt{3}\,)^{|t|-2}\,, \quad |t| > 2\,.$$

Then Lemma 1 gives

$$\sum_{n\in\mathbb{Z}} |L_4(t-n)|^2 \leq K(2-\sqrt{3}\,)^{2(N-|t|-2)}\,, \quad |t| < N-2\,,$$

where $K := 6/[1-(2-\sqrt{3}\,)^2]$. Following the method in [24] it is easy to calculate that the optimal lower Riesz bound of $\{L_4(t-n)\}_{n\in\mathbb{Z}}$ is 17/35. Using the inequalities (5) and (6) we obtain that for any $f \in V_{N_4}$,

$$|E_N f(t)|^2 \leq K\left(\frac{35}{17}\|f\|^2_{L^2(\mathbb{R})} - \sum_{|n|<N}|f(n)|^2\right)(2-\sqrt{3}\,)^{2(N-|t|-2)}\,, \quad |t| < N-2\,.$$

In general, consider the space $V_{N_{m+1}}$ of splines of odd degree m in $L^2(\mathbb{R})$ with nodes at the integers. For any $f \in V_{N_{m+1}}$ the sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) L_{m+1}(t-n)\,, \quad t \in \mathbb{R}\,,$$

holds, where L_{m+1} denotes the fundamental cardinal spline of degree m (see [8]). In this particular case,

$$\det \mathsf{G}(z) = \mathsf{G}(z) = \frac{z}{m!} E_m(z)\,,$$

where E_m denotes the Euler-Fröbenius polynomial of degree $m-1$. This polynomial can be written as

$$E_m(z) = \prod_{k=1}^{(m-1)/2} (z - x_k)(z - 1/x_k)\,,$$

where $-1 < x_{(m-1)/2} < \ldots < x_1 < 0$. Therefore, taking $\rho = |x_{(m-1)/2}|$ we have that $E_N f(t) = o\big(\rho^N\big)$ when $N \to \infty$ uniformly in compact subsets of $\mathbb{R}$.

2. Derivative sampling. We consider the reconstruction of a cubic spline from its samples and the samples of its derivative at $2n + a$, $n \in \mathbb{Z}$. We assume without loss of generality that $a \in [0, 1)$. Take the systems $\mathcal{L}_1 f(t) = f(t + a)$ and $\mathcal{L}_2 f(t) = f'(t + a)$. In this case we have

$$\begin{aligned}\det \mathbf{G}(z) = &\Big(\frac{2}{3} - 2a + \frac{13}{6}a^2 - a^3 + \frac{1}{6}a^4\Big) z^5 \\ &+ \Big(-\frac{2}{3} - 2a + \frac{5}{3}a^2 + \frac{2}{3}a^3 - \frac{1}{3}a^4\Big) z^3 \\ &+ \Big(\frac{1}{6}a^2 + \frac{1}{3}a^3 + \frac{1}{6}a^4\Big) z\,.\end{aligned}$$

When $a = 0$, $\det \mathbf{G}(1) = 0$ and then the necessary condition $\det \mathbf{G}(z) \neq 0$, $|z| = 1$, is not satisfied. For $a \in (0, 1)$ the zeros of $\det \mathbf{G}(z)$ are $-\sqrt{\lambda}$, $-\sqrt{\mu}$, 0, $\sqrt{\mu}$, $\sqrt{\lambda}$, where $\mu < 1 < \lambda$ are the zeros of $a^2 + 2a^3 + a^4 + (10a^2 - 4 - 12a + 4a^3 - 2a^4)x + (4 - 12a + 13a^2 - 6a^3 + a^4)x^2$. If $a \le 1/2$ then $1/\lambda$ is a decreasing function of a and $1/\lambda > \mu$. If $a = 1/2$ then $1/\lambda = \mu$. If $a \ge 1/2$ then μ is a increasing function of a and $\mu > 1/\lambda$. Hence,

$$\rho^2 = \begin{cases} 1/\lambda & a \in (0, 1/2] \\ \mu & a \in (1/2, 1) \end{cases}.$$

The minimum value of ρ is $\sqrt{(19 - 4\sqrt{22})/3} \approx 0.28$ and it occurs when $a = 1/2$ (see Figure 1).

On the other hand, for any $f \in V_{N_4}$ the sampling formula (4) reads

$$f(t) = \sum_{n\in\mathbb{Z}} \big[f(2n + 1/2) S_1(t - 2n) + f'(2n + 1/2) S_2(t - 2n)\big]\,, \quad t \in \mathbb{R}\,,$$

where

$$S_1(t) := \frac{1}{\sqrt{22}} \sum_{n\in\mathbb{Z}} b^{|n+1|} s_1(t - 2n)\,, \quad S_2(t) := \frac{1}{6\sqrt{22}} \sum_{n\in\mathbb{Z}} b^{|n+1|} s_2(t - 2n)\,,$$

$b := (19 - 4\sqrt{22})/3$, and

$$s_1(t) := N_4(t+1) - 5N_4(t) - 5N_4(t-1) + N_4(t-2)\,,$$
$$s_2(t) := N_4(t+1) - 23N_4(t) + 23N_4(t-1) - N_4(t-2)\,.$$

Bearing in mind that the support of s_1 is $[-1,6]$ and that

$$\sum_{n\in\mathbb{Z}} |s_1(t-2n)| \le \sum_{n\in\mathbb{Z}} N_4(t+n) + 5N_4(t+n) = 6\,,$$

for $|t| > 4$ we obtain

$$|S_1(t)| \le \frac{1}{\sqrt{22}}\, b^{|t/2|-2} \sum_{n\in\mathbb{Z}} |s_1(t-2n)| \le \frac{6}{\sqrt{22}} b^{|t/2|-2}\,.$$

Hence, Lemma 1 gives

$$\sum_{|n|\ge N} |S_1(t-2n)|^2 \le \frac{36\, b^{2N-|t|-4}}{11(1-b^2)}\ , \quad |t| < 2N-4\,.$$

Analogously we obtain

$$\sum_{|n|\ge N} |S_2(t-2n)|^2 \le \frac{16\, b^{2N-|t|-4}}{11(1-b^2)}\ , \quad |t| < 2N-4\,.$$

By using Corollary 1 in [10] we obtain that 136/2835 is a lower bound for the Riesz basis $\{S_j(t-2n)\}_{j=1,2,n\in\mathbb{Z}}$. Therefore, from (5) and (6), for $|t| < 2N-4$, we obtain

$$|E_N f(t)|^2 \le K\Big(C\|f\|^2_{L^2(\mathbb{R})} - \sum_{|n|<N} \big[|f(2n+1/2)|^2 + |f'(2n+1/2)|^2\big]\Big) b^{2N-|t|-4}\,,$$

where $b = (19 - 4\sqrt{22})/3 \approx 0.079$, $C = 2835/136$ and $K = 52/(11 - 11b^2)$.

3. Uniform sampling. We consider the reconstruction of a cubic spline from its samples at $n+a$, $n \in \mathbb{Z}$. We assume without loss of generality that $a \in [0,1/2]$. We take the system $\mathcal{L}_1 f(t) = f(t+a)$. For $a = 1/2$, the necessary condition $\det \mathbf{G}(z) \ne 0$ at $|z| = 1$ fails. For $a \in [0,1/2)$ the rate ρ is the absolute value of the zero of $a^3 + (1+3a+3a^2-3a^3)z + (4-6a^2+3a^3)z^2 + (1-a)^3 z^3$ in $(-1, \sqrt{3}-2]$. Since ρ is an increasing function of a, its minimum occurs at $a = 0$ (see Figure 1). The corresponding sampling formula is again (11).

ACKNOWLEDGEMENT

This work has been supported by the grant BFM2003–01034 from the D.G.I. of the Spanish *Ministerio de Ciencia y Tecnología.*

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