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A general sampling theory in the functional Hilbert space induced by a Hilbert space valued kernel

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Abstract

Let $\mathbb{H}$ be a separable Hilbert space and Ω a fixed subset of $\mathbb{R}$. Consider an $\mathbb{H}$-valued function $K:\Omega \longrightarrow \mathbb{H}$ and $x \in \mathbb{H}$. Then, the function $f_x:\Omega \longrightarrow \mathbb{C}$ given by $f_x(t) := \langle x, K(t)\rangle_{\mathbb{H}}$ is well-defined. Denote by $\mathcal{H}_K$ the set of functions obtained in this way. Although a variety of sampling results for $\mathcal{H}_K$ is known in the literature, there exist some simple examples where they do not apply because the implicit interpolation condition that appears does not adjust the former pattern we need. The main aim of this paper is to obtain a more general sampling result including most of these special cases. In this way, the concept of interpolation condition is redefined and we study how to combine two of them in order to obtain a new sampling result. Some examples are obtained in this new framework.

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1 Statement of the problem

For the past few years a significant mathematical literature on the topic of sampling theorems associated with differential or difference problems has flourished [1, 4, 5, 6, 7, 8]. See also [15] and the references therein. In turn, we might consider the Weiss-Kramer sampling theorem as the *leitmotiv* of all these results [11, 13]. Roughly speaking, the common situation for these sampling problems is the following:

Let f be a function defined on $\mathbb{C}$ by $f(t)=\int_I F(x)\,K(x,t)\,dx$, $F\in L^2(I)$, (or $f(t)=\sum_n F(n)\,K(n,t)$, $F\in\ell^2$). The kernel K, which belongs to $L^2(I)$ (or ℓ^2) for each fixed $t\in\mathbb{C}$, satisfies the differential (difference) equation appearing in a differential (difference) problem (P) which has the sequence of eigenvalues $\{t_n\}$. Moreover, whenever we substitute in K the spectral parameter t by $\{t_n\}$ we obtain the sequence of orthogonal eigenfunctions associated with (P) which constitutes an orthogonal basis for $L^2(I)$ (ℓ^2). Under these circumstances,

f is an entire function which can be recovered from its samples $\{f(t_n)\}$ by means of a sampling formula $f(t) = \sum_n f(t_n)\, S_n(t)$, where the sampling functions $\{S_n\}$ are given by $S_n(t) = \|K(\cdot, t_n)\|^{-2} \langle K(\cdot, t), K(\cdot, t_n)\rangle$ (the inner product in $L^2(I)$ or ℓ^2).

All the results above can be formulated in an abstract way following the approach in Saitoh's book [12]. Namely, let $\mathbb{H}$ be a separable Hilbert space, and Ω a fixed subset of $\mathbb{R}$. Given an $\mathbb{H}$-valued function $K : \Omega \to \mathbb{H}$, for $x \in \mathbb{H}$, the function $f(t) := \langle x, K(t)\rangle_{\mathbb{H}}$ is well-defined as a function $f : \Omega \to \mathbb{C}$. We denote by $\mathcal{H}_K$ the set of functions obtained in this way and by T the linear transform

$$\begin{array}{rccc} T: & \mathbb{H} & \longrightarrow & \mathcal{H}_K \\ & x & \longmapsto & f \end{array} \tag{1}$$

Hereafter we refer the function K as the *kernel* of the transform T. Note that the continuity of the kernel K implies that the functions in $\mathcal{H}_K$ are continuous in Ω, a natural setting for sampling purposes. If we define in $\mathcal{H}_K$ the norm $\|f\|_{\mathcal{H}_K} = \inf\{\|x\|_{\mathbb{H}} : f = T(x)\}$ we obtain a reproducing kernel Hilbert space (RKHS hereafter) whose reproducing kernel is given by $k(t, s) := \langle K(s), K(t)\rangle_{\mathbb{H}}$ i.e., for each $s \in \Omega$ the function k_s defined as $k_s(t) := k(t, s)$ belongs to $\mathcal{H}_K$, and the reproducing property

$$f(s) = \langle f, k_s\rangle_{\mathcal{H}_K} = \langle f, k(\cdot, s)\rangle_{\mathcal{H}_K}\,, \quad s \in \Omega\,, \quad f \in \mathcal{H}_K \tag{2}$$

holds. Recall that the Moore-Aronszajn procedure [2] leads to the same RKHS via the *positive definite (or positive matrix)* function k. Under these circumstances it is known that the linear operator T is one-to-one if and only if T is an isometry between $\mathbb{H}$ and $\mathcal{H}_K$, or, equivalently, if and only if the set of functions $\{K(t)\}_{t\in\Omega}$ is complete in $\mathbb{H}$ [12]. An important property of $\mathcal{H}_K$ is that convergence in its norm implies pointwise convergence. In fact, by the reproducing property, we have that

$$|f(t) - f_n(t)| = |\big\langle f - f_n, k(\cdot, t)\big\rangle| \le \|f - f_n\|_{\mathcal{H}_K} \|K(t)\|_{\mathbb{H}}\,.$$

Notice that the last inequality also implies uniform convergence in subsets of Ω where the function $k(t, t) = \|K(t)\|^2$ is bounded. The RKHS $\mathcal{H}_K$ has been largely studied in the mathematical literature (see the superb monograph [12] and references therein).

A sampling result for $\mathcal{H}_K$ can be easily established (see [9] or [10]). Namely, let $\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ be a pair of biorthonormal Riesz bases for a Hilbert space $\mathbb{H}$. Assume that, for each fixed $t \in \Omega$, $K(t)$ can be written as $K(t) = \sum_{n=1}^\infty \overline{S_n(t)}\, x_n^*$, where the functions $S_n \in \mathcal{H}_K$ satisfy, for some fixed sequence $\{t_n\}_{n=1}^\infty$ in Ω, the interpolation property: $S_n(t_m) = a_n \delta_{n,m}$ for some constants $\{a_n\}_{n=1}^\infty \subset \mathbb{C} \setminus \{0\}$. Then, any function $f \in \mathcal{H}_K$ can be expanded as

$$f(t) = \sum_{n=1}^{\infty} f(t_n) \frac{S_n(t)}{a_n}\,, \quad t \in \Omega\,, \tag{3}$$

where the convergence of the series is absolute and uniform on subsets of Ω where the function $\|K(t)\|$ is bounded.

Recall that a Riesz basis $\{w_n\}_{n=1}^\infty$ for $\mathbb{H}$ is the image of an orthonormal basis by means of a bounded invertible operator on $\mathbb{H}$. Any Riesz basis $\{w_n\}_{n=1}^\infty$ has a unique biorthonormal (dual) Riesz basis $\{w_n^*\}_{n=1}^\infty$, i.e., such that $\langle w_n, w_m^*\rangle_{\mathbb{H}} = \delta_{n,m}$, and the expansions

$$x = \sum_{n=1}^{\infty} \langle x, w_n^*\rangle_{\mathbb{H}} w_n = \sum_{n=1}^{\infty} \langle x, w_n\rangle_{\mathbb{H}} w_n^*$$

hold for every $x \in \mathbb{H}$ (see [3] or [14] for more details and proofs).

Perhaps the most important examples of RKHS $\mathcal{H}_K$ that verify the mentioned result are the classical Paley-Wiener spaces of bandlimited functions, i.e., square integrable functions in $\mathbb{R}$ such that their Fourier transforms are zero outside a bounded set in $\mathbb{R}$. For instance, any function of the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x) e^{itx} dx \,, \quad t \in \mathbb{R} \,,$$

where $F \in L^2[-\pi, \pi]$, can be expanded as the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc}(t-n) \,, \quad t \in \mathbb{R} \,,$$

where sinc stands for cardinal sine function (or sinc function) defined as $\operatorname{sinc} t = \sin \pi t / \pi t$ for $t \neq 0$ and $\operatorname{sinc} 0 = 1$.

The sampling series (3) might also contain samples from functions which are related to f in some sense. Thus, this sampling result can be generalized in the following way: Let $\{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty} \cup \{y_n^*\}_{n=1}^{\infty}$ be a pair of biorthonormal Riesz bases for the Hilbert space $\mathbb{H}$. For each fixed $t \in \Omega$, $K(t)$ can be written as

$$K(t) = \sum_{n=1}^{\infty} \left[\overline{S_n(t)} x_n^* + \overline{T_n(t)} y_n^* \right] ,$$

where $S_n(t)$ and $T_n(t)$ denote the evaluation at $t \in \Omega$ of the functions $S_n = T(x_n) \in \mathcal{H}_K$ and $T_n = T(y_n) \in \mathcal{H}_K$ obtained by means of the linear transform (1) .

Assume that there exist two kernels $K_1, K_2 : \Omega \longrightarrow \mathbb{H}$ each defining a function in the way K does, i.e., $f_j(t) := \langle x, K_j(t) \rangle_{\mathbb{H}}$, $j = 1, 2$. Let T_1 and T_2 be the corresponding linear transforms. These kernels can be written as

$$K_j(t) = \sum_{n=1}^{\infty} \left[\overline{S_n^j(t)} x_n^* + \overline{T_n^j(t)} y_n^* \right] , \quad j = 1, 2 \,,$$

where $S_n^j(t) = T_j(x_n)[t]$ and $T_n^j(t) = T_j(y_n)[t]$ for $t \in \Omega$, $j = 1, 2$. Suppose there exist two sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ in Ω such that the functions $\{S_n^j\}_{n=1}^{\infty}$ and $\{T_n^j\}_{n=1}^{\infty}$, $j = 1, 2$, satisfy the interpolation conditions

$$\begin{array}{lll} S_n^1(s_m) = a_n \delta_{n,m}; & T_n^1(s_m) = b_n \delta_{n,m}, & \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C} \\ S_n^2(t_m) = c_n \delta_{n,m}; & T_n^2(t_m) = d_n \delta_{n,m}, & \{c_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty} \subset \mathbb{C} \,, \end{array}$$

where $\Delta_n := a_n d_n - b_n c_n \neq 0$ for all $n \in \mathbb{N}$. Suppose that f and the functions f_1 and f_2 are related in the sense that $\ker T \subseteq \ker T_1 \cap \ker T_2$. This implies that $\ker T = \{0\}$ so that $\mathcal{H}_K$ becomes a RKHS under the inner product $\langle f, g \rangle_{\mathcal{H}_K} := \langle x, y \rangle_{\mathbb{H}}$ where $Tx = f$ and $Ty = g$. Under these conditions, the technique used in [9] gives the following result:

Theorem 1 *The sequence $\{S_n\}_{n=1}^{\infty} \cup \{T_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}_K$ and, expansions with respect to this basis allow to recover any function f in $\mathcal{H}_K$ from the samples $\{f_1(s_n)\}_{n=1}^{\infty}$*

and $\{f_2(t_n)\}_{n=1}^{\infty}$ *of* f_1 *and* f_2*, by means of the sampling formula*

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty}\Big[\frac{d_n f_1(s_n) - b_n f_2(t_n)}{\Delta_n} S_n(t) + \frac{a_n f_2(t_n) - c_n f_1(s_n)}{\Delta_n} T_n(t)\Big] \\ &= \sum_{n=1}^{\infty}\Big[f_1(s_n)\frac{d_n S_n(t) - c_n T_n(t)}{\Delta_n} + f_2(t_n)\frac{a_n T_n(t) - b_n S_n(t)}{\Delta_n}\Big], \quad t \in \Omega\,. \end{aligned} \tag{4}$$

The convergence of the series is absolute and uniform on subsets of Ω *where the function* $\|K(t)\|$ *is bounded.*

Using a matrix notation, formula (4) can be written as

$$f(t) = \sum_{n=1}^{\infty} \begin{pmatrix} f_1(s_n) & f_2(t_n) \end{pmatrix} \begin{pmatrix} a_n & c_n \\ b_n & d_n \end{pmatrix}^{-1} \begin{pmatrix} S_n(t) \\ T_n(t) \end{pmatrix}, \tag{5}$$

from which it is not difficult to derive a more general result involving the samples of N functions related to f.

Multi-channel sampling in Paley-Wiener spaces can be easily derived from this approach [9]. As a different example of Theorem 1 we can obtain a *Hermite-type interpolation series* for $\mathcal{H}_K$, i.e., a sampling series which involves samples of any function $f \in \mathcal{H}_K$ and its first derivative, and in addition, the sampling functions generalize the classical Hermite interpolation polynomials. Indeed, let $\{t_n\}_{n=1}^{\infty}$ be a sequence of distinct nonzero real numbers such that $\sum_{n=1}^{\infty} |t_n|^{-2} < \infty$. There exists an entire function $P(t)$ with simple zeros at the sequence $\{t_n\}_{n=1}^{\infty}$. Specifically, the function $P(t)$ is given by the canonical product

$$P(t) = \begin{cases} \displaystyle\prod_{n=1}^{\infty}\Big(1 - \frac{t}{t_n}\Big)\exp(t/t_n) & \text{if } \displaystyle\sum_{n=1}^{\infty}|t_n|^{-1} = \infty \\ \displaystyle\prod_{n=1}^{\infty}\Big(1 - \frac{t}{t_n}\Big) & \text{if } \displaystyle\sum_{n=1}^{\infty}|t_n|^{-1} < \infty\,. \end{cases}$$

Consider $\{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty} \cup \{y_n^*\}_{n=1}^{\infty}$ a pair of biorthonormal Riesz bases for $\mathbb{H}$ and $Q(t) := P(t)^2$, which has double zeros at $\{t_n\}_{n=1}^{\infty}$. Take the functions

$$S_n(t) = \frac{Q(t)}{(t - t_n)^2} \quad \text{and} \quad T_n(t) = \frac{Q(t)}{t - t_n}$$

and define the kernels $K(t), K_1(t)$ and $K_2(t)$, $t \in \mathbb{R}$, by

$$K(t) = \sum_{n=1}^{\infty}\Big[\frac{Q(t)}{(t - t_n)^2} x_n^* + \frac{Q(t)}{t - t_n} y_n^*\Big],$$

$K_1(t) = K(t)$ and $K_2(t) = K'(t)$. It is easy to check the interpolation conditions:

$$S_n(t_m) = \frac{Q''(t_n)}{2}\delta_{n,m}\,; \quad T_n(t_m) = 0$$

$$S_n'(t_m) = \frac{Q'''(t_n)}{6}\delta_{n,m}\,; \quad T_n'(t_m) = \frac{Q''(t_n)}{2}\delta_{n,m}\,.$$

Taking into account that $Q''(t_n) \neq 0$ for all $n \in \mathbb{N}$, any function $f \in \mathcal{H}_K$ can be expanded as the series

$$f(t) = \sum_{n=1}^{\infty} \Big[f(t_n) \Big(1 - \frac{Q'''(t_n)}{3Q''(t_n)}(t - t_n) \Big) + f'(t_n)(t - t_n) \Big] \frac{2Q(t)}{Q''(t_n)(t - t_n)^2}\,, \quad t \in \mathbb{R}\,.$$

1.1 An easy anomalous example

Although Theorem 1 is a quite general sampling result, the following example exhibits a situation where it does not work. Indeed, consider the usual orthonormal basis for $L^2[-\pi, \pi]$ given by

$$\Big\{ \frac{1}{\sqrt{2\pi}} \Big\} \cup \Big\{ \frac{1}{\sqrt{\pi}} \cos nx \Big\}_{n=1}^{\infty} \cup \Big\{ \frac{1}{\sqrt{\pi}} \sin nx \Big\}_{n=1}^{\infty}$$

and the kernels

$$K(t) = K_1(t) = \cos tx + \sin tx \ \text{ and } K_2(t) = \cos tx\,,$$

from which we define, for each $\phi \in L^2[-\pi, \pi]$, the transforms

$$f(t) = T\phi(t) = \big\langle \phi, K(t) \big\rangle \ \text{ and } \ f_i(t) = T_i\phi(t) = \big\langle \phi, K_i(t) \big\rangle\,, \quad i \in \{1, 2\}\,.$$

Notice that, if $\phi \in L^2[-\pi, \pi]$, we obtain that $\big\langle \phi, \cos(tx) \big\rangle$ is an even function of t and that $\big\langle \phi, \sin(tx) \big\rangle$ is an odd one. Consequently, $\phi \in \ker T$ implies that $T_2\phi(t) = \big\langle \phi, \cos(tx) \big\rangle = -\big\langle \phi, \sin(tx) \big\rangle$ is both an odd and an even function of t. Therefore, $\phi = 0$ so that T is one-to-one.

The corresponding sampling functions are given by

$$\begin{aligned} S_0(t) &= \Big\langle \frac{1}{\sqrt{2\pi}}, K(t) \Big\rangle = \sqrt{2\pi}\,\operatorname{sinc} t \\ S_n(t) &= \Big\langle \frac{1}{\sqrt{\pi}} \cos nx, K(t) \Big\rangle = \frac{2t(-1)^n \sin \pi t}{\sqrt{\pi}(t^2 - n^2)} \quad (n \in \mathbb{N}) \\ T_n(t) &= \Big\langle \frac{1}{\sqrt{\pi}} \sin nx, K(t) \Big\rangle = \frac{2n(-1)^n \sin \pi t}{\sqrt{\pi}(t^2 - n^2)} \quad (n \in \mathbb{N})\,. \end{aligned}$$

In this case, $S_0^1 = S_0^2 = S_0$, $S_n^1 = S_n^2 = S_n$, $T_n^1 = T_n$ and $T_n^2 = 0$ for $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$, the following interpolation condition holds:

$$\begin{pmatrix} S_n^1(m) & T_n^1(m) \\ S_n^2(m) & T_n^2(m) \end{pmatrix} = \delta_{m,n} \begin{pmatrix} \sqrt{\pi} & \sqrt{\pi} \\ \sqrt{\pi} & 0 \end{pmatrix}\,.$$

However, what is happening with S_0^1 and S_0^2? Even if we decide to define $T_0^1 = T_0^2 := 0$, the interpolation matrix that we used in (5) will be singular and we will not be able to apply Theorem 1.

This example gives us a suitable generalization that, in practice, can be very useful. As we only need one coefficient for S_0 (T_0 is not defined), we can consider the matrix

$$\begin{pmatrix} S_0^1(m) \\ S_0^2(m) \end{pmatrix} = \delta_{m,0} \begin{pmatrix} \sqrt{2\pi} \\ \sqrt{2\pi} \end{pmatrix} \tag{6}$$

where $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The coefficient of $S_0(t)$ can be chosen in two ways for the sampling formula, $f_1(0)/\sqrt{2\pi}$ or $f_2(0)/\sqrt{2\pi}$, the choice being at our disposal; we might choose the simplest, or perhaps the only one available, etc.

2 A general sampling result in $\mathcal{H}_K$

The aim in this section is to prove a new sampling result which also applies for anomalous examples. To this end, we need to introduce new interpolation conditions which will be combined in an appropriate way to obtain the desired sampling result. These topics are the subject of the next three subsections:

2.1 Interpolation condition of type (L, M)

Consider $L, M \in \mathbb{N}$ such that $L \geq M$ and define $\mathsf{M} := \{1, 2, \ldots, M\}$ and $\mathsf{L} := \{1, 2, \ldots, L\}$.

Consider a linear independent system for $\mathbb{H}$ written as

$$\{x_{1,n}\}_{n\in\mathbb{I}} \cup \{x_{2,n}\}_{n\in\mathbb{I}} \cup \cdots \cup \{x_{M,n}\}_{n\in\mathbb{I}}$$

where $\mathbb{I} \subseteq \mathbb{N}$ can be a finite set. Suppose that we have L linear transforms $T_1, T_2, \ldots, T_L$ with associate kernels $K_1(t), K_2(t), \ldots, K_L(t)$ and defined as follows

$$f_\ell(t) = T_\ell(x)[t] = \langle x, K_\ell(t) \rangle\,, \quad \ell \in \mathsf{L}\,.$$

Denote

$$S^\ell_{m,n}(t) = T_\ell(x_{m,n})[t]\,, \quad t \in \Omega\,,$$

where $\ell \in \mathsf{L}$, $m \in \mathsf{M}$ and $n \in \mathbb{I}$.

Definition 1 *We say that an interpolation condition of type (L, M) is satisfied by these elements if there exist L sets of points $\{t^\ell_n\}_{n\in\mathbb{I}}$ in Ω, $\ell \in \mathsf{L}$, such that, for any fixed $\ell \in \mathsf{L}$ and $m \in \mathsf{M}$,*

$$S^\ell_{m,n}(t^\ell_k) = a^n_{\ell,m}\delta_{n,k}\,, \qquad n, k \in \mathbb{I}\,,$$

holds, and the coefficients $a^n_{\ell,m} \in \mathbb{C}$ verify that the rank of the matrices

$$A^n := \begin{pmatrix} a^n_{1,1} & a^n_{1,2} & \cdots & a^n_{1,M} \\ a^n_{2,1} & a^n_{2,2} & \cdots & a^n_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{L,1} & a^n_{L,2} & \cdots & a^n_{L,M} \end{pmatrix}$$

is just M for all $n \in \mathbb{I}$.

Definition 2 *Denote by ψ any increasing function from M into L, which we shall call a* choice function. *For any matrix B with L rows $b(1), b(2), \ldots, b(L)$, we define the choice of M rows of B by means of ψ as*

$$B_\psi := \begin{pmatrix} b[\psi(1)] & b[\psi(2)] & \cdots & b[\psi(M)] \end{pmatrix}^\top\,.$$

As we can see, B_ψ is a matrix obtained from B by choosing the rows of B given by $\psi(1), \psi(2), \ldots, \psi(M)$.

In these terms, the rank of A^n is M if and only if there exists a choice function ψ_n such that $A^n_{\psi_n}$ is not singular.

Definition 3 *We shall call such a* ψ_n*, an* A^n-regular choice function*.*

Any A^n-regular choice function ψ is related to A^n in such a way that A^n_ψ is regular. However, this choice function can be applied to any matrix with L rows and any number of columns.

If we have a vector $v = \begin{pmatrix} v_1 & v_2 & \cdots & v_L \end{pmatrix}^\top$, then we can choose the same rows from both A^n and v. The product of the matrix and the vector obtained in such a way is given by

$$A^n_\psi v_\psi = \begin{pmatrix} a^n_{\psi(1),1} & a^n_{\psi(1),2} & \cdots & a^n_{\psi(1),M} \\ a^n_{\psi(2),1} & a^n_{\psi(2),2} & \cdots & a^n_{\psi(2),M} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{\psi(M),1} & a^n_{\psi(M),2} & \cdots & a^n_{\psi(M),M} \end{pmatrix} \begin{pmatrix} v_{\psi(1)} \\ v_{\psi(2)} \\ \vdots \\ v_{\psi(M)} \end{pmatrix} .$$

2.2 Compatibility

We began this article by showing an example in which two interpolation conditions were used. For the first one, $\mathbb{I}_1 = \{0\}$, the linear independent system was just one vector and we had only one transform, i.e., it was an interpolation condition of type $(1,1)$. In the second one, $\mathbb{I}_2 = \mathbb{N}$, the partition of the linear independent system had two elements and there were two transforms, so it was an interpolation condition of type $(2,2)$. The goal of this section is to establish which properties must be verified by two interpolation conditions in order for a sampling theorem to be possible. It is the topic *compatible interpolation conditions*.

The fact of working with two interpolation conditions at least makes the notation used hard. For the sake of clarity, hereafter, an index k_j denotes that the indexed element corresponds to the j-th interpolation condition of those we are using.

Consider two interpolation conditions of types (L_1, M_1) and (L_2, M_2), respectively. This means that we have two linear independent systems in $\mathbb{H}$:

$$\begin{aligned} \mathfrak{S}_1 :=& \{x^1_{1,n_1}\}_{n_1\in\mathbb{I}_1} \cup \{x^1_{2,n_1}\}_{n_1\in\mathbb{I}_1} \cup \cdots \cup \{x^1_{M_1,n_1}\}_{n_1\in\mathbb{I}_1} \\ \mathfrak{S}_2 :=& \{x^2_{1,n_2}\}_{n_2\in\mathbb{I}_2} \cup \{x^2_{2,n_2}\}_{n_2\in\mathbb{I}_2} \cup \cdots \cup \{x^2_{M_2,n_2}\}_{n_2\in\mathbb{I}_2} \end{aligned}$$

where $\mathbb{I}_1, \mathbb{I}_2 \subseteq \mathbb{N}$ (possibly finite) and $M_1, M_2 \in \mathbb{N}$. Moreover, we suppose the j-th interpolation condition, $j \in \{1,2\}$, has $L_j \in \mathbb{N}$ transforms T_{ℓ_j}, $\ell_j \in \mathsf{L}_j$ and $M_j \leq L_j$, defined from each kernel $K_{\ell_j}(t)$, $\ell_j \in \mathsf{L}_j$, by

$$f_{\ell_j}(t) = T_{\ell_j}(x)[t] = \big\langle x, K_{\ell_j}(t) \big\rangle \,, \qquad \ell_j \in \mathsf{L}_j \,.$$

Denote

$$S^{\ell_i}_{m_j,n_j} = T_{\ell_i}(x^j_{m_j,n_j})$$

for $\ell_i \in \mathsf{L}_i$, $m_j \in \mathsf{M}_j$, $n_j \in \mathbb{I}_j$ and $i, j \in \{1,2\}$, i.e., the image of $\mathfrak{S}_1 \cup \mathfrak{S}_2$ by each of the transforms of both the interpolation conditions is calculated.

As they are interpolation conditions, there exist sequences $\{t^{\ell_j}_{k_j}\}_{k_j\in\mathbb{I}_j}$, where $\ell_j \in \mathsf{L}_j$ and $j \in \{1,2\}$, such that

$$S^{\ell_j}_{m_j,n_j}(t^{\ell_j}_{k_j}) = a^{n_j}_{\ell_j,m_j}\delta_{n_j,k_j}$$

holds for $m_j \in \mathsf{M}_j$, $\ell_j \in \mathsf{L}_j$, $n_j, k_j \in \mathbb{I}_j$ and $j \in \{1,2\}$ in such a way the coefficients $a^{n_j}_{\ell_j,m_j} \in \mathbb{C}$ satisfy that the rank of the matrix

$$A^{n_j} := \begin{pmatrix} a^{n_j}_{1,1} & a^{n_j}_{1,2} & \cdots & a^{n_j}_{1,M_j} \\ a^{n_j}_{2,1} & a^{n_j}_{2,2} & \cdots & a^{n_j}_{2,M_j} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n_j}_{L_j,1} & a^{n_j}_{L_j,2} & \cdots & a^{n_j}_{L_j,M_j} \end{pmatrix}$$

is just M_j for all $n_j \in \mathbb{I}_j$, $j \in \{1,2\}$.

Definition 4 *Two interpolation conditions are compatible if the following* compatibility condition *holds:*

$$S^{\ell_2}_{m_1,n_1}(t^{\ell_2}_{k_2}) = 0 \quad , \quad S^{\ell_1}_{m_2,n_2}(t^{\ell_1}_{k_1}) = 0\,,$$

for $\ell_j \in \mathsf{L}_j$, $m_j \in \mathsf{M}_j$, $n_j, k_j \in \mathbb{I}_j$, $j \in \{1,2\}$.

Observe that this condition implies that $\mathfrak{S}_1 \cup \mathfrak{S}_2$ is a linear independent system. Indeed, suppose there exists $x^2_{m_2,n_2}$ in $\mathfrak{S}_2$ such that

$$x^2_{m_2,n_2} = \sum_{m_1=1}^{M_1} \alpha_{m_1} x^1_{m_1,n_1} + \sum_{k=1}^{N} \beta_k x_k$$

where $x_k \in \mathfrak{S}_1 \cup \mathfrak{S}_2$ is not in $\{x^2_{m_2,n_2}, x^1_{1,n_1}, x^1_{2,n_1}, \dots, x^1_{M_1,n_1}\}$ and there exists an index $k_1 \in \mathsf{M}_1 = \{1,2,\dots,M_1\}$ such that $\alpha_{k_1} \neq 0$. For all $\ell_1 \in \mathsf{L}_1$, we obtain that

$$0 = [T_{\ell_1} x^2_{m_2,n_2}](t^{\ell_1}_{n_1}) = \sum_{m_1=1}^{M_1} \alpha_{m_1} a^{n_1}_{\ell_1,m_1}\,,$$

i.e., A^{n_1} has a zero column or it has at least two linear dependent columns so that its rank cannot be M_1.

2.3 The sampling result

Consider the pair of dual Riesz bases for $\mathbb{H}$ given by

$$\bigcup_{r=1}^{R} \bigcup_{m_r=1}^{M_r} \{x_{m_r,n_r}\}_{n_r \in \mathbb{I}_r} \qquad \text{and} \qquad \bigcup_{r=1}^{R} \bigcup_{m_r=1}^{M_r} \{x^*_{m_r,n_r}\}_{n_r \in \mathbb{I}_r} \tag{7}$$

and suppose that $R \in \mathbb{N}$ interpolation conditions are satisfied, and each is compatible with every other (see Definition 1 and Definition 4).

Suppose that the following condition is satisfied, as well:

$$\ker T \subseteq \bigcap_{r=1}^{R} \bigcap_{\ell_r=1}^{L_r} \ker T_{\ell_r}\,. \tag{8}$$

We show the transform T is one-to-one. First, observe that the kernel K can be written as

$$K(t) = \sum_{r=1}^{R} \sum_{m_r=1}^{M_r} \sum_{n_r \in \mathbb{I}_r} \overline{S_{m_r,n_r}(t)} x^*_{m_r,n_r}$$

where $S_{m_r,n_r}(t) = \langle x_{m_r,n_r}, K(t) \rangle$. Analogously, for $\ell_q \in \mathsf{L}_q$ and $1 \le q \le R$, the kernel K_{ℓ_q} can be expanded as

$$K_{\ell_q}(t) = \sum_{r=1}^{R} \sum_{m_r=1}^{M_r} \sum_{n_r \in \mathbb{I}_r} \overline{S^{\ell_q}_{m_r,n_r}(t)} x^*_{m_r,n_r} .$$

Suppose that $x \in \mathbb{H}$ verifies $Tx = 0$. Then, for all $q \in \{1, 2, \ldots, R\}$ and $\ell_q \in \mathsf{L}_q$, we have that

$$f_{\ell_q} = T_{\ell_q} x = 0$$

and therefore, for $p_q \in \mathbb{I}_q$,

$$0 = f_{\ell_q}(t^{\ell_q}_{p_q}) = \Big\langle x, K_{\ell_q}(t^{\ell_q}_{p_q}) \Big\rangle = \Big\langle x, \sum_{r=1}^{R} \sum_{m_r=1}^{M_r} \sum_{n_r \in \mathbb{I}_r} \overline{S^{\ell_q}_{m_r,n_r}(t^{\ell_q}_{p_q})} x^*_{m_r,n_r} \Big\rangle .$$

Thus, compatibility implies

$$0 = \Big\langle x, \sum_{m_q=1}^{M_q} \sum_{n_q \in \mathbb{I}_q} \overline{S^{\ell_q}_{m_q,n_q}(t^{\ell_q}_{p_q})} x^*_{m_q,n_q} \Big\rangle ,$$

and by using the q-th interpolation condition, we have

$$0 = \Big\langle x, \sum_{m_q=1}^{M_q} \overline{a^{p_q}_{\ell_q,m_q}} x^*_{m_q,p_q} \Big\rangle = \sum_{m_q=1}^{M_q} a^{p_q}_{\ell_q,m_q} \langle x, x^*_{m_q,p_q} \rangle , \qquad 1 \le \ell_q \le L_q .$$

Thus, we have a homogeneous linear system with L_q equations and M_q unknowns whose unique solution is the trivial one. Consequently, we obtain that $\langle x, x^*_{m_q,p_q} \rangle = 0$ for all $q \in \{1, 2, \ldots R\}$, $m_q \in \mathsf{M}_q$ and $p_q \in \mathbb{I}_q$. Since $\{x^*_{m_r,n_r} : 1 \le r \le R,\ m_r \in \mathsf{M}_r,\ n_r \in \mathbb{I}_r\}$ is a Riesz basis, $x = 0$. Observe that the following sequences

$$\bigcup_{r=1}^{R} \bigcup_{m_r=1}^{M_r} \{S_{m_r,n_r}\}_{n_r \in \mathbb{I}_r} \qquad \text{and} \qquad \bigcup_{r=1}^{R} \bigcup_{m_r=1}^{M_r} \{S^*_{m_r,n_r}\}_{n_r \in \mathbb{I}_r} \tag{9}$$

are dual Riesz bases of $\mathcal{H}_K$.

If we do the same for any $x \in \mathbb{H}$ such that $f = Tx$, we obtain the consistent linear system

$$f_{\ell_q}(t^{\ell_q}_{p_q}) = \sum_{m_q=1}^{M_q} a^{p_q}_{\ell_q,m_q} \langle x, x^*_{m_q,p_q} \rangle , \qquad 1 \le \ell_q \le L_q ,$$

which has a unique solution. For each $p_q \in \mathbb{I}_q$ we can find a choice function ψ_{p_q} such that $A^{p_q}_{\psi_{p_q}}$ is regular. Thus, we can write the coefficients $\langle x, x^*_{m_q,p_q} \rangle$ with respect to the samples $f_{\ell_q}(t^{\ell_q}_{p_q})$ by means of

$$\Big(\langle x, x^*_{1,p_q} \rangle \quad \langle x, x^*_{2,p_q} \rangle \quad \cdots \quad \langle x, x^*_{M_q,p_q} \rangle \Big)^{\top} = \big(A^{p_q}_{\psi_{p_q}}\big)^{-1} F^{p_q}_{\psi_{p_q}} ,$$

where $F^{p_q}_{\psi_{p_q}}$ is obtained from the vector

$$F^{p_q} := \Big(f_1(t^1_{p_q}) \quad f_2(t^2_{p_q}) \quad \cdots \quad f_{L_q}(t^{L_q}_{p_q}) \Big)^\top \tag{10}$$

by using the choice function ψ_{p_q}.

Now, expanding f by using the Riesz basis given by $\{S_{m_r,n_r} : n_r \in \mathbb{I}_r\}_{m_r=1}^{M_r}$ and taking into account that T is an isometry, we have that

$$\begin{aligned} f(t) &= \sum_{r=1}^{R} \sum_{m_r=1}^{M_r} \sum_{n_r \in \mathbb{I}_r} \langle f, S^*_{m_r,n_r} \rangle_{\mathcal{H}_K} S_{m_r,n_r}(t) \\ &= \sum_{r=1}^{R} \sum_{n_r \in \mathbb{I}_r} \sum_{m_r=1}^{M_r} \langle x, x^*_{m_r,n_r} \rangle_{\mathbb{H}} S_{m_r,n_r}(t) \\ &= \sum_{r=1}^{R} \sum_{n_r \in \mathbb{I}_r} \big(F^{n_r}_{\psi_{n_r}}\big)^\top \Big[\big(A^{n_r}_{\psi_{n_r}}\big)^{-1} \Big]^\top \mathbb{S}_{n_r}(t) \,, \end{aligned}$$

where F^{n_r} is given by (10) and

$$\mathbb{S}_{n_r}(t) = \big(S_{1,n_r}(t) \quad S_{2,n_r}(t) \quad \cdots \quad S_{M_r,n_r}(t) \big)^\top \,, \tag{11}$$

which is just the sampling formula we are looking for. Convergence is, as we know, in the $\mathcal{H}_K$-norm sense and, also, absolute in Ω and uniform in those subsets of Ω where $\|K(t)\|$ is bounded. Consequently, we have proved the next result:

Theorem 2 *Consider the dual Riesz bases given by* (7). *Suppose we have $R \in \mathbb{N}$ interpolation conditions of types (L_r, M_r), where $1 \le r \le R$, each two of them being compatible (in the sense of Definition 4). Assume condition* (8) *is satisfied. Then, the sets in* (9) *are dual Riesz bases for $\mathcal{H}_K$ and for each set of A^{n_r}-regular choice functions $\big\{\psi_{n_r} : \mathsf{M}_r \longrightarrow \mathsf{L}_r \mid n_r \in \mathbb{I}_r \,,\ 1 \le r \le R\big\}$, we have that any function $f \in \mathcal{H}_K$ can be recovered from its samples*

$$\big\{ f_{\ell_r}(t^{\ell_r}_{n_r}) : n_r \in \mathbb{I}_r \,, \quad \ell_r \in \mathsf{L}_r \big\}_{r=1}^{R}$$

by the following sampling formula

$$f(t) = \sum_{r=1}^{R} \sum_{n_r \in \mathbb{I}_r} \big(F^{n_r}_{\psi_{n_r}}\big)^\top \Big[\big(A^{n_r}_{\psi_{n_r}}\big)^{-1} \Big]^\top \mathbb{S}_{n_r}(t) \,, \quad t \in \Omega \,,$$

where F^{n_r} and $\mathbb{S}_{n_r}$ are given by (10) *and* (11), *respectively. Convergence is absolute and, also, uniform in those subsets of Ω where $\|K(t)\|$ is bounded.*

Notice that the number of transforms we have for each interpolation condition of type (L, M) is M at least. However, it is not important how many of them we have at most. In fact, only the possibility of finding a regular choice of rows of A^n is needed. Thus, if L is a finite or infinite set of indices, we have an interpolation condition of type $(\operatorname{card} \mathsf{L}, M)$ if there exist some points $\{t^\ell_n : n \in \mathbb{I}\}_{\ell \in \mathsf{L}}$ such that, for any fixed $\ell \in \mathsf{L}$ and $m \in \mathsf{M}$,

$$S^\ell_{m,n}(t^\ell_k) = a^n_{\ell,m} \delta_{n,k} \,, \quad n, k \in \mathbb{I} \,,$$

holds, and for each $n \in \mathbb{I}$ there exists an A^n-regular choice function ψ_n where

$$A^n = \begin{pmatrix} a^n_{\ell,1} & a^n_{\ell,2} & \cdots & a^n_{\ell,M} \end{pmatrix}_{\ell \in \mathsf{L}}$$

represents a function from L into $\mathbb{C}^M$ for each $n \in \mathbb{I}$. If these more general interpolation conditions are used, Theorem 2 still remains valid.

3 Sampling by using linear operators in $\mathbb{H}$

Let $\mathbb{H}$ be a separable Hilbert space. Consider two dual Riesz bases of $\mathbb{H}$ written as

$$\{x_{1,n}\}_{n=1}^\infty \cup \{x_{2,n}\}_{n=1}^\infty \cup \cdots \cup \{x_{M,n}\}_{n=1}^\infty \,,$$
$$\{x^*_{1,n}\}_{n=1}^\infty \cup \{x^*_{2,n}\}_{n=1}^\infty \cup \cdots \cup \{x^*_{M,n}\}_{n=1}^\infty \,,$$

where $M \in \mathbb{N}$. Given a kernel $K : \Omega \subset \mathbb{R} \longrightarrow \mathbb{H}$, we define the linear transform T as in (1). Suppose we have a family of bounded linear operators $\{L_\lambda : \mathbb{H} \longrightarrow \mathbb{H}\}_{\lambda \in \Lambda}$. Related to $f = Tx \in \mathcal{H}_K$ we define the functions $f_\lambda(t) := \big\langle L_\lambda x, K(t) \big\rangle_{\mathbb{H}}$, where $\lambda \in \Lambda$. For any fixed $n \in \mathbb{N}$ and $1 \le m \le M$, we denote $S^n_{\lambda,m}(t) := \big\langle L_\lambda x_{m,n}, K(t) \big\rangle_{\mathbb{H}}$ and, for any $n \in \mathbb{N}$, we write $S_{m,n}(t) := \big\langle x_{m,n}, K(t) \big\rangle_{\mathbb{H}}$.

In the sequel, we assume that $x \in \ker T$ implies $L_\lambda x \in \ker T$ for all $\lambda \in \Lambda$ (which occurs when T commutes with L_λ for all $\lambda \in \Lambda$), and that there exists a family of sequences $\big\{\{t^\lambda_k\}_{k \in \mathbb{N}} : \lambda \in \Lambda\big\} \subset \Omega$ such that $S^n_{\lambda,m}(t^\lambda_k) = a^n_{\lambda,m}\delta_{n,k}$ for $n, k \in \mathbb{N}$, $1 \le m \le M$ and $\lambda \in \Lambda$. For each $n \in \mathbb{N}$, define the function

$$\begin{array}{rccl} A^n : & \Lambda & \longrightarrow & \mathbb{C}^M \\ & \lambda & \longmapsto & (a^n_{\lambda,1}, a^n_{\lambda,2}, \ldots, a^n_{\lambda,M}) \end{array}$$

and suppose that there exists a sequence $\{\psi_n : \mathsf{M} \longrightarrow \Lambda\}_{n \in \mathbb{N}}$ of A^n-regular choice functions. Then, the following result holds:

Theorem 3 *Under the hypotheses as above, any $f \in \mathcal{H}_K$ can be recovered from its samples $\big\{\{f_\lambda(t^\lambda_n)\}_{n \in \mathbb{N}} : \lambda \in \Lambda\big\}$ by means of the sampling formula*

$$f(t) = \sum_{n \in \mathbb{N}} \big(F^n_{\psi_n}\big)^\top \Big[\big(A^n_{\psi_n}\big)^{-1}\Big]^\top \mathbb{S}_n(t) \,, \quad t \in \Omega \,,$$

where $F^n(\lambda) := f_\lambda(t^\lambda_n)$ and $\mathbb{S}_n(t) = \begin{pmatrix} S_{1,n}(t) & S_{2,n}(t) & \cdots & S_{M,n}(t) \end{pmatrix}^\top$. Convergence is absolute and uniform in subsets of Ω where $\|K(t)\|$ is bounded.

Proof: Defining $K_\lambda(t) := L^*_\lambda[K(t)]$ for $\lambda \in \Lambda$, where L^*_λ denotes the adjoint operator of L_λ, we have

$$f_\lambda(t) := T_\lambda(x)[t] = \big\langle L_\lambda x, K(t) \big\rangle_{\mathbb{H}} = \big\langle x, K_\lambda(t) \big\rangle_{\mathbb{H}} \,, \quad t \in \Omega \,.$$

If $x \in \ker T$ then, by assumption, $L_\lambda x \in \ker T$ for all $\lambda \in \Lambda$, i.e., $0 = \big\langle L_\lambda x, K(t) \big\rangle_{\mathbb{H}} = \big\langle x, K_\lambda(t) \big\rangle_{\mathbb{H}}$ for all $\lambda \in \Lambda$. As a consequence,

$$\ker T \subseteq \bigcap_{\lambda \in \Lambda} \ker T_\lambda \,,$$

and Theorem 2 implies the desired result. ■

4 Some illustrative examples

In this section we go back to our anomalous example in order to handle it into the new setting. We also give another example which involves two interpolation conditions of type $(2,2)$.

4.1 The introductory example revisited

Consider the orthonormal basis of $L^2[-\pi,\pi]$ given by

$$\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos nx\right\}_{n=1}^{\infty} \cup \left\{\frac{1}{\sqrt{\pi}}\sin nx\right\}_{n=1}^{\infty}$$

and the kernels $K_1(t) = K(t) = \cos tx + \sin tx$, $K_2(t) = \cos tx - \sin tx$ and $K_3(t) = \cos tx$. Notice that, if $\phi \in L^2[-\pi,\pi]$, we obtain that $\langle\phi, \cos(tx)\rangle$ is an even function of t and that $\langle\phi, \sin(tx)\rangle$ is an odd one. Consequently, $\phi \in \ker T$ implies that $T_3\phi(t) = \langle\phi, \cos(tx)\rangle = -\langle\phi, \sin(tx)\rangle$ is both an odd and an even function of t. Thus, T is injective and trivially $\{0\} = \ker T \subseteq \ker T_1 \cap \ker T_2 \cap \ker T_3$.

Sampling functions are given by

$$\begin{aligned}
S_0(t) &= \left\langle \frac{1}{\sqrt{2\pi}}, K(t)\right\rangle = \sqrt{2\pi}\,\mathrm{sinc}\, t \\
S_n(t) &= \left\langle \frac{1}{\sqrt{\pi}}\cos nx, K(t)\right\rangle = \frac{2t(-1)^n \sin \pi t}{\sqrt{\pi}(t^2-n^2)} \quad (n \in \mathbb{N}) \\
T_n(t) &= \left\langle \frac{1}{\sqrt{\pi}}\sin nx, K(t)\right\rangle = \frac{2n(-1)^n \sin \pi t}{\sqrt{\pi}(t^2-n^2)} \quad (n \in \mathbb{N})
\end{aligned}$$

Easy calculations show that $S_n^1 = S_n^2 = S_n^3 = S_n$, that $T_n^3 = 0$ and that $-T_n^2 = T_n^1 = T_n$ for $n \in \mathbb{N}$ and $S_0^1 = S_0^2 = S_0^3 = S_0$. Thus, we have two interpolation conditions of types $(3,1)$ and $(3,2)$, respectively. The first one verifies that:

$$\begin{pmatrix} S_0^1(m) \\ S_0^2(m) \\ S_0^3(m) \end{pmatrix} = \begin{pmatrix} \sqrt{2\pi} \\ \sqrt{2\pi} \\ \sqrt{2\pi} \end{pmatrix} \delta_{0,m} \quad m \in \mathbb{N} \cup \{0\}\,,$$

and the second one, that:

$$\begin{pmatrix} S_n^1(m) & T_n^1(m) \\ S_n^2(m) & T_n^2(m) \\ S_n^3(m) & T_n^3(m) \end{pmatrix} = \begin{pmatrix} \sqrt{\pi} & \sqrt{\pi} \\ \sqrt{\pi} & -\sqrt{\pi} \\ \sqrt{\pi} & 0 \end{pmatrix} \delta_{n,m} \quad m \in \mathbb{N} \cup \{0\}\,,$$

so we have a couple of compatible interpolation conditions.

We define $f(t) := \langle F, K(t)\rangle$ and $f_k(t) := \langle F, K_k(t)\rangle$ for $k = 1,2,3$ and $F \in L^2[-\pi,\pi]$. Finally, Theorem 2 yields the following sampling result:

Corollary 4 *Any function f defined as*

$$f(t) = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} F(x)[\cos tx + \sin tx]dx\,, \quad t \in \mathbb{R}\,,$$

where $F \in L^2[-\pi,\pi]$*, can be recovered from the samples* $f(0)$ *and* $\{f_k(n)\}_{n=1}^{\infty}$*,* $k=1,2,3$*, of its related functions* f_1, f_2, f_3 *by means of the following sampling formula:*

$$f(t) = f(0)\,\operatorname{sinc} t + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \big[\mathcal{G}_1(n) S_n(t) + \mathcal{G}_2(n) T_n(t)\big]\,, \quad t \in \mathbb{R}\,,$$

where $\big(\mathcal{G}_1(n)\ \mathcal{G}_2(n)\big)$ *is any row of the matrix*

$$\begin{pmatrix} f_3(n) & f_3(n)-f_2(n) \\ f_3(n) & f_1(n)-f_3(n) \\ \frac{f_2(n)+f_1(n)}{2} & \frac{f_1(n)-f_2(n)}{2} \end{pmatrix}. \tag{12}$$

Convergence is absolute and uniform in subsets of Ω *where* $\|K(t)\|$ *is bounded.*

This result allows us to recover any classical band-limited function to $[-\pi,\pi]$ by means of the samples of the related functions f_1, f_2, f_3 since band-limited functions can be written as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x)[\cos tx + \sin tx]dx\,, \quad t \in \mathbb{R}\,,$$

where $F \in L^2[-\pi,\pi]$. Notice that the above integral representation involves the Hartley transform of F (see [16]).

4.2 Another example

In this example, we use two compatible interpolation conditions of type $(2,2)$. To this end, consider $\{x_n, y_n\}_{n=1}^{\infty} \cup \{\widetilde{x}_n, \widetilde{y}_n\}_{n=1}^{\infty}$, a Riesz basis for $\mathbb{H}$ whose dual Riesz basis is given by $\{x_n^*, y_n^*\}_{n=1}^{\infty} \cup \{\widetilde{x}_n^*, \widetilde{y}_n^*\}_{n=1}^{\infty}$. Suppose we have five kernels $K, K_1, K_2, \widetilde{K}_1, \widetilde{K}_2$ each of them defining a transform, as usual: $f(t) = (Tx)[t] := \langle x, K(t)\rangle$, $f_j(t) = (T_j x)[t] := \langle x, K_j(t)\rangle$, and $\widetilde{f}_j(t) = (\widetilde{T}_j x)[t] := \langle x, \widetilde{K}_j(t)\rangle$ where $t \in \Omega$, $x \in \mathbb{H}$ and $j \in \{1,2\}$, denoting a tilde over an element that it is related to the second interpolation condition. Assume that the following condition holds:

$$\ker T \subseteq \ker T_1 \cap \ker T_2 \cap \ker \widetilde{T}_1 \cap \ker \widetilde{T}_2\,.$$

Denote

$$\begin{aligned} S_n &:= T x_n\,, \quad T_n := T y_n\,, \\ \widetilde{S}_n &:= T \widetilde{x}_n\,, \quad \widetilde{T}_n := T \widetilde{y}_n\,, \end{aligned}$$

and suppose there exist sequences $\{s_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$, $\{\widetilde{s}_n\}_{n=1}^{\infty}$ and $\{\widetilde{t}_n\}_{n=1}^{\infty}$ such that

$$\begin{pmatrix} (T_1(x_n))(s_m) & (T_1(y_n))(s_m) \\ (T_2(x_n))(t_m) & (T_2(y_n))(t_m) \end{pmatrix} = \begin{pmatrix} a_{1,1}^n & a_{1,2}^n \\ a_{2,1}^n & a_{2,2}^n \end{pmatrix} \delta_{m,n} =: A^n \delta_{m,n}$$

for the first interpolation condition, and

$$\begin{pmatrix} (\widetilde{T}_1(\widetilde{x}_n))(s_m) & (\widetilde{T}_1(\widetilde{y}_n))(s_m) \\ (\widetilde{T}_2(\widetilde{x}_n))(t_m) & (\widetilde{T}_2(\widetilde{y}_n))(t_m) \end{pmatrix} = \begin{pmatrix} \widetilde{a}_{1,1}^n & \widetilde{a}_{1,2}^n \\ \widetilde{a}_{2,1}^n & \widetilde{a}_{2,2}^n \end{pmatrix} \delta_{m,n} =: \widetilde{A}^n \delta_{m,n}$$

for the second one, being the matrices A^n and $\widetilde{A}^n$ invertible. For these interpolation conditions the compatibility condition reads:

$$\begin{pmatrix} (\widetilde{T}_1(x_n))(\widetilde{s}_m) & (\widetilde{T}_1(y_n))(\widetilde{s}_m) \\ (\widetilde{T}_2(x_n))(\widetilde{t}_m) & (\widetilde{T}_2(y_n))(\widetilde{t}_m) \end{pmatrix} = \begin{pmatrix} (T_1(\widetilde{x}_n))(s_m) & (T_1(\widetilde{y}_n))(s_m) \\ (T_2(\widetilde{x}_n))(t_m) & (T_2(\widetilde{y}_n))(t_m) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

Finally, as a consequence of Theorem 2 we deduce the following sampling result for the preceding compatible interpolation conditions:

Corollary 5 *Any function f in the Hilbert space $\mathcal{H}_K$ can be recovered from the sequences of samples $\{f_1(s_n)\}_{n=1}^{\infty}$, $\{f_2(t_n)\}_{n=1}^{\infty}$, $\{\widetilde{f}_1(\widetilde{s}_n)\}_{n=1}^{\infty}$ and $\{\widetilde{f}_2(\widetilde{t}_n)\}_{n=1}^{\infty}$ by means of the following sampling formula*

$$f(t) = \sum_{n=1}^{\infty} \Big[f_1(s_n) \frac{a_{2,2}^n S_n(t) - a_{2,1}^n T_n(t)}{a_{1,1}^n a_{2,2}^n - a_{1,2}^n a_{2,1}^n} + f_2(t_n) \frac{a_{1,1}^n T_n(t) - a_{1,2}^n S_n(t)}{a_{1,1}^n a_{2,2}^n - a_{1,2}^n a_{2,1}^n} \Big] +$$
$$+ \sum_{n=1}^{\infty} \Big[\widetilde{f}_1(s_n) \frac{\widetilde{a}_{2,2}^n \widetilde{S}_n(t) - \widetilde{a}_{2,1}^n \widetilde{T}_n(t)}{\widetilde{a}_{1,1}^n \widetilde{a}_{2,2}^n - \widetilde{a}_{1,2}^n \widetilde{a}_{2,1}^n} + \widetilde{f}_2(t_n) \frac{\widetilde{a}_{1,1}^n \widetilde{T}_n(t) - \widetilde{a}_{1,2}^n \widetilde{S}_n(t)}{\widetilde{a}_{1,1}^n \widetilde{a}_{2,2}^n - \widetilde{a}_{1,2}^n \widetilde{a}_{2,1}^n} \Big] .$$

The convergence of the series above is absolute and uniform in subsets of Ω where $\|K(t)\|$ is bounded.

4.3 A comment on the choice of the samples

Theorem 2 allows us to combine several interpolation conditions whose types are not necessarily equal. Corollary 4 gives a sampling formula for the example of subsection 1.1 which shows some advantages of our approach. Indeed, for each $n \in \mathbb{N}$, we can choose any row of the matrix in (12). Thus, if the samples of f_2 are lost for $n \in \mathbf{N} \subseteq \mathbb{N}$, we can still recover f by using the second row of that matrix for $n \in \mathbf{N}$.

On the other hand, fixed $n \in \mathbb{N}$, the rows of (12) are obtained as solutions of a consistent linear system with three equations and two unknowns, so we can choose any two of them in order to solve the system. This means that every row of (12) is equal to every other. As a consequence, we can choose any element of the first column and any element (not necessarily in the same row) of the second one. This remark allows us to avoid cancellation errors by choosing the appropriate elements of (12). For instance, suppose we have that $f_1(n_0)f_2(n_0) > 0$ and that $f_1(n_0)f_3(n_0) < 0$ for $n_0 \in \mathbb{N}$. Then, we can choose the third element of the first column, $\frac{1}{2}[f_1(n_0) + f_2(n_0)]$, and the second element of the second one, $f_1(n) - f_3(n)$.

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