Numerical Functional Analysis and Optimization, 29(1–2):126–144, 2008

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560701873845

ALIASING ERROR OF SAMPLING SERIES IN WAVELET SUBSPACES

A. G. García,[1] J. M. Kim,[2] K. H. Kwon,[2] and G. Pérez-Villalón[3]

[1]*Departamento de Matemáticas, Universidad Carlos III de Madrid, Leganés, Madrid, Spain*
[2]*Division of Applied Mathematics, KAIST, Daejeon, Korea*
[3]*Departamento de Matemática Aplicada, EUITT, Universidad Politécnica de Madrid, Madrid, Spain*

□ *Alising error arises whenever a sampling formula, valid for a prescribed space, is applied to a function in a bigger space. In this work, we estimate the aliasing error of classic and average sampling expansions in wavelet subspaces of a multiresolution analysis.*

Keywords Aliasing error; Sampling; Wavelet subspace.

AMS Subject Classification 42C15; 42C40; 94A20.

1. INTRODUCTION

The Shannon sampling theorem states that any function f in the classic Paley–Wiener space $PW_\pi := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \operatorname{supp}\hat{f} \subseteq [-\pi, \pi]\}$, where $\hat{f}$ stands for the Fourier transform of f, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}(t-n), \quad t \in \mathbb{R},$$

where $\operatorname{sinc}(t) = \sin \pi t/\pi t$ denotes the cardinal sinc function. The space PW_π can be seen as the subspace V_0 of the Shannon multiresolution

Address correspondence to K. H. Kwon, Division of Applied Mathematics, KAIST, Daejeon 305-701, Korea; E-mail: khkwon@amath.kaist.ac.kr

analysis, whose scaling function is precisely $\phi(t) = \mathrm{sinc}(t)$. Although Shannon's sampling theory has had an enormous impact, it has a number of drawbacks, as pointed out by Unser in [11]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of finite duration signal; the band-limiting operation generates Gibbs oscillations, and finally, the sinc function has a very slow decay, which makes computation in the signal domain very inefficient. Moreover, many applied problems impose different *a priori* constraints on the type of functions. For these reasons, the sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces.

In [12], Walter extended, under appropriate hypotheses, the Shannon sampling theorem to the subspace V_0 of a general multiresolution analysis $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$: For any $f \in V_0$, the sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), \quad t \in \mathbb{R}$$

holds, where $\widehat{S}(\xi) := \hat{\phi}(\xi)/\big(\sum_{n\in\mathbb{Z}} \phi(n)e^{-in\xi}\big)$ and ϕ denotes the scaling function. Later on, Unser and Aldroubi introduced in [10], under suitable conditions, the average sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathscr{L}f)(n)\, S_{\mathscr{L}}(t-n), \quad t \in \mathbb{R},$$

which uses the average samples $\{(\mathscr{L}f)(n)\}_{n\in\mathbb{Z}}$ obtained from $f \in V_0$ by means of a linear time-invariant system $\mathscr{L}f := f * \mathsf{h}$ defined on V_0. Notice that, in practice, the measurements of a function f in V_0 are taken not from the function itself but from some filtered version $\mathscr{L}f$.

Whenever these sampling formulas are applied to a function f that does not belong to V_0, the so-called aliasing error arises:

$$E^A f(t) := f(t) - \sum_{n\in\mathbb{Z}} f(n)S(t-n) \quad \text{or}$$
$$E^A_{\mathscr{L}} f(t) := f(t) - \sum_{n\in\mathbb{Z}} (\mathscr{L}f)(n)S_{\mathscr{L}}(t-n), \quad t \in \mathbb{R}.$$

Concerning this error in Shannon's setting, a classic result by Brown [1] states that if $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then

$$\left| f(t) - \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc}(t-n) \right| \le \frac{2}{\sqrt{2\pi}} \int_{|\xi|>\pi} |\hat{f}(\xi)|\, d\xi, \quad t \in \mathbb{R}. \tag{1.1}$$

In addition, the function $f(t) = \text{sinc}(2t - 1)$ is an extremal solution for (1.1), i.e., there exists a value of t for which (1.1) becomes an equality. Notice that if $f \in V_1 = PW_{2\pi}$, then (1.1) can be written as

$$|E^A f(t)| \le \frac{2}{\sqrt{2\pi}} \|P_{W_0} f\|_{L^1(\mathbb{R})},$$

where P_{W_0} denotes the orthogonal projection onto W_0, the orthogonal complement of V_0 in V_1. The aliasing error in Shannon's setting has been largely studied: see [7] and references therein. Besides, Walter [12] has proved a similar result for functions in the subspace V_1 of a general multiresolution analysis. Specifically, for any $f \in V_1$, there exists a constant C such that

$$|E^A f(t)| \le C\|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}. \tag{1.2}$$

Notice that $\|P_{W_0} f\|_{L^2(\mathbb{R})}$ can be expressed in terms of the wavelet coefficients of the function f.

On the other hand, Janssen generalized in [9] Walter's sampling formula by using shifted samples $\{f(n + \sigma)\}_{n\in\mathbb{Z}}$, where $\sigma \in [0, 1)$. As to the corresponding aliasing error $E^A f$, he proved the inequalities

$$K_0\|P_{W_0} f\|_{L^2(\mathbb{R})} \le \|E^A f\|_{L^2(\mathbb{R})} \le K_\infty \|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad f \in V_1.$$

In addition, he found the smallest possible value for the constant K_0 and the largest possible value for K_∞. Later on, in [5] the authors dealt with the aliasing error function $E^A f$ for $f \in V_1$. In so doing, they calculate its Fourier transform, $\widehat{E^A f}$, in terms of the Fourier transform of $P_{W_0} f$. Besides recovering Janssen's inequalities, this technique also allows one to derive a precise bound like (1.2), exhibiting the extremal solutions in some cases. Some results concerning the aliasing error for functions $f \in V_2$ are also provided. See also references [10] and [13] for the general wavelet setting.

In the current paper, we study the aliasing error arising when the classic sampling formula is applied to a function f in the wavelet subspace V_n, $n \ge 1$, of a multiresolution analysis. Estimations both in L^2 and L^∞ norms are provided. The alising error arising when we apply the average sampling formula for V_0 to a function $f \in V_1$ is also included. In particular, this paper improves the results in [5] in different directions: Apart from work with a non-necessarily orthonormal scaling function ϕ, some of the results in [5] are derived under weaker hypotheses.

The paper is organized as follows: In Section 2, the needed preliminaries are included; in particular, a variance of the Poisson summation formula used with profusion in the sequel. Section 3 is devoted to study the aliasing error in classic sampling for wavelet subspaces in a multiresolution analysis. Finally, in Section 4, the aliasing error in average sampling is carried out.

2. PRELIMINARIES

On $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we take the Fourier transform to be normalized as

$$\mathscr{F}[\phi](\xi) = \hat{\phi}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{-it\xi} dt$$

so that $\mathscr{F}[\cdot]$ becomes a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. Let

$$Z_\phi(t, \xi) := \sum_{n\in\mathbb{Z}} \phi(t+n) e^{-in\xi}$$

be the Zak transform of $\phi(t)$ in $L^2(\mathbb{R})$ (cf. [9]). We first introduce a variance of the Poisson summation formula.

Lemma 2.1. *Let $\phi \in L^2(\mathbb{R})$ be such that $\hat{\phi} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, for any $t \in \mathbb{R}$, the series $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}$ converges absolutely in $L^1[0, 2\pi]$ and*

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)} \sim \frac{1}{\sqrt{2\pi}} Z_\phi(t, \xi), \tag{2.1}$$

which means that $\frac{1}{\sqrt{2\pi}} Z_\phi(t, \xi)$ is the Fourier series expansion of $\sum_{n\in\mathbb{Z}} e^{it(\xi+2n\pi)} \hat{\phi}(\xi + 2n\pi)$. If moreover for any fixed t in $\mathbb{R}$, $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}$ converges in $L^2[0, 2\pi]$ or equivalently $\{\phi(t+n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)} = \frac{1}{\sqrt{2\pi}} Z_\phi(t, \xi) \quad \text{in } L^2[0, 2\pi]. \tag{2.2}$$

Proof. Because $\hat{\phi} \in L^1(\mathbb{R})$ and

$$\sum_{n\in\mathbb{Z}} \|\hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}\|_{L^1[0,2\pi]} = \sum_{n\in\mathbb{Z}} \int_0^{2\pi} |\hat{\phi}(\xi + 2n\pi)| d\xi = \int_{-\infty}^{+\infty} |\hat{\phi}(\xi)| d\xi,$$

the series $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}$ converges absolutely in $L^1[0, 2\pi]$. Hence

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)} \sim \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \left\langle \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}, e^{-ik\xi} \right\rangle_{L^2[0,2\pi]} e^{-ik\xi},$$

where

$$\left\langle \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi+2n\pi)e^{it(\xi+2n\pi)}, e^{-ik\xi} \right\rangle_{L^2[0,2\pi]} = \int_0^{2\pi} \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi+2n\pi)e^{it(\xi+2n\pi)}e^{ik\xi}d\xi$$

$$= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \hat{\phi}(\xi+2n\pi)e^{i((t+k)\xi+2n\pi)}d\xi = \int_{-\infty}^{+\infty} \hat{\phi}(\xi)e^{i(t+k)\xi}d\xi = \sqrt{2\pi}\phi(t+k).$$

Hence (2.1) holds from which the second claim follows immediately. □

Lemma 2.1 generalizes Lemma 1 in [5] and a claim in the Appendix in [2]. For any $\phi \in L^2(\mathbb{R})$, let

$$\Phi(t) := \sum_{n\in\mathbb{Z}} |\phi(t-n)|^2 \quad \text{and} \quad G_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi+2n\pi)|^2.$$

Then $\Phi(t) = \Phi(t+1) \in L^1[0,1]$, $G_\phi(\xi) = G_\phi(\xi+2\pi) \in L^1[0,2\pi]$, and $\|\phi(t)\|^2_{L^2(\mathbb{R})} = \|\Phi(t)\|_{L^1[0,1]} = \|G_\phi(\xi)\|_{L^1[0,2\pi]}$. For any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, let

$$\hat{\mathbf{c}}^*(\xi) := \sum_{n\in\mathbb{Z}} c(n)e^{-in\xi}$$

be the discrete Fourier transform of $\mathbf{c}$. Then $\hat{\mathbf{c}}^* \in L^2[0,2\pi]$ and $\int_0^{2\pi} |\hat{\mathbf{c}}^*(\xi)|^2 d\xi = 2\pi\|\mathbf{c}\|^2$ where $\|\mathbf{c}\|^2 := \sum_{n\in\mathbb{Z}} |c(n)|^2$. For any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ and $\mathbf{d} = \{d(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, the discrete convolution product of $\mathbf{c}$ and $\mathbf{d}$ is defined by

$$\mathbf{c} * \mathbf{d} = \left\{ (\mathbf{c} * \mathbf{d})(n) := \sum_{k\in\mathbb{Z}} c(k)d(n-k) \right\}_{n\in\mathbb{Z}}.$$

Then $\hat{\mathbf{c}}^* \hat{\mathbf{d}}^*$ belongs to $L^1[0,2\pi]$ and its Fourier series is $\sum_{n\in\mathbb{Z}} (\mathbf{c} * \mathbf{d})(n)e^{-in\xi}$ so that $\mathbf{c} * \mathbf{d} \in \ell^\infty(\mathbb{Z})$ and

$$\int_0^{2\pi} |\hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi)|^2 d\xi = 2\pi\|\mathbf{c} * \mathbf{d}\|^2. \tag{2.3}$$

Lemma 2.2. *If* $\phi \in L^2(\mathbb{R})$ *is such that* $H_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi+2n\pi)| \in L^2[0,2\pi]$, *then* $\phi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ *and* $\sup_{\mathbb{R}} \Phi(t) < \infty$. *In particular,* $Z_\phi(t,\cdot) \in L^2[0,2\pi]$ *for each* t *in* $\mathbb{R}$.

Proof. Because $H_\phi \in L^2[0,2\pi] \subset L^1[0,2\pi]$ and $\|H_\phi(\xi)\|_{L^1[0,2\pi]} = \|\hat{\phi}(\xi)\|_{L^1(\mathbb{R})}$, $\hat{\phi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the series $\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi)e^{it(\xi+2n\pi)}$ converges in $L^2[0,2\pi]$. Hence we have by Lemma 2.1

$$\sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi)e^{it(\xi+2n\pi)} = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \phi(t+n)e^{-in\xi} \quad \text{in } L^2[0,2\pi]$$

so that

$$\Phi(t) = \sum_{n\in\mathbb{Z}} |\phi(t+n)|^2 = \left\| \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi)e^{it(\xi+2n\pi)} \right\|^2_{L^2[0,2\pi]} \leq \|H_\phi\|^2_{L^2[0,2\pi]}.$$

Hence, $\sup_{\mathbb{R}} \Phi(t) \leq \|H_\phi\|^2_{L^2[0,2\pi]} < \infty$ and so $Z_\phi(t,\cdot) \in L^2[0,2\pi]$, $t \in \mathbb{R}$. □

Finally in this section, let us recall the following sampling theorem (cf. [8, 9, 14]), which extends the classic Shannon sampling theorem in the Paley–Wiener space to the general shift invariant space. For any measurable function f on $\mathbb{R}$, let

$$\|f\|_0 := \sup_{|E|=0} \inf_{\mathbb{R}\setminus E} |f(t)| \quad \text{and} \quad \|f\|_\infty := \inf_{|E|=0} \sup_{\mathbb{R}\setminus E} |f(t)|$$

be the essential infimum and the essential supremum of $|f|$, respectively, where $|E|$ is the Lebesgue measure of E.

Proposition 2.3. *Let V_0 be the closed subspace of $L^2(\mathbb{R})$ of which $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis. Assume $\phi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\sup_{\mathbb{R}} \Phi(t) < \infty$. Then*

$$0 < \|Z_\phi(\sigma,\xi)\|_0 \leq \|Z_\phi(\sigma,\xi)\|_\infty < \infty \tag{2.4}$$

holds for some σ in $[0,1)$ if and only if there is a Riesz basis $\{S_\sigma(t-n) : n \in \mathbb{Z}\}$ of V_0 such that for each $f \in V_0$

$$f = \sum_{n\in\mathbb{Z}} f(\sigma+n)S_\sigma(\cdot - n) \quad \text{in } V_0. \tag{2.5}$$

Moreover, in this case, we have

$$\widehat{S_\sigma}(\xi) = Z_\phi(\sigma,\xi)^{-1}\hat{\phi}(\xi) \quad \text{a.e. on } \mathbb{R}. \tag{2.6}$$

Note that in Proposition 2.3, any function in V_0 is of the form

$$f(t) = (\mathbf{c} * \phi)(t) := \sum_{n\in\mathbb{Z}} c(n)\phi(t-n)$$

for some $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$ to a continuous function, and V_0 is a reproducing kernel Hilbert space. Let $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ be the Fourier coefficients of $Z_\phi(\sigma,\cdot)^{-1} \in L^\infty[0,2\pi]$ so that $Z_\phi(\sigma,\xi)^{-1} = \sum_{n\in\mathbb{Z}} c(n)e^{-in\xi}$. Then $S_\sigma(t) = (\mathbf{c} * \phi)(t)$ and so we have by (2.3)

$$\sum_{n\in\mathbb{Z}}|S_\sigma(t+n)|^2 = \sum_{n\in\mathbb{Z}}|(\mathbf{c}*\phi)(t+n)|^2 = \frac{1}{2\pi}\int_0^{2\pi}|\hat{\mathbf{c}}^*(\xi)Z_\phi(t,\xi)|^2 d\xi.$$

Hence

$$\sum_{n\in\mathbb{Z}}|S_\sigma(t+n)|^2 \le \|Z_\phi(\sigma,\xi)\|_0^{-2}\sum_{n\in\mathbb{Z}}|\phi(t+n)|^2, \quad t\in\mathbb{R} \tag{2.7}$$

so that $\sup_{\mathbb{R}}\sum_{n\in\mathbb{Z}}|S_\sigma(t+n)|^2 < \infty$. Therefore, the sampling series (2.5) converges not only in $L^2(\mathbb{R})$ but also absolutely and uniformly on $\mathbb{R}$.

3. ALIASING ERROR IN CLASSIC SAMPLING

From now on, let ϕ be a scaling function of a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ (cf. [3, 4]), where the V_j's are closed subspaces of $L^2(\mathbb{R})$ satisfying

(i) $V_j \subset V_{j+1}$, $j\in\mathbb{Z}$;
(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iii) $f(t)\in V_j$ if and only if $f(2t)\in V_{j+1}$, $j\in\mathbb{Z}$;
(iv) $\{\phi(t-n) : n\in\mathbb{Z}\}$ is a Riesz basis for V_0.

We also assume that $H_\phi\in L^2[0,2\pi]$ and the condition (2.4) holds for some σ in $[0,1)$ so that there is a regular shifted sampling expansion (2.5) on V_0. Then $\phi\in L^2(\mathbb{R})\cap C(\mathbb{R})$ and $\sup_{\mathbb{R}}\Phi(t)<\infty$ by Lemma 2.2. Moreover, for any $f\in V_j$ $(j\in\mathbb{Z})$, $f(2^{-j}t) = (\mathbf{c}*\phi)(t)\in V_0$ for some $\mathbf{c}$ in $\ell^2(\mathbb{Z})$ so that

$$\|\hat{f}(\xi)\|_{L^1(\mathbb{R})} = \|2^j\hat{f}(2^j\xi)\|_{L^1(\mathbb{R})} = \|\hat{\mathbf{c}}^*(\xi)\hat{\phi}(\xi)\|_{L^1(\mathbb{R})} = \|\hat{\mathbf{c}}^*(\xi)H_\phi(\xi)\|_{L^1[0,2\pi]} < \infty,$$

by using the Cauchy–Schwarz inequality. Hence $\hat{f}\in L^1(\mathbb{R})$ for any $f\in V_j$ $(j\in\mathbb{Z})$ so that $V_j\subset L^2(\mathbb{R})\cap C(\mathbb{R}), j\in\mathbb{Z}$. Now for any integer $j\ge 1$, let

$$V_j(\sigma) := \{f\in V_j : \{f(\sigma+n)\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})\}$$

and

$$E^A f(t) := f(t) - \sum_{n\in\mathbb{Z}} f(\sigma+n)S_\sigma(t-n), \quad f\in V_j(\sigma)$$

be the aliasing error of f, which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$ to a continuous function as $\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |S_\sigma(t-n)|^2 < \infty$. Then $E^A f \in V_j$ so that the function $\widehat{E^A f}(\xi) = \hat{f}(\xi) - Z_f(\sigma,\xi) Z_\phi(\sigma,\xi)^{-1} \hat{\phi}(\xi)$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Lemma 3.1. *For any fixed integer $j \geq 1$, if $Z_\phi(2^j\sigma,\cdot) \in L^\infty[0,2\pi]$, then $V_j(\sigma) = V_j$. In particular, if $\sigma = 0$, then $V_j(0) = V_j$ for any $j \geq 1$.*

Proof. Assume $Z_\phi(2^j\sigma,\cdot) \in L^\infty[0,2\pi]$. Let $f \in V_j$. Then $f(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(2^j t - k)$ for some $\mathbf{c} = \{c(k)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Hence, we have by (2.3)

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |f(\sigma+n)|^2 &= \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} c(k)\phi(2^j\sigma + 2^j n - k) \right|^2 \\
&\leq \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} c(k)\phi(2^j\sigma + n - k) \right|^2 \\
&= \frac{1}{2\pi} \|\hat{\mathbf{c}}^*(\xi) Z_\phi(2^j\sigma,\xi)\|^2_{L^2[0,2\pi]} \\
&\leq \frac{1}{2\pi} \|Z_\phi(2^j\sigma,\xi)\|^2_\infty \|\hat{\mathbf{c}}^*(\xi)\|^2_{L^2[0,2\pi]} < \infty
\end{aligned}$$

so that $f \in V_j(\sigma)$. Hence $V_j = V_j(\sigma)$. If $\sigma = 0$, then $Z_\phi(0,\cdot) \in L^\infty[0,2\pi]$ by (2.4). Hence $V_j(0) = V_j$, $j \geq 1$. □

Corollary 3.2. *If $H_\phi \in L^\infty[0,2\pi]$, then $Z_\phi(t,\cdot) \in L^\infty[0,2\pi]$ for any t in $\mathbb{R}$ so that $V_j(\sigma) = V_j$ for any $j \geq 1$.*

Proof. By Lemma 2.1, for any t in $\mathbb{R}$, $Z_\phi(t,\xi) = \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it(\xi+2n\pi)}$ in $L^2[0,2\pi]$. Hence $|Z_\phi(t,\xi)| \leq \sqrt{2\pi} H_\phi(\xi)$ a.e. on $\mathbb{R}$ so that $Z_\phi(t,\cdot) \in L^\infty[0,2\pi]$ for any t in $\mathbb{R}$. □

We also note that if $\mathbf{c} \in \ell^2(\mathbb{Z})$ is such that $\hat{\mathbf{c}}^* \in L^\infty[0,2\pi]$, then $f(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(2^j t - k) \in V_j(\sigma)$ for any $j \geq 1$.

Let $\psi \in V_1$ be a wavelet associated with the scaling function ϕ, that is, $\{\psi(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of W_0, the orthogonal complement of V_0 in V_1. We may express any $f \in V_1$ uniquely as

$$f(t) = \sum_{n\in\mathbb{Z}} c(n)\phi(2t-n) = g(t) + h(t),$$

where $g(t)=\sum_{n\in\mathbb{Z}} a(n)\phi(t-n)\in V_0$, $h(t)=\sum_{n\in\mathbb{Z}} b(n)\psi(t-n)\in W_0$, and $\mathbf{a}=\{a(n)\}_{n\in\mathbb{Z}}$, $\mathbf{b}=\{b(n)\}_{n\in\mathbb{Z}}$, $\mathbf{c}=\{c(n)\}_{n\in\mathbb{Z}}$ are in $\ell^2(\mathbb{Z})$. Then

$$\hat{f}(\xi)=m_f\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)=u_f(\xi)\hat{\phi}(\xi)+v_f(\xi)\hat{\psi}(\xi), \tag{3.1}$$

where $m_f(\xi)=\frac{1}{2}\hat{\mathbf{c}}^*(\xi)$, $u_f(\xi)=\hat{\mathbf{a}}^*(\xi)$, and $v_f(\xi)=\hat{\mathbf{b}}^*(\xi)$. In particular, m_ϕ and m_ψ are in $L^\infty[0,2\pi]$ so that ϕ and ψ belong to $V_1(\sigma)$. Note also that for any $f=g+h$ with $g\in V_0$ and $h\in W_0$, $f\in V_1(\sigma)$ if and only if $h\in V_1(\sigma)$.

Lemma 3.3. *For any $f\in V_1(\sigma)$,*

$$Z_f(\sigma,\xi)=m_f\left(\frac{\xi}{2}\right)Z_\phi\left(2\sigma,\frac{\xi}{2}\right)+m_f\left(\frac{\xi}{2}+\pi\right)Z_\phi\left(2\sigma,\frac{\xi}{2}+\pi\right)\quad \text{in } L^2[0,2\pi] \tag{3.2}$$

Proof. See Lemma 2 in [5]. □

In [5], (3.2) was proved under a stronger condition on ϕ, say, $H_\phi\in L^\infty[0,2\pi]$. But we can easily see that (3.2) holds also when $H_\phi\in L^2[0,2\pi]$ by using Lemma 2.1.

Proposition 3.4. *For any $f\in V_1(\sigma)$,*

$$\widehat{E^A f}(\xi)=v_f(\xi)\hat{\phi}\left(\frac{\xi}{2}\right)Z_\phi\left(2\sigma,\frac{\xi}{2}+\pi\right)Z_\phi(\sigma,\xi)^{-1}M\left(\frac{\xi}{2}\right) \tag{3.3}$$

where v_f is the one given in (3.1) *and*

$$M(\xi):=m_\psi(\xi)m_\phi(\xi+\pi)-m_\phi(\xi)m_\psi(\xi+\pi). \tag{3.4}$$

Proof. For any $f=g+h\in V_1$ with $g\in V_0$ and $h\in W_0$, we have by applying Lemma 2.1 to $Z_h(\sigma,\xi)$ and $Z_\psi(\sigma,\xi)$

$$\begin{aligned}\widehat{E^A f}(\xi)&=\widehat{E^A h}(\xi)=\hat{h}(\xi)-Z_h(\sigma,\xi)\hat{S}(\xi)\\&=v_f(\xi)m_\psi\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)-v_f(\xi)Z_\psi(\sigma,\xi)\widehat{S}(\xi)\\&=v_f(\xi)\hat{\phi}\left(\frac{\xi}{2}\right)Z_\phi(\sigma,\xi)^{-1}\left[m_\psi\left(\frac{\xi}{2}\right)Z_\phi(\sigma,\xi)-m_\phi\left(\frac{\xi}{2}\right)Z_\psi(\sigma,\xi)\right].\end{aligned}$$

Then (3.3) comes immediately by applying (3.2) to ϕ and ψ. □

Proposition 3.4 was proved in [5] (see Theorem 1 in [5]) assuming that ϕ is an orthonormal scaling function satisfying $\phi(t)=O((1+|t|)^{-s})$

with $s > \frac{1}{2}$ and $H_\phi \in L^\infty[0, 2\pi]$. In such a case, we may take the wavelet ψ with $m_\psi(\xi) = e^{i\xi}\overline{m_\phi(\xi+\pi)}$ as in [5]. Then

$$M(\xi) = e^{i\xi}(|m_\phi(\xi)|^2 + |m_\phi(\xi+\pi)|^2) = e^{i\xi} \quad \text{a.e. on } \mathbb{R}$$

so that the equation (3.3) becomes the equation (13) in [5]:

$$\widehat{E^A f}(\xi) = e^{i\frac{\xi}{2}} v_f(\xi)\hat{\phi}\left(\frac{\xi}{2}\right) Z_\phi\left(2\sigma, \frac{\xi}{2}+\pi\right) Z_\phi(\sigma, \xi)^{-1}.$$

Theorem 3.5. *Assume* $Z_\phi(2\sigma, \cdot) \in L^\infty[0, 2\pi]$ *so that* $V_1(\sigma) = V_1$. *Then we have for any* $f \in V_1$,

$$\begin{aligned} |E^A f(t)| &\le \frac{1}{\pi} \| Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} Z_\phi(2t, \xi) M(\xi) \|_{L^2[0,2\pi]} \| v_f(\xi) \|_{L^2[0,2\pi]} \\ &\le \sqrt{\frac{2}{\pi}} \| Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} \\ &\qquad \times H_\phi(\xi) M(\xi) \|_{L^2[0,2\pi]} \| v_f(\xi) \|_{L^2[0,2\pi]}, \quad t \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Moreover, the equality holds in the first inequality of (3.5) *at* $t = \sigma + k + \frac{1}{2}$ $(k \in \mathbb{Z})$ *for any* $f \in V_1$ *satisfying*

$$\overline{v_f(2\xi)} = \lambda e^{i(2k+1)\xi} Z_\phi(2\sigma, \xi) Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} M(\xi) \quad (\lambda \in \mathbb{C}). \tag{3.6}$$

Proof. Because $\widehat{E^A f} \in L^1(\mathbb{R})$, we have by (3.3) and the Poisson summation formula

$$\begin{aligned} E^A f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{E^A f}(\xi) e^{it\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_f(\xi)\hat{\phi}\left(\frac{\xi}{2}\right) Z_\phi\left(2\sigma, \frac{\xi}{2}+\pi\right) Z_\phi(\sigma, \xi)^{-1} M\left(\frac{\xi}{2}\right) e^{it\xi} d\xi \quad (3.7) \\ &= \frac{1}{\pi} \int_0^{2\pi} Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} Z_\phi(2t, \xi) M(\xi) v_f(2\xi) d\xi. \end{aligned}$$

Then (3.5) follows from (3.7) as $|Z_\phi(2t, \xi)| \le \sqrt{2\pi} H_\phi(\xi)$ a.e. on $\mathbb{R}$. Now the equality holds in the first inequality of (3.5) if and only if for some constant λ

$$\overline{v_f(2\xi)} = \lambda Z_\phi(2\sigma, \xi+\pi) Z_\phi(\sigma, 2\xi)^{-1} Z_\phi(2t, \xi) M(\xi), \tag{3.8}$$

which requires that the right–hand side of (3.8) must be π-periodic. For $t = \sigma + k + \frac{1}{2}$ $(k \in \mathbb{Z})$, $Z_\phi(2\sigma + 2k + 1, \xi) = e^{i(2k+1)\xi} Z_\phi(2\sigma, \xi)$ so that (3.8) becomes (3.6), of which the right-hand side is indeed π-periodic as $M(\xi + \pi) = -M(\xi)$ (cf. (3.4)). □

When ϕ is an orthonormal scaling function and $m_\psi(\xi) = e^{i\xi}\overline{m_\phi(\xi+\pi)}$ so that $M(\xi) = e^{i\xi}$ a.e. on $\mathbb{R}$, the first inequality (3.5) and the equation (3.7) become the equations (19) and (21) in [5], respectively. In [5], the extremal solutions of the inequality (3.5) were found (see Corollary 3 in [5]) when $\arg\hat{\phi}(\xi)$ is 2π-periodic. For a L^2-estimate of the aliasing error, we have:

Theorem 3.6. *Assume $Z_\phi(2\sigma,\cdot) \in L^\infty[0,2\pi]$ so that $V_1(\sigma) = V_1$. Then we have for any $f \in V_1$,*

$$K_0\|v_f\|^2_{L^2[0,2\pi]} \le \|E^A(f)\|^2_{L^2(\mathbb{R})} \le K_\infty\|v_f\|^2_{L^2[0,2\pi]} \tag{3.9}$$

where $K_0 := \|K(\xi)\|_0$, $K_\infty := \|K(\xi)\|_\infty$, and

$$K(\xi) := |M(\xi)|^2|Z_\phi(\sigma,2\xi)|^{-2}[G_\phi(\xi)|Z_\phi(2\sigma,\xi+\pi)|^2 + G_\phi(\xi+\pi)|Z_\phi(2\sigma,\xi)|^2].$$

Moreover, K_0 and K_∞ are the optimal constants for (3.9).

Proof. By using (3.3) and $M(\xi+\pi) = -M(\xi)$, we have

$$\begin{aligned}\|E^A(f)\|^2_{L^2(\mathbb{R})} &= \|\widehat{E^A(f)}\|^2_{L^2(\mathbb{R})}\\ &= 2\int_0^{2\pi} |M(\xi)|^2|Z_\phi(\sigma,2\xi)|^{-2}G_\phi(\xi)|Z_\phi(2\sigma,\xi+\pi)|^2|v_f(2\xi)|^2 d\xi\\ &= 2\int_0^{\pi} K(\xi)|v_f(2\xi)|^2 d\xi\end{aligned}$$

from which (3.9) follows. Note that $K(\xi) = K(\xi+\pi) \in L^\infty[0,\pi]$. If $K_\infty = 0$, i.e., $K(\xi) = 0$ a.e. in $[0,\pi]$, then $K_0 = K_\infty = 0$ are trivially optimal. When $K_\infty > 0$, choose any λ with $0 < \lambda < K_\infty$. Then there is a subset of D of $[0,\pi]$ with $|D| > 0$ and $K(\xi) > \lambda$ a.e. on D.

Let $f \in V_1$ be such that $\hat{f}(\xi) = v_f(\xi)\hat{\psi}(\xi)$, where $v_f(\xi) = \chi_{2D}(\xi)$ the characteristic function of $2D$. Then

$$\|E^A f\|^2_{L^2(\mathbb{R})} = 2\int_0^{\pi} K(\xi)|v_f(2\xi)|^2 d\xi > 2\lambda\int_0^{\pi}|v_f(2\xi)|^2 d\xi = \lambda\|v_f\|^2_{L^2[0,2\pi]}$$

so that K_∞ is optimal. Similarly we can see that K_0 is also optimal. □

Aliasing error estimates in Theorems 3.5 and 3.6 can also be expressed via $\|P_{W_0}f\|_{L^2(\mathbb{R})}$ where $P_{W_0}f(=h)$ is the orthogonal projection of $f \in V_1$ onto

W_0. In fact, as $\{\psi(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of W_0, there are positive constants A and B such that $A \le G_\psi(\xi) \le B$ a.e. in $[0, 2\pi]$. Then we have

$$A\|v_f\|^2_{L^2[0,2\pi]} \le \|P_{W_0}f\|^2_{L^2(\mathbb{R})} \le B\|v_f\|^2_{L^2[0,2\pi]}.$$

Moreover, if $\{\psi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal basis of W_0, then $\|P_{W_0}f\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|v_f\|^2_{L^2[0,2\pi]}$.

We now estimate the aliasing error for signals in $V_j(\sigma)$ for arbitrary $j \ge 1$.

Proposition 3.7. *For any integer $j \ge 1$ and any $f \in V_j(\sigma)$*

$$\widehat{E^A f}(\xi) = 2^{-\frac{j-1}{2}}\hat{\phi}\left(\frac{\xi}{2^j}\right)\sum_{k=1}^{2^{j-1}}\left[A_k\left(\frac{\xi}{2^j}\right)u_g\left(\frac{\xi + 2(k-1)\pi}{2^{j-1}}\right)\right.$$
$$\left. + B_k\left(\frac{\xi}{2^j}\right)v_g\left(\frac{\xi + 2(k-1)\pi}{2^{j-1}}\right)\right] \tag{3.10}$$

where $g(t) := 2^{-\frac{j-1}{2}} f\left(\frac{t}{2^{j-1}}\right) \in V_1$ *so that* $\hat{g}(\xi) = u_g(\xi)\hat{\phi}(\xi) + v_g(\xi)\hat{\psi}(\xi)$ (*cf.* (3.1)) *and*

$$A_k(\xi) := m_\phi(\xi)\delta_{1,k}$$
$$- Z_\phi(\sigma, 2^j\xi)^{-1} Z_\phi\left(2^{j-1}\sigma, 2\xi + 2^{2-j}(k-1)\pi\right)\prod_{l=0}^{j-1} m_\phi(2^l\xi) \tag{3.11}$$

$$B_k(\xi) := m_\psi(\xi)\delta_{1,k}$$
$$- Z_\phi(\sigma, 2^j\xi)^{-1} Z_\psi\left(2^{j-1}\sigma, 2\xi + 2^{2-j}(k-1)\pi\right)\prod_{l=0}^{j-1} m_\phi(2^l\xi) \tag{3.12}$$

for $1 \le k \le 2^{j-1}$.

Proof. From $g(t) = 2^{-\frac{j-1}{2}} f\left(\frac{t}{2^{j-1}}\right)$ and $\hat{g}(\xi) = u_g(\xi)\hat{\phi}(\xi) + v_g(\xi)\hat{\psi}(\xi)$, we obtain

$$\hat{f}(\xi) = 2^{-\frac{j-1}{2}}\left[u_g\left(\frac{\xi}{2^{j-1}}\right)\hat{\phi}\left(\frac{\xi}{2^{j-1}}\right) + v_g\left(\frac{\xi}{2^{j-1}}\right)\hat{\psi}\left(\frac{\xi}{2^{j-1}}\right)\right]$$
$$= 2^{-\frac{j-1}{2}}\left[u_g\left(\frac{\xi}{2^{j-1}}\right)m_\phi\left(\frac{\xi}{2^j}\right) + v_g\left(\frac{\xi}{2^{j-1}}\right)m_\psi\left(\frac{\xi}{2^j}\right)\right]\hat{\phi}\left(\frac{\xi}{2^j}\right). \tag{3.13}$$

We then have by Lemma 2.1

$$
\begin{aligned}
Z_f(\sigma,\xi) &= \sqrt{2\pi}\sum_{n\in\mathbb{Z}}\hat{f}(\xi+2n\pi)e^{i\sigma(\xi+2n\pi)}\\
&= 2^{-\frac{j-1}{2}}\sqrt{2\pi}\sum_{n\in\mathbb{Z}}\left[u_g\left(\frac{\xi+2n\pi}{2^{j-1}}\right)\hat{\phi}\left(\frac{\xi+2n\pi}{2^{j-1}}\right)\right.\\
&\qquad\left. + v_g\left(\frac{\xi+2n\pi}{2^{j-1}}\right)\hat{\psi}\left(\frac{\xi+2n\pi}{2^{j-1}}\right)\right]e^{i\sigma(\xi+2n\pi)}\\
&= 2^{-\frac{j-1}{2}}\sqrt{2\pi}\sum_{k=1}^{2^{j-1}}\left[u_g\left(\frac{\xi+2(k-1)\pi}{2^{j-1}}\right)\sum_{n\in\mathbb{Z}}\hat{\phi}\left(\frac{\xi+2(k-1)\pi}{2^{j-1}}+2n\pi\right)\right.\\
&\quad + v_g\left(\frac{\xi+2(k-1)\pi}{2^{j-1}}\right)\\
&\quad\left.\times\sum_{n\in\mathbb{Z}}\hat{\psi}\left(\frac{\xi+2(k-1)\pi}{2^{j-1}}+2n\pi\right)\right]e^{i2^{j-1}\sigma\left(\frac{\xi+2(k-1)\pi}{2^{j-1}}+2n\pi\right)}\\
&= 2^{-\frac{j-1}{2}}\sum_{k=1}^{2^{j-1}}\left[u_g\left(\frac{\xi+2(k-1)\pi}{2^{j-1}}\right)Z_\phi\left(2^{j-1}\sigma,\frac{\xi+2(k-1)\pi}{2^{j-1}}\right)\right.\\
&\quad\left. + v_g\left(\frac{\xi+2(k-1)\pi}{2^{j-1}}\right)Z_\psi\left(2^{j-1}\sigma,\frac{\xi+2(k-1)\pi}{2^{j-1}}\right)\right]. \qquad (3.14)
\end{aligned}
$$

Therefore we obtain (3.10) together with (3.11) and (3.12) by applying (3.13) and (3.14) to

$$
\begin{aligned}
\widehat{E^A f}(\xi) &= \hat{f}(\xi) - Z_f(\sigma,\xi)Z_\phi(\sigma,\xi)^{-1}\hat{\phi}(\xi)\\
&= \hat{f}(\xi) - Z_f(\sigma,\xi)Z_\phi(\sigma,\xi)^{-1}\left[\prod_{l=1}^{j} m_\phi\left(\frac{\xi}{2^l}\right)\right]\hat{\phi}\left(\frac{\xi}{2^j}\right).
\end{aligned}
$$

□

The method of deriving (3.10) expressing the Fourier transform of the aliasing error $E^A f$ is much easier than the one in [5], where authors employed the subspace $M_\sigma := \{f\in V_1 : f(\sigma+n)=0, n\in\mathbb{Z}\}$ to derive the corresponding expressions for $j=1$ and 2 (see Theorems 1 and 4 in [5]).

Note that $A_k(\xi)=A_k(\xi+2\pi)$ and $B_k(\xi)=B_k(\xi+2\pi)$ belong to $L^2[0,2\pi]$ for $1\le k\le 2^{j-1}$. On the other hand, by applying (3.2) to ϕ and ψ, we have

$$
\begin{aligned}
A_k(\xi) &= m_\phi(\xi)\delta_{1,k} - Z_\phi\left(\sigma,2^j\xi\right)^{-1}\\
&\qquad\times\left[m_\phi\left(\xi+2^{1-j}(k-1)\pi\right)Z_\phi\left(2^j\sigma,\xi+2^{1-j}(k-1)\pi\right)\right.
\end{aligned}
$$

$$+ m_\phi(\xi + \pi + 2^{1-j}(k-1)\pi) \times Z_\phi\big(2^j\sigma, \xi + \pi + 2^{1-j}(k-1)\pi\big) \prod_{l=0}^{j-1} m_\phi(2^l\xi)$$

and

$$\begin{aligned} B_k(\xi) = {} & m_\psi(\xi)\delta_{1,k} - Z_\phi(\sigma, 2^j\xi)^{-1} \\ & \times \big[m_\psi\big(\xi + 2^{1-j}(k-1)\pi\big) Z_\phi\big(2^j\sigma, \xi + 2^{1-j}(k-1)\pi\big) \\ & + m_\psi(\xi + \pi + 2^{1-j}(k-1)\pi) \\ & \times Z_\phi\big(2^j\sigma, \xi + \pi + 2^{1-j}(k-1)\pi\big) \prod_{l=0}^{j-1} m_\phi\big(2^l\xi\big). \end{aligned}$$

Hence A_k and B_k belong to $L^\infty[0, 2\pi]$ for $1 \le k \le 2^{j-1}$ provided that $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$.

Theorem 3.8. *Assume $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$ so that $V_j(\sigma) = V_j$ for an integer $j \ge 1$. Then we have for any $f \in V_j$ and for any t in $\mathbb{R}$*

$$\begin{aligned} |E^A f(t)| \le {} & \frac{1}{\pi} 2^{-\frac{j-1}{2}} \sum_{k=1}^{2^{j-1}} [\|A_k(\xi) Z_\phi(2^j t, \xi)\|_{L^2[0,2\pi]} \|u_g(\xi)\|_{L^2[0,2\pi]} \\ & + \|B_k(\xi) Z_\phi(2^j t, \xi)\|_{L^2[0,2\pi]} \|v_g(\xi)\|_{L^2[0,2\pi]}] \\ \le {} & \sqrt{\frac{2^j}{\pi}} \sum_{k=1}^{2^{j-1}} [\|H_\phi(\xi) A_k(\xi)\|_{L^2[0,2\pi]} \|u_g(\xi)\|_{L^2[0,2\pi]} \\ & + \|H_\phi(\xi) B_k(\xi)\|_{L^2[0,2\pi]} \|v_g(\xi)\|_{L^2[0,2\pi]}] \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|E^A f\|^2_{L^2(\mathbb{R})} \le {} & 2^{j+1} \|G_\phi(\xi)\|_\infty \sum_{k=1}^{2^{j-1}} [\|A_k(\xi)\|_\infty \|u_g(\xi)\|_{L^2[0,2\pi]} \\ & + \|B_k(\xi)\|_\infty \|v_g(\xi)\|_{L^2[0,2\pi]}]. \end{aligned} \tag{3.16}$$

Proof. Because $\widehat{E^A f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have by (3.10) and Lemma 2.1

$$\begin{aligned} E^A f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{E^A f}(\xi) e^{it\xi} d\xi \\ &= \frac{2^N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{E^A f}\big(2^N\xi\big) e^{it2^N\xi} d\xi \end{aligned}$$

$$= \frac{2^{\frac{j+1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{k=1}^{2^{j-1}} \big[A_k(\xi) u_g\big(2\xi + 2^{2-j}(k-1)\pi\big)$$
$$+ B_k(\xi) v_g\big(2\xi + 2^{2-j}(k-1)\pi\big)\big]\hat{\phi}(\xi) e^{it2^N\xi} d\xi$$
$$= \frac{2^{\frac{j+1}{2}}}{\sqrt{2\pi}} \int_{0}^{2\pi} \sum_{k=1}^{2^{j-1}} \big[A_k(\xi) u_g\big(2\xi + 2^{2-j}(k-1)\pi\big)$$
$$+ B_k(\xi) v_g\big(2\xi + 2^{2-j}(k-1)\pi\big)\big] \cdot \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi) e^{it2^N(\xi+2n\pi)} d\xi$$
$$= \frac{2^{\frac{j+1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{k=1}^{2^{j-1}} \big[A_k(\xi) u_g\big(2\xi + 2^{2-j}(k-1)\pi\big)$$
$$+ B_k(\xi) v_g\big(2\xi + 2^{2-j}(k-1)\pi\big)\big] Z_\phi(2^N t, \xi) d\xi$$

from which the first inequality of (3.15) follows. Then the second inequality of (3.15) follows as $|Z_\phi(t,\xi)| \leq \sqrt{2\pi} H_\phi(\xi)$ for any t in $\mathbb{R}$. Similarly (3.16) also comes from (3.10). □

In Theorem 3.8, we assume $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$ only to guarantee the finiteness of the upper bounds in (3.15) and (3.16). For example, if we assume $H_\phi \in L^4[0, 2\pi]$, then we can easily see that $H_\phi\, A_k$ and $H_\phi\, B_k$ belong to $L^2[0, 2\pi]$ for $1 \leq k \leq 2^{j-1}$ so that the estimate (3.15) remains to hold for any $f \in V_j(\sigma)$ even if we do not assume $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$.

Note that $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$ for any $j \geq 1$ if either $\sigma = 0$ or $H_\phi \in L^\infty[0, 2\pi]$ as $\|Z_\phi(\sigma, \cdot)\|_\infty < \infty$ by (2.4) and $|Z_\phi(t,\xi)| \leq \sqrt{2\pi} H_\phi(\xi)$ for any t in $\mathbb{R}$ by (2.2). It is also worth noting that if $Z_\phi(2^j\sigma, \cdot) \in L^\infty[0, 2\pi]$ for some integer $j \geq 1$, then $Z_\phi(2^k\sigma, \cdot) \in L^\infty[0, 2\pi]$ for $0 \leq k \leq j$. It follows immediately from (3.2) applied to ϕ as $m_\phi \in L^\infty[0, 2\pi]$.

4. ALIASING ERROR IN AVERAGE SAMPLING

Suppose that $\mathscr{L}$ is a linear time-invariant system defined on V_0 as

$$\big(\mathscr{L}f\big)(t) := [f * \mathsf{h}](t) = \int_{-\infty}^{\infty} f(x)\mathsf{h}(t-x)\,dx, \quad t \in \mathbb{R},$$

where the impulse response h of $\mathscr{L}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. As a consequence, the sequence $\{\mathscr{L}\phi(n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ (see [6, Lemma 1]) and $Z_{\mathscr{L}\phi}(0, \xi)$ belongs to $L^2[0, 2\pi]$. Moreover, for each $f \in L^2(\mathbb{R})$ we have that $\widehat{\mathscr{L}f}(\xi) = \widehat{\mathsf{h} * f}(\xi) = \sqrt{2\pi}\,\hat{\mathsf{h}}(\xi)\hat{f}(\xi)$ a.e. in $\mathbb{R}$.

Unser and Aldroubi derived in [10] a sampling formula that allows one to recover any function f in V_0 from the average samples $\{(\mathscr{L}f)(n)\}_{n\in\mathbb{Z}}$.

The following theorem, proof of which can be found in [6], gives a necessary and sufficient condition for this formula to hold.

Theorem 4.1. *Assume that* $\phi \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, $\sup_{\mathbb{R}} \Phi(t) < \infty$ *and* $\|Z_{\mathscr{L}\phi}(0,\cdot)\|_\infty < \infty$. *Then, there exists a Riesz basis* $\{S_{\mathscr{L}}(t-n) : n \in \mathbb{Z}\}$ *for* V_0 *such that the sampling formula*

$$f(t) = \sum_{n\in\mathbb{Z}} \big(\mathscr{L}f\big)(n)\, S_{\mathscr{L}}(t-n)\,, \quad t \in \mathbb{R}, \tag{4.1}$$

holds for each $f \in V_0$ *if and only if* $\|Z_{\mathscr{L}\phi}(0,\cdot)\|_0 > 0$. *Moreover, in this case* $\widehat{S_{\mathscr{L}}}(\xi) = Z^{-1}_{\mathscr{L}\phi}(0,\xi)\hat{\phi}(\xi)$. *The convergence of the series in* (4.1) *is in the* $L^2(\mathbb{R})$*-sense, absolute and uniform on* $\mathbb{R}$.

A similar formulation can be given in terms of the shifted samples $\{(\mathscr{L}f)(\sigma+n)\}_{n\in\mathbb{Z}}$. Throughout this section, we assume the hypotheses in Theorem 4.1 and that the sampling formula (4.1) holds. In the following, we assume $H_\phi \in L^\infty[0,2\pi]$. Then we can easily see that $\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |f(t+n)|^2 < \infty$ so that $\{\mathscr{L}f(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for any $f \in V_1$ (cf. [6, Lemma 1]).

For each $f \in V_1$, we define the average aliasing error as

$$E^A_{\mathscr{L}}f(t) := f(t) - \sum_{n\in\mathbb{Z}} \big(\mathscr{L}f\big)(n)\, S_{\mathscr{L}}(t-n), \quad t \in \mathbb{R}.$$

Proposition 4.2. *Assume that* $H_\phi \in L^\infty[0,2\pi]$. *For any* $f \in V_1$,

$$\widehat{E^A_{\mathscr{L}}f}(\xi) = v_f(\xi) N_{\mathscr{L}}(\xi/2)\hat{\phi}(\xi/2), \tag{4.2}$$

where $v_f(\xi)$ *is the function given in* (3.1) *and*

$$N_{\mathscr{L}}(\xi) := m_\psi(\xi) - Z_{\mathscr{L}\psi}(0,2\xi) Z^{-1}_{\mathscr{L}\phi}(0,2\xi) m_\phi(\xi).$$

Proof. First, we check that we can apply the Poisson summation formula to $\mathscr{L}f$ for any $f \in V_1$. Indeed,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} |\widehat{\mathscr{L}f}(\xi+2n\pi)| \\ &\quad = \sum_{n\in\mathbb{Z}} |\hat{\mathsf{h}}(\xi+2n\pi)\hat{f}(\xi+2n\pi)| \le \|\hat{\mathsf{h}}\|_{L^\infty(\mathbb{R})} \sum_{n\in\mathbb{Z}} |\hat{f}(\xi+2n\pi)| \\ &\quad \le \frac{1}{\sqrt{2\pi}} \|\mathsf{h}\|_{L^1(\mathbb{R})} \sum_{n\in\mathbb{Z}} \left| m_f\left(\frac{\xi}{2}+n\pi\right)\hat{\phi}\left(\frac{\xi}{2}+n\pi\right)\right| \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \|\mathsf{h}\|_{L^1(\mathbb{R})} \left[\left| m_f\left(\frac{\xi}{2}\right) \right| \sum_{n\in\mathbb{Z}} \left| \hat{\phi}\left(\frac{\xi}{2} + 2n\pi\right) \right| \right.$$
$$\left. + \left| m_f\left(\frac{\xi}{2} + \pi\right) \right| \sum_{n\in\mathbb{Z}} \left| \hat{\phi}\left(\frac{\xi}{2} + \pi + 2n\pi\right) \right| \right].$$

Because we have assumed that $\mathsf{h} \in L^1(\mathbb{R})$ and $H_\phi(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)| \in L^\infty[0, 2\pi]$, we have that the series $\sum_{n\in\mathbb{Z}} |\widehat{\mathscr{L}f}(\xi + 2n\pi)|$ belongs to $L^2[0, 2\pi]$. Then, using Lemma 2.1, we have that, for each $f \in V_1$,

$$Z_{\mathscr{L}f}(0, \xi) = \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \widehat{\mathscr{L}f}(\xi + 2n\pi), \quad \text{a.e. } \xi \in [0, 2\pi].$$

Now, for $f \in V_1$, we consider the orthogonal projection h onto W_0, i.e., $f = g + h$ with $g \in V_0$ and $h \in W_0$. We have

$$\widehat{E^A_{\mathscr{L}}f}(\xi) = \widehat{E^A_{\mathscr{L}}h}(\xi) = \hat{h}(\xi) - \sum_{n\in\mathbb{Z}} (\mathscr{L}h)(n)\, \widehat{S_{\mathscr{L}}}(\xi) e^{-in\xi} = \hat{h}(\xi) - Z_{\mathscr{L}h}(0, \xi) \widehat{S}_{\mathscr{L}}(\xi)$$

Using that $\hat{h}(\xi) = v_f(\xi)\hat{\psi}(\xi)$ and the Poisson summation formula, we obtain

$$\begin{aligned}
\widehat{E^A_{\mathscr{L}}f}(\xi) &= \hat{h}(\xi) - \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \widehat{\mathscr{L}h}(\xi + 2n\pi) \widehat{S}_{\mathscr{L}}(\xi) \\
&= v_f(\xi)\hat{\psi}(\xi) - 2\pi \sum_{n\in\mathbb{Z}} \hat{\mathsf{h}}(\xi + 2n\pi) \hat{h}(\xi + 2n\pi) \widehat{S}_{\mathscr{L}}(\xi) \\
&= v_f(\xi)[\hat{\psi}(\xi) - 2\pi \sum_{n\in\mathbb{Z}} \hat{\mathsf{h}}(\xi + 2n\pi) \hat{\psi}(\xi + 2n\pi) \widehat{S}_{\mathscr{L}}(\xi)] \\
&= v_f(\xi)[\hat{\psi}(\xi) - Z_{\mathscr{L}\psi}(0, \xi) \widehat{S}_{\mathscr{L}}(\xi)] \\
&= v_f(\xi)[\hat{\psi}(\xi) - Z_{\mathscr{L}\psi}(0, \xi) Z^{-1}_{\mathscr{L}\phi}(0, \xi) \hat{\phi}(\xi)] \\
&= v_f(\xi)\hat{\phi}(\xi/2)[m_\psi(\xi/2) - Z_{\mathscr{L}\psi}(0, \xi) Z^{-1}_{\mathscr{L}\phi}(0, \xi) m_\phi(\xi/2)],
\end{aligned}$$

which concludes the proof. □

Theorem 4.3. *Assume that $H_\phi \in L^\infty[0, 2\pi]$. For any $f \in V_1$,*

$$|E^A_{\mathscr{L}}f(t)| \leq \frac{1}{\pi} \|N_{\mathscr{L}}\, Z_\phi(2t, \cdot)\|_{L^2[0,2\pi]} \|v_f\|_{L^2[0,2\pi]}, \quad t \in \mathbb{R}.$$

Moreover, the following uniform bound holds

$$|E^A_{\mathscr{L}}f(t)| \leq \sqrt{\frac{2}{\pi}}\, \|N_{\mathscr{L}}\, H_\phi\|_{L^2[0,2\pi]}\, \|v_f\|_{L^2[0,2\pi]}, \quad t \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}\|\widehat{E^A_{\mathscr{L}}f}\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} \left|\widehat{E^A_{\mathscr{L}}f}(\xi)\right| d\xi = \int_{-\infty}^{\infty} |v_f(\xi)\hat{\phi}(\xi/2)N_{\mathscr{L}}(\xi/2)|\, d\xi \\ &= 2\int_{-\infty}^{\infty} |v_f(2\xi)\hat{\phi}(\xi)N_{\mathscr{L}}(\xi)|\, d\xi = 2\int_0^{2\pi} |v_f(2\xi)N_{\mathscr{L}}(\xi)| H_\phi(\xi)\, d\xi.\end{aligned}$$

Because we have assumed that $H_\phi \in L^\infty[0,2\pi]$ and $N_{\mathscr{L}}, v_f \in L^2[0,2\pi]$, the Fourier transform of $E^A_{\mathscr{L}}f$ belongs to $L^1(\mathbb{R})$. Using the inverse Fourier transform and the Poisson summation formula, we obtain

$$\begin{aligned}E^A_{\mathscr{L}}f(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{E^A_{\mathscr{L}}f}(\xi)e^{it\xi}\, d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v_f(\xi)\hat{\phi}(\xi/2)N_{\mathscr{L}}(\xi/2)e^{it\xi}\, d\xi \\ &= \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v_f(2\xi)\hat{\phi}(\xi)N_{\mathscr{L}}(\xi)e^{2it\xi}\, d\xi \\ &= \frac{2}{\sqrt{2\pi}}\int_0^{2\pi} v_f(2\xi)N_{\mathscr{L}}(\xi)\sum_{n\in\mathbb{Z}}\hat{\phi}(\xi+2n\pi)e^{2it(\xi+2n\pi)}\, d\xi \\ &= \frac{1}{\pi}\int_0^{2\pi} v_f(2\xi)N_{\mathscr{L}}(\xi)Z_\phi(2t,\xi)\, d\xi.\end{aligned}$$

Then

$$\begin{aligned}|E^A_{\mathscr{L}}f(t)| &\le \frac{1}{\pi}\|v_f(2\xi)\|_{L^2[0,2\pi]}\|N_{\mathscr{L}}(\xi)Z_\phi(2t,\xi)\|_{L^2[0,2\pi]} \\ &= \frac{1}{\pi}\|N_{\mathscr{L}}(\xi)Z_\phi(2t,\xi)\|_{L^2[0,2\pi]}\|v_f(\xi)\|_{L^2[0,2\pi]}.\end{aligned}$$

Having in mind that

$$\begin{aligned}|Z_\phi(2t,\xi)| &= \sqrt{2\pi}\left|\sum_{n\in\mathbb{Z}}\hat{\phi}(\xi+2n\pi)e^{i2t(\xi+2n\pi)}\right| \\ &\le \sqrt{2\pi}\sum_{n\in\mathbb{Z}}|\hat{\phi}(\xi+2n\pi)| = \sqrt{2\pi}H_\phi(\xi),\end{aligned}$$

the uniform bound follows. □

Theorem 4.4. *Assume that $H_\phi \in L^\infty[0,2\pi]$. For any $f \in V_1$,*

$$K_0\|v_f\|^2_{L^2[0,2\pi]} \le \|E^A_{\mathscr{L}}f\|^2_{L^2(\mathbb{R})} \le K_\infty\|v_f\|^2_{L^2[0,2\pi]},$$

where

$$\begin{aligned}K_0 &:= \|\,|N_{\mathscr{L}}(\xi)|^2G_\phi(\xi) + |N_{\mathscr{L}}(\xi+\pi)|^2G_\phi(\xi+\pi)\|_0 \\ K_\infty &:= \|\,|N_{\mathscr{L}}(\xi)|^2G_\phi(\xi) + |N_{\mathscr{L}}(\xi+\pi)|^2G_\phi(\xi+\pi)\|_\infty.\end{aligned}$$

The constants K_0 and K_∞ are the optimal constants for these inequalities.

Proof. We have

$$\begin{aligned}\|E^A_{\mathscr{L}}f\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |\widehat{E^A_{\mathscr{L}}f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |v_f(\xi)\hat{\phi}(\xi/2)N_{\mathscr{L}}(\xi/2)|^2 d\xi \\ &= 2\int_{-\infty}^{\infty} |v_f(2\xi)\hat{\phi}(\xi)N_{\mathscr{L}}(\xi)|^2 d\xi = 2\int_0^{2\pi} |v_f(2\xi)N_{\mathscr{L}}(\xi)|^2 G_\phi(\xi)\,d\xi \\ &= 2\int_0^{\pi} |v_f(2\xi)N_{\mathscr{L}}(\xi)|^2 G_\phi(\xi)\,d\xi + 2\int_{\pi}^{2\pi} |v_f(2\xi)N_{\mathscr{L}}(\xi)|^2 G_\phi(\xi)\,d\xi \\ &= 2\int_0^{\pi} |v_f(2\xi)|^2\big(|N_{\mathscr{L}}(\xi)|^2 G_\phi(\xi) + |N_{\mathscr{L}}(\xi+\pi)|^2 G_\phi(\xi+\pi)\big)\,d\xi,\end{aligned}$$

from which the inequalities are easily obtained. The optimality of the constants K_0 and K_∞ can be proved similarly as in the proof of Theorem 3.6. □

ACKNOWLEDGMENTS

This work was partially supported by KOSEF(R01-2006-000-10424-0) and by the grant MTM2006–09737 from the D.G.I. of the Spanish Ministerio de Ciencia y Tecnología.

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