Sampling associated with resolvent-type kernels and Lagrange-type interpolation series

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Abstract

In this paper a new class of Kramer kernels is introduced, motivated by the resolvent of a symmetric operator with compact resolvent. The article gives a necessary and sufficient condition to ensure that the associated sampling formula can be expressed as a Lagrange-type interpolation series. Finally, an illustrative example, taken from the Hamburger moment problem theory, is included.

Keywords: Kramer kernel; Resolvent-type kernel; Lagrange-type interpolation series; Zero-removing property; Indeterminate Hamburger moment problem.
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1 Introduction

The classical Kramer sampling theorem provides a method for obtaining orthogonal sampling theorems [9, 15, 17, 24]. This theorem has played a very significant role in sampling theory, interpolation theory, signal analysis and, generally, in mathematics; see the survey articles [5, 6].

Nowadays, an abstract version of the Kramer sampling theorem can be stated as follows (see, for instance, [10, 16]): Let $K : \Omega \longrightarrow \mathcal{H}$ be a mapping, where Ω denotes an open subset of $\mathbb{R}$ (or $\mathbb{C}$) and $\mathcal{H}$ is a separable Hilbert space. Assume that there exists a sequence of distinct numbers $\{t_n\} \subset \Omega$, with n belonging to an indexing set $\mathbb{I}$

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contained in $\mathbb{Z}$, such that $\{K(t_n)\}$ is a complete orthogonal sequence for $\mathcal{H}$. Then for any f of the form $f(t) = \langle K(t), x\rangle_{\mathcal{H}},\ t \in \Omega$, where $x \in \mathcal{H}$, we have

$$f(t) = \lim_{N\to\infty} \sum_{\substack{|n|\le N\\ n\in\mathbb{I}}} f(t_n) S_n(t)\,, \quad t \in \Omega\,, \tag{1}$$

with

$$S_n(t) = \frac{\langle K(t), K(t_n)\rangle_{\mathcal{H}}}{\|K(t_n)\|^2_{\mathcal{H}}}\,, \quad t \in \Omega\,. \tag{2}$$

The series in (1) converges absolutely and uniformly on subsets of Ω where the function $t \mapsto \|K(t)\|_{\mathcal{H}}$ is bounded.

Notice that the sampling formula (1) works in the reproducing kernel Hilbert space (written shortly as RKHS) $\mathcal{H}_K$ introduced by Saitoh in [18] for the mapping K, whenever the Kramer sampling property holds, i.e., there exists a sequence $\{t_n\} \subset \Omega$ such that $\{K(t_n)\}$ is a complete orthogonal sequence for $\mathcal{H}$. In other words, there exist sequences $\{t_n\}$ in Ω, $\{a_n\}$ in $\mathbb{R} \setminus \{0\}$ and an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ such that $K(t_n) = a_n\, e_n$ for each $n \in \mathbb{I}$.

The Kramer sampling theorem can be stated in a more general setting involving Riesz bases [11] by assuming the existence of sequences $\{t_n\}$ in Ω, $\{a_n\}$ in $\mathbb{R} \setminus \{0\}$ and a Riesz basis $\{x_n\}$ for $\mathcal{H}$ such that $K(t_n) = a_n\, x_n$ for each $n \in \mathbb{I}$. From now on we say that K is a Kramer kernel. Recall that a Riesz basis in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthonormal (dual) Riesz basis $\{y_n\}_{n=1}^\infty$, i.e., $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that, for every $x \in \mathcal{H}$, the expansions

$$x = \sum_{n=1}^\infty \langle x, y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n=1}^\infty \langle x, x_n\rangle_{\mathcal{H}}\, y_n \quad \text{in } \mathcal{H}$$

hold (see [23] for more details and proofs).

The very frequent case where the kernel $K : \mathbb{C} \longrightarrow \mathcal{H}$ is analytic and, consequently, the sampled space $\mathcal{H}_K$ is a RKHS of entire functions, was treated in [8, 14]. For this analytic case, it was proved in [10, 11] a necessary and sufficient condition ensuring that the sampling formula (1) can be written as a Lagrange-type interpolation series, i.e., for each $n \in \mathbb{I}$

$$S_n(t) = \frac{G(t)}{(t - t_n)G'(t_n)}\,, \quad t \in \mathbb{C}\,,$$

where g denotes an entire function having only simple zeros at $\{t_n\}$. Roughly speaking, the aforesaid necessary and sufficient condition concerns the stability of the functions belonging to the space $\mathcal{H}_K$, on removing a finite number of zeros.

The Kramer sampling theorem has been the cornerstone for a significant mathematical literature on the topic of sampling theorems associated with differential or difference problems which has flourished for the past few years. As a small but significant sample of examples see, for instance, [2, 3, 9, 13, 19, 20, 24, 25] and references therein.

In this paper we introduce a new family of kernels K_σ for which the Kramer property holds. These kernels are motivated on the resolvent of a symmetric operator with compact resolvent. Morever, we give a necessary and sufficient condition ensuring that the associated sampling formula (1) can be written as a Lagrange-type interpolation series. Finally, we include an illustrative example taken from the indeterminate Hamburger moment problem theory [1, 21].

2 Sampling associated with resolvent-type kernels

2.1 By way of motivation

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be a symmetric (formally self–adjoint) linear operator, densely defined on $\mathcal{H}$. Assume that there exists its inverse operator $\mathcal{T} = \mathcal{A}^{-1}$, and that it is a compact operator defined on $\mathcal{H}$. We know from the spectral theorem for symmetric compact operators defined on a Hilbert space that $\mathcal{T}$ has discrete spectrum [22]. Moreover, if $\{\mu_n\}_{n=1}^\infty$ is the sequence of eigenvalues of $\mathcal{T}$, then $\lim_{n\to\infty} |\mu_n| = 0$. We may assume that $|\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_n| \geq \ldots$. Moreover, the eigenspace associated with each eigenvalue μ_n is finite-dimensional; we will assume that $k_n = \dim \ker(\mu_n I - \mathcal{T}) = 1$ for all $n \in \mathbb{N}$. Note that 0 is not an eigenvalue of $\mathcal{T}$, so the sequence $\{e_n\}_{n=1}^\infty$ of eigenvectors of $\mathcal{T}$ is a complete orthonormal system for $\mathcal{H}$. The sequences $\{z_n = \mu_n^{-1}\}_{n=1}^\infty$ and $\{e_n\}_{n=1}^\infty$ are, respectively, the sequence of eigenvalues and the sequence of associated eigenvectors of the operator $\mathcal{A}$. Since $\lim_{n\to\infty} |\mu_n| = 0$, we have $0 < |z_1| \leq |z_2| \leq \ldots \leq |z_n| \leq \ldots$ and $\lim_{n\to\infty} |z_n| = \infty$.

The resolvent operator $R_z := (z\, I - \mathcal{A})^{-1}$ is a meromorphic function in $\mathbb{C}$ with simple poles at $\{z_n\}_{n=1}^\infty$. For each $x \in \mathcal{H}$ the following expansion holds in $\mathcal{H}$ [22]:

$$R_z(x) = \sum_{m=1}^\infty \frac{\langle x, e_m \rangle_\mathcal{H}}{z - z_m}\, e_m \quad \text{in } \mathcal{H}\,. \tag{3}$$

Let G denote an entire function having simple zeros at $\{z_n\}_{n=1}^\infty$; this is allowed by Weierstrass' theorem [23, p. 54]. Thus, for a fixed $a \in \mathcal{H}$ the $\mathcal{H}$-valued mapping defined by

$$\begin{array}{rccl} K_a : & \mathbb{C} & \longrightarrow & \mathcal{H} \\ & z & \longrightarrow & K_a(z) := G(z) R_z(a)\,, \end{array} \tag{4}$$

it is an entire mapping, and defining

$$\mathcal{H}_a := \big\{ f : \mathbb{C} \longrightarrow \mathbb{C} \;:\; f(z) = \langle K_a(z), x \rangle_\mathcal{H} \;\text{ where } x \in \mathcal{H} \big\}\,,$$

we obtain a RKHS of entire functions (see [18]). Since $K_a(z_m) = G'(z_m)\, \langle a, e_m \rangle_\mathcal{H}\, e_m$ for each $m \in \mathbb{N}$; assuming that $\langle a, e_m \rangle_\mathcal{H} \neq 0$ for all $m \in \mathbb{N}$, the mapping K_a satisfies the Kramer property at the eigenvalues sequence $\{z_m\}_{m=1}^\infty$. As a consequence, following (1) and (2), one obtains that any $f \in \mathcal{H}_a$ can be recovered through the Lagrange-type interpolation series:

$$f(z)=\sum_{n=1}^{\infty} f(z_n)\,\frac{G(z)}{(z-z_n)G'(z_n)}\,,\quad z\in\mathbb{C}\,. \tag{5}$$

Now, the resolvent sampling kernel K_a given in (4) can be generalized in the following way: Consider

- an entire $\mathcal{H}$-valued function $\sigma:\mathbb{C}\longrightarrow\mathcal{H}$,
- an arbitrary sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ such that $\lim_{n\to\infty}|z_n|=\infty$,
- an entire function $G(z)$ having only simple zeros at $\{z_n\}_{n=1}^{\infty}$,
- an arbitrary Riesz basis $\{x_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ with dual basis $\{y_n\}_{n=1}^{\infty}$,

and define the kernel $K_\sigma:\mathbb{C}\longrightarrow\mathcal{H}$ as

$$K_\sigma(z):=\sum_{m=1}^{\infty}\frac{G(z)}{z-z_m}\langle\sigma(z),y_m\rangle_{\mathcal{H}}\,x_m\,,\quad z\in\mathbb{C}\,. \tag{6}$$

By using [14, Theorem 2.3] we deduce that K_σ defines an entire $\mathcal{H}$-valued mapping since, for each $m\in\mathbb{N}$, the function $\frac{G(z)}{z-z_m}\langle\sigma(z),y_m\rangle_{\mathcal{H}}$ is an entire function, and the function $z\mapsto\|K_\sigma(z)\|_{\mathcal{H}}$ is bounded on compact subsets of $\mathbb{C}$. To prove this, due to the Riesz basis condition on $\{x_n\}_{n=1}^{\infty}$ (see [23, p. 27]), there exists a constant $B>0$ such that

$$\|K_\sigma(z)\|_{\mathcal{H}}\le B\sum_{m=1}^{\infty}\left|\frac{G(z)\langle\sigma(z),y_m\rangle}{z-z_m}\right|^2,\quad z\in\mathbb{C}\,.$$

Next, we prove that the series is uniformly bounded on compact subsets of the complex plane. Indeed, given M a compact in $\mathbb{C}$ there exists a closed disk D_R centered at the origin with radius $R>0$ such that $M\subseteq D_R$. Apart from a possible finite number of points $\{z_k\}$, k in $\mathbb{I}_R$, a finite subset of $\mathbb{N}$, we have the result that $|z-z_m|\ge\big||z|-|z_m|\big|\ge |z_m|-R$ for all $z\in M$ and $m\in\mathbb{N}\setminus\mathbb{I}_R$. Thus,

$$\begin{aligned}\sum_{m=1}^{\infty}\left|\frac{G(z)\langle\sigma(z),y_m\rangle}{z-z_m}\right|^2 &\le \sum_{m\in\mathbb{I}_R}\left|\frac{G(z)\langle\sigma(z),y_m\rangle}{z-z_m}\right|^2+|G(z)|^2\sum_{m\in\mathbb{N}\setminus\mathbb{I}_R}\frac{|\langle\sigma(z),y_m\rangle|^2}{(|z_m|-R)^2}\\ &\le \sum_{m\in\mathbb{I}_R}\left|\frac{G(z)\langle\sigma(z),y_m\rangle}{z-z_m}\right|^2+C\,|G(z)|^2\,\|\sigma(z)\|^2\,,\end{aligned}$$

where C denotes a constant, and both summands are bounded on the compact M. For the second summand, note that the sequence $\{1/(|z_m|-R)^2\}$ is bounded, and that $\sum_{m\in\mathbb{N}}|\langle\sigma(z),y_m\rangle|^2\le C'\|\sigma(z)\|^2$ for some positive constant C' since the sequence $\{y_m\}_{m=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$.

Besides, for each z_n we have $K_\sigma(z_n) = G'(z_n)\langle\sigma(z_n), y_n\rangle_\mathcal{H}\, x_n$. If we assume that $\langle\sigma(z_n), y_n\rangle_\mathcal{H} \neq 0$ for all $n \in \mathbb{N}$, we obtain that K_σ is an analytic kernel satisfying the Kramer sampling property for the data $\{z_n\}_{n=1}^\infty \subset \mathbb{C}$, $\{G'(z_n)\langle\sigma(z_n), y_n\rangle_\mathcal{H}\}_{n=1}^\infty \subset \mathbb{C}\setminus\{0\}$ and the Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$.

Definition 1. *We say that the entire $\mathcal{H}$-valued function K_σ defined as in* (6)*, and satisfying that $\langle\sigma(z_n), y_n\rangle_\mathcal{H} \neq 0$ for all $n \in \mathbb{N}$, is a resolvent-type sampling kernel.*

Next, we derive the sampling theory associated with K_σ:

2.2 The sampling result

Let K_σ be a resolvent-type kernel satisfying the Kramer property for the sequence $\{z_n\}_{n=1}^\infty$. Define the mapping $\mathcal{T}_\sigma$ by

$$\begin{aligned} \mathcal{T}_\sigma :\ \mathcal{H} &\longrightarrow \mathbb{C}^\mathbb{C} \\ x &\longmapsto \mathcal{T}_\sigma(x)\,, \end{aligned}$$

where $[\mathcal{T}_\sigma(x)](z) := \langle K_\sigma(z), x\rangle_\mathcal{H}$, $z \in \mathbb{C}$. Note that $\mathcal{T}_\sigma(x)$ defines an entire function [22]. The mapping $\mathcal{T}_\sigma$ is anti-linear, i.e.,

$$\mathcal{T}_\sigma(\alpha x + \beta y) = \overline{\alpha}\,\mathcal{T}_\sigma(x) + \overline{\beta}\,\mathcal{T}_\sigma(y) \quad \text{for all } x, y \in \mathcal{H} \text{ and } \alpha, \beta \in \mathbb{C}\,.$$

Since the sequence $\big\{K_\sigma(z_n)\big\}_{n=1}^\infty$ forms a complete system in $\mathcal{H}$, the mapping $\mathcal{T}_\sigma$ is one-to-one (see [18, p. 21]). Thus, if we denote by $\mathcal{H}_\sigma$ the range space of $\mathcal{T}_\sigma$, i.e., $\mathcal{H}_\sigma := \mathcal{T}_\sigma(\mathcal{H})$. endowed with the norm $\|f\|_{\mathcal{H}_\sigma} := \|x\|_\mathcal{H}$ such that $f = \mathcal{T}_\sigma(x)$, we obtain a Hilbert space of entire functions.

Moreover, the space $\mathcal{H}_\sigma$ is a reproducing kernel Hilbert space since the point-evaluation functional $E_z(f) := f(z)$ is continuous for each $z \in \mathbb{C}$. Its reproducing kernel k_σ is given by

$$k_\sigma(z, \omega) = \langle K_\sigma(z), K_\sigma(\omega)\rangle_\mathcal{H}\,, \quad z, w \in \mathbb{C}\,,$$

that is, for each $\omega \in \mathbb{C}$ the function $k_a(\cdot, \omega)$ belongs to $\mathcal{H}_\sigma$, and the reproducing property

$$f(\omega) = \langle f, k_a(\cdot, \omega)\rangle_{\mathcal{H}_\sigma} \quad \text{for } \omega \in \mathbb{C} \text{ and } f \in \mathcal{H}_\sigma\,,$$

holds.

The sampling theorem allowing the recovery of any function in $\mathcal{H}_\sigma$ from its samples at the sequence $\{z_n\}_{n=1}^\infty$ reads as follows:

Theorem 1. *Any function $f \in \mathcal{H}_\sigma$ can be recovered from its samples $\{f(z_n)\}_{n=1}^\infty$ by means of the sampling formula*

$$f(z) = \sum_{n=1}^\infty f(z_n)\, \frac{\langle\sigma(z), y_n\rangle_\mathcal{H}}{\langle\sigma(z_n), y_n\rangle_\mathcal{H}}\, \frac{G(z)}{(z - z_n)G'(z_n)}\,, \quad z \in \mathbb{C}\,. \tag{7}$$

The convergence of the series in (7) *is absolute and uniform in compact subsets of* $\mathbb{C}$.

Proof. Assume that, for $x \in \mathcal{H}$, we have $f(z) = \langle K_\sigma(z), x\rangle_\mathcal{H},\ z \in \mathbb{C}$. Expanding $x \in \mathcal{H}$ with respect to the Riesz basis $\{y_n\}_{n=1}^\infty$ for $\mathcal{H}$ we obtain $x = \sum_{n=1}^\infty \langle x, x_n\rangle_\mathcal{H}\, y_n$ in $\mathcal{H}$, and consequently

$$f = \mathcal{T}_\sigma(x) = \sum_{n=1}^\infty \overline{\langle x, x_n\rangle_\mathcal{H}}\, \mathcal{T}_\sigma(y_n) \quad \text{in } \mathcal{H}_\sigma\,. \tag{8}$$

By using the biorthonormality, i.e., $\langle x_n, y_m\rangle = \delta_{n,m}$, we get $\mathcal{T}_\sigma(y_n)(z) = \frac{G(z)}{z-z_n}\,\langle \sigma(z), y_n\rangle_\mathcal{H}$, $z \in \mathbb{C}$. Now, for each $n \in \mathbb{N}$ we obtain $f(z_n) = G'(z_n)\langle \sigma(z_n), y_n\rangle_\mathcal{H}\,\langle x_n, x\rangle_\mathcal{H}$. Substituting in (8) we deduce (7) with convergence in $\mathcal{H}_\sigma$. Since $\mathcal{H}_\sigma$ is a RKHS, the convergence in $\mathcal{H}_\sigma$ implies pointwise convergence which is uniform on subsets of $\mathbb{C}$ where the function $z \mapsto \|K_\sigma(z)\|_\mathcal{H}$ is bounded; in particular, on compact subsets of $\mathbb{C}$. This pointwise convergence is absolute due to the unconditional convergence of a Riesz basis expansion. □

In the particular case where $\sigma(z) = a \in \mathcal{H}$, a constant vector such that $\langle a, e_n\rangle_\mathcal{H} \neq 0$ for all $n \in \mathbb{N}$, we obtain, as a consequence, the sampling formula (5) for the RKHS $\mathcal{H}_a$.

2.3 Lagrange-type interpolation series

A challenge problem is give a neccesary and sufficient condition on the function σ such that the sampling formula (7) can be written as a Lagrange-type interpolation series (10). Observe that it is equivalent to the existence of an entire function $A : \mathbb{C} \longrightarrow \mathbb{C}$ without zeros, such that, for each $n \in \mathbb{N}$ we have

$$\frac{\langle \sigma(z), y_n\rangle_\mathcal{H}}{\langle \sigma(z_n), y_n\rangle_\mathcal{H}} = \frac{A(z)}{A(z_n)}\,, \quad z \in \mathbb{C}\,. \tag{9}$$

In this case, the sampling formula (7) reduces to a Lagrange-type interpolation series (10) where $H(z) = A(z)\,G(z),\ z \in \mathbb{C}$.

As it was proved in [11, Theorem 4], a necessary and sufficient condition assuring that the sampling formula associated with an analytic Kramer kernel K can be written as a Lagrange-type interpolation series is that the zero-removing property holds in $\mathcal{H}_K$; this property reads:

Definition 2. *A set $\mathcal{A}$ of entire functions has the zero-removing property if for any $g \in \mathcal{A}$ and any zero w of g the function $g(z)/(z-w)$ belongs to $\mathcal{A}$.*

As a corollary of the aforementioned result [11, Theorem 4]) we obtain:

Corollary 2. *The sampling formula (7) in $\mathcal{H}_\sigma$ can be written as a Lagrange-type interpolation series*

$$f(z) = \sum_{n=1}^\infty f(z_n)\frac{H(z)}{(z-z_n)H'(z_n)}\,, \quad z \in \mathbb{C}\,, \tag{10}$$

where H denotes an entire function having only simple zeros at $\{z_n\}_{n=1}^\infty$ if and only if the space $\mathcal{H}_\sigma$ satisfies the zero-removing property.

Now, we are ready to prove when the sampling formula (7) can be expressed as a Lagrange-type interpolation series, or, equivalently, when the zero-removing property in $\mathcal{H}_\sigma$ holds:

Theorem 3. *In the RKHS of entire functions $\mathcal{H}_\sigma$ associated with a resolvent-type sampling kernel K_σ (see (6)) the zero-removing property holds if and only if the $\mathcal{H}$-valued function σ has the form $\sigma(z) = F(z)u$ where $F : \mathbb{C} \longrightarrow \mathbb{C}$ is an entire function without zeros and $u \in \mathcal{H}$ with $\langle u, y_n\rangle_{\mathcal{H}} \neq 0$ for each $n \in \mathbb{N}$.*

Proof. Assume that $\sigma(z) = F(z)\,u$, with $\langle u, y_n\rangle_{\mathcal{H}} \neq 0$ for each $n \in \mathbb{N}$ and F entire function without zeros. For $f \in \mathcal{H}_\sigma$, the sampling formula (7) reads

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} f(z_n) \frac{\big\langle F(z)\,u, y_n\big\rangle_{\mathcal{H}}}{\big\langle F(z_n)\,u, y_n\big\rangle_{\mathcal{H}}} \frac{G(z)}{(z-z_n)G'(z_n)} \\ &= \sum_{n=1}^{\infty} f(z_n) \frac{F(z)}{F(z_n)} \frac{G(z)}{(z-z_n)G'(z_n)}, \qquad z \in \mathbb{C}. \end{aligned} \tag{11}$$

Taking $H(z) := F(z)G(z)$, $z \in \mathbb{C}$, we have $H'(z_n) = F(z_n)G'(z_n)$, and substituting in (11) we obtain the Lagrange-type interpolation series

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{H(z)}{(z-z_n)H'(z_n)}, \qquad z \in \mathbb{C}.$$

By using Corollary 2, the zero-removing property in $\mathcal{H}_\sigma$ holds.

Conversely, assume that the zero-removing property in $\mathcal{H}_\sigma$ holds. In this case, it is easy to deduce that $\sigma(z) \neq 0$ for all $z \in \mathbb{C}$. Indeed, if $\sigma(z_0) = 0$ then $K_\sigma(z_0) = 0$ and, consequently, every function in $\mathcal{H}_\sigma$ has a zero at z_0. Let f be a nonzero entire function in $\mathcal{H}_\sigma$ and let r denote the order of its zero at z_0. The function $f(z)/(z-z_0)^r$ belongs to $\mathcal{H}_\sigma$ and, however it does not vanish at z_0, a contradiction.

For each $n \in \mathbb{N}$ the function $S_n(z) := \langle K_\sigma(z), y_n\rangle_{\mathcal{H}} = \frac{G(z)}{(z-z_n)} \langle \sigma(z), y_n\rangle_{\mathcal{H}}$, $z \in \mathbb{C}$, belongs to $\mathcal{H}_\sigma$ and it has zeros at $\{z_m\}_{m\neq n}$. Since the zero-removing property holds, for $m \neq n$, the functions

$$T_{n,m}(z) := \frac{S_n(z)}{z-z_m} = \frac{G(z)}{(z-z_n)(z-z_m)} \langle \sigma(z), y_n\rangle_{\mathcal{H}}\,, \quad z \in \mathbb{C}\,,$$

belong to $\mathcal{H}_\sigma$. The sampling formula (7) for $T_{n,m}(z)$ gives

$$T_{n,m}(z) = \sum_{j=1}^{\infty} T_{n,m}(z_j) \frac{\langle \sigma(z), y_j\rangle_{\mathcal{H}}}{\langle \sigma(z_j), y_j\rangle_{\mathcal{H}}} \frac{G(z)}{(z-z_j)G'(z_j)}\,. \tag{12}$$

Evaluating the function $T_{n,m}$ at the sequence $\{z_j\}_{j=1}^{\infty}$ we get

$$
T_{n,m}(z_j) = \frac{S_n(z_j)}{z_j - z_m} = \begin{cases} \dfrac{G'(z_n)}{z_n - z_m} \langle \sigma(z_n), y_n \rangle_{\mathcal{H}} & j = n \\[2ex] \dfrac{G'(z_m)}{z_m - z_n} \langle \sigma(z_m), y_n \rangle_{\mathcal{H}} & j = m \\[2ex] 0 & j \neq m, n \end{cases}
$$

from which expansion (12) reads

$$
\begin{aligned}
T_{n,m}(z) &= \frac{G'(z_n)}{z_n - z_m} \big\langle \sigma(z_n), y_n \big\rangle_{\mathcal{H}} \frac{\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}}}{\big\langle \sigma(z_n), y_n \big\rangle_{\mathcal{H}}} \frac{G(z)}{(z - z_n) G'(z_n)} \\
&+ \frac{G'(z_m)}{z_m - z_n} \big\langle \sigma(z_m), y_n \big\rangle_{\mathcal{H}} \frac{\big\langle \sigma(z), y_m \big\rangle_{\mathcal{H}}}{\big\langle \sigma(z_m), y_m \big\rangle_{\mathcal{H}}} \frac{G(z)}{(z - z_m) G'(z_m)} \\
&= \frac{G(z)}{z_n - z_m} \left[\frac{\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}}}{z - z_n} - \frac{\big\langle \sigma(z_m), y_n \big\rangle_{\mathcal{H}} \big\langle \sigma(z), y_m \big\rangle_{\mathcal{H}}}{\big\langle \sigma(z_m), y_m \big\rangle_{\mathcal{H}} (z - z_m)} \right]. \qquad (13)
\end{aligned}
$$

Hence,

$$
\frac{\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}}}{(z - z_n)(z - z_m)} = \frac{1}{z_n - z_m} \left[\frac{\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}}}{z - z_n} - \frac{\big\langle \sigma(z_m), y_n \big\rangle_{\mathcal{H}} \big\langle \sigma(z), y_m \big\rangle_{\mathcal{H}}}{\big\langle \sigma(z_m), y_m \big\rangle_{\mathcal{H}} (z - z_m)} \right],
$$

that is

$$
\frac{\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}}}{(z - z_m)(z - z_n)} - \frac{\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}}}{(z - z_n)(z_n - z_m)} = -\frac{\big\langle \sigma(z_m), y_n \big\rangle_{\mathcal{H}} \big\langle \sigma(z), y_m \big\rangle_{\mathcal{H}}}{\big\langle \sigma(z_m), y_m \big\rangle_{\mathcal{H}} (z - z_m)(z_n - z_m)}.
$$

or

$$
\frac{\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}}}{z - z_n} \left[\frac{z - z_n}{(z - z_m)(z_m - z_n)} \right] = \frac{\big\langle \sigma(z_m), y_n \big\rangle_{\mathcal{H}} \big\langle \sigma(z), y_m \big\rangle_{\mathcal{H}}}{\big\langle \sigma(z_m), y_m \big\rangle_{\mathcal{H}} (z - z_m)(z_m - z_n)}.
$$

Therefore

$$
\big\langle \sigma(z), y_n \big\rangle_{\mathcal{H}} = \frac{\big\langle \sigma(z_m), y_n \big\rangle_{\mathcal{H}}}{\big\langle \sigma(z_m), y_m \big\rangle_{\mathcal{H}}} \big\langle \sigma(z), y_m \big\rangle_{\mathcal{H}}. \qquad (14)
$$

Expanding $\sigma(z)$ with respect to the Riesz basis $\{x_n\}_{n=1}^{\infty}$ we have

$$
\sigma(z) = \sum_{j=1}^{\infty} \big\langle \sigma(z), y_j \big\rangle_{\mathcal{H}} x_j \quad \text{in } \mathcal{H}.
$$

Having in mind (14) we observe that the coefficients $\big\langle \sigma(z), y_j \big\rangle_{\mathcal{H}}$ satisfy

$$
\big\langle \sigma(z), y_j \big\rangle_{\mathcal{H}} = a_{m,j} \big\langle \sigma(z), y_m \big\rangle_{\mathcal{H}}
$$

where

$$a_{m,j} = \begin{cases} \dfrac{\langle \sigma(z_m), y_j \rangle_{\mathcal{H}}}{\langle \sigma(z_m), y_m \rangle_{\mathcal{H}}} & j \neq m. \\ \\ 1 & j = m \end{cases}$$

Notice that the sequence $\{a_{m,j}\}_{j=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$ for each $m \in \mathbb{N}$. As a consequence of (14) we obtain

$$\sigma(z) = \sum_{j=1}^{\infty} \langle \sigma(z), y_j \rangle_{\mathcal{H}} x_j = \langle \sigma(z), y_m \rangle_{\mathcal{H}} \sum_{j=1}^{\infty} a_{m,j} x_j = F_m(z)\, u_m \,,$$

where $u_m \neq 0$ belongs to $\mathcal{H}$, and $F_m(z) = \langle \sigma(z), y_m \rangle_{\mathcal{H}}$, $z \in \mathbb{C}$, is an entire function without zeros; recall that $\sigma(z) \neq 0$ for any $z \in \mathbb{C}$. Fixing any $m \in \mathbb{N}$ we conclude the proof of the theorem. Note that $\langle u, y_n \rangle_{\mathcal{H}} \neq 0$ for all $n \in \mathbb{N}$; in case that $\langle u, y_k \rangle \neq 0$ for some $k \in \mathbb{N}$ we derive that $f(z_k) = 0$ for every $f \in \mathcal{H}_\sigma$ and, consequently, the ZR property does not hold in $\mathcal{H}_\sigma$. □

3 An illustrative example

Given two sequences $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ of, respectively, real and positive numbers consider the semi-infinite Jacobi matrix

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \ddots \\ 0 & 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{15}$$

whose domain $D(\mathcal{A})$ is the set of sequences of finite support. The Hamburger moment problem associated with $\mathcal{A}$ reads as follows: Given the real numbers $s_n = \langle \delta_0, \mathcal{A}^n \delta_0 \rangle_{\ell^2}$, $n \geq 0$, where δ_0 stands for the sequence $(1, 0, 0, \dots)$, we are interested in the search of positive Borel measures μ supported on $(-\infty, \infty)$ satisfying

$$s_n = \int_{-\infty}^{\infty} x^n d\mu(x)\,,\ n \geq 0\,.$$

If such a measure exists and is unique, the moment problem is determinate. If a measure μ exists, but it is not unique, the moment problem is called indeterminate (see, for instance, [21] or the classical reference [1]).

The operator $\mathcal{A}$ is closable since it is symmetric and densely defined; we denote again by $\mathcal{A}$ its closure. The domain of the adjoint of $\mathcal{A}$ is given by $D(\mathcal{A}^*) = \{z \in \ell^2(\mathbb{N}_0) \mid \mathcal{A}z \in \ell^2(\mathbb{N}_0)\}$ [21, p.105]. If $\mathcal{A}$ is not a self-adjoint operator (the associated Hamburger moment problem is indeterminate) its (von Neumann) self-adjoint extensions, $\mathcal{A} \subset \mathcal{S}_t \subset \mathcal{A}^*$, can be parametrized by $t \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and their domains are [21, p.125]

$$\mathcal{D}(\mathcal{S}_t) = \begin{cases} \mathcal{D}(\mathcal{A}) + \operatorname{span}\{t\Pi(0) + \Theta(0)\} & \text{if } t \in \mathbb{R}\,, \\ \mathcal{D}(\mathcal{A}) + \operatorname{span}\{\Pi(0)\} & \text{if } t = \infty\,, \end{cases}$$

where

$$\Pi(z) := \{P_0(z), P_1(z), P_2(z), \dots\} \quad \text{and} \quad \Theta(z) := \{Q_0(z), Q_1(z), Q_2(z), \dots\}\,,$$

denote the polynomial solutions $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ of the second order difference equation

$$a_n\gamma_{n+1} + b_n\gamma_n + a_{n-1}\gamma_{n-1} = z\gamma_n\,, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad (a_{-1} = 1) \tag{16}$$

corresponding to the initial data $\gamma_{-1} = 0$, $\gamma_0 = 1$ and $\gamma_{-1} = -1$, $\gamma_0 = 0$ respectively. Equivalently (see [21, p.126]), for a sequence $\Gamma = \{\gamma_n\}$ we have

$$\Gamma \in D(\mathcal{S}_t) \Leftrightarrow \begin{cases} \lim\limits_{n\to\infty} W(\Gamma, t\Pi(0) + \Theta(0))(n) = 0 & \text{if } t \in \mathbb{R}, \\ \lim\limits_{n\to\infty} W(\Gamma, \Pi(0))(n) = 0 & \text{if } t = \infty. \end{cases}$$

where $W(\Gamma, \Gamma')(n) = a_n(\gamma_{n+1}\gamma'_n - \gamma_n\gamma'_{n+1})$ denotes the Wronskian of the sequences $\Gamma = \{\gamma_n\}$ and $\Gamma' = \{\gamma'_n\}$.

The eigenvalue problem $(zI - \mathcal{S}_t)\Gamma = 0$ is equivalent to the discrete Sturm-Liouville problem

$$\begin{cases} a_n\gamma_{n+1} + b_n\gamma_n + a_{n-1}\gamma_{n-1} = z\gamma_n\,, & n \in \mathbb{N}_0 \\ \gamma_{-1} = 0\,,\ \lim_{n\to\infty} W(\Gamma, t\Pi(0) + \Theta(0))(n) = 0. \end{cases} \tag{17}$$

whenever $t \in \mathbb{R}$, or

$$\begin{cases} a_n\gamma_{n+1} + b_n\gamma_n + a_{n-1}\gamma_{n-1} = z\gamma_n\,, & n \in \mathbb{N}_0 \\ \gamma_{-1} = 0\,,\ \lim_{n\to\infty} W(\Gamma, \Pi(0))(n) = 0. \end{cases} \tag{18}$$

in the case $t = \infty$. As a consequence, z will be an eigenvalue of $\mathcal{S}_t$ if and only if

$$\lim_{n\to\infty} W(\Pi(z), t\Pi(0) + \Theta(0))(n) = 0 \quad \text{whenever } t \in \mathbb{R}\,,$$

or

$$\lim_{n\to\infty} W(\Pi(z), \Pi(0))(n) = 0 \quad \text{whenever } t = \infty\,.$$

It is known [21, p.127] that each self–adjoint extension $\mathcal{S}_t$ of $\mathcal{A}$ has a pure point spectrum $\{z_i^t = z_i(\mathcal{S}_t)\}_{i=0}^{\infty}$. The corresponding eigenfunctions $\{\Pi_i^t\}_{i=0}^{\infty}$ are given by

$$\Pi_i^t = \Pi(z_i^t) = \{P_0(z_i^t), P_1(z_i^t), \dots, P_n(z_i^t), \dots\}, \quad i \in \mathbb{N}_0\,,$$

and they form an orthogonal basis in $\ell^2(\mathbb{N}_0)$ [4, 12]. Consequently, the resolvent operator $R_z^t = (zI - \mathcal{S}_t)^{-1}$, where $z \notin \rho(\mathcal{S}_t)$, is a compact operator [7, p.423].

Consider the canonical product $G_t(z)$ of the sequence of eigenvalues $\{z_i^t\}_{i=0}^\infty$; this canonical product always exists because, in particular, $\sum_{i=0}^\infty |z_i^t|^{-2} < \infty$ (see [21, p. 128]). Specifically, the canonical product is given by

$$G_t(z) = \begin{cases} \prod_{n=0}^\infty (1 - \frac{z}{z_n^t}) \exp(z/z_n^t) & \text{if } \sum_{n=0}^\infty |z_n^t|^{-1} = \infty \\ \prod_{n=0}^\infty (1 - \frac{z}{z_n^t}) & \text{if } \sum_{n=0}^\infty |z_n^t|^{-1} < \infty \end{cases}$$

whenever $z_0^t \neq 0$, and

$$G_t(z) = \begin{cases} z\prod_{n=1}^\infty (1 - \frac{z}{z_n^t}) \exp(z/z_n^t) & \text{if } \sum_{n=0}^\infty |z_n^t|^{-1} = \infty \\ z\prod_{n=1}^\infty (1 - \frac{z}{z_n^t}) & \text{if } \sum_{n=0}^\infty |z_n^t|^{-1} < \infty \end{cases}$$

in the case $z_0^t = 0$.

Thus, for a fixed $t \in \overline{\mathbb{R}}$, we define the kernel $K^t : \mathbb{C} \longrightarrow \ell^2(\mathbb{N}_0)$ as

$$K^t(z)(m) := \sum_{i=0}^\infty \frac{G_t(z)}{z - z_i^t} \langle \delta_0, \frac{\Pi_i^t}{\|\Pi_i^t\|} \rangle_{\ell^2} \frac{\Pi_i^t(m)}{\|\Pi_i^t\|} = \sum_{i=0}^\infty \frac{G_t(z)}{z - z_i^t} \frac{\Pi_i^t(m)}{\|\Pi_i^t\|^2}, \quad m \in \mathbb{N}_0 .$$

Note that K^t corresponds to the particular choice $\sigma(z) = \delta_0$ for all $z \in \mathbb{C}$, and that $P_0(z_i^t) = 1$ for each $i \in \mathbb{N}_0$. As a consequence of Theorem 1, any function f defined as

$$f(z) = \langle K^t(z), \{c_n\} \rangle_{\ell^2} = \sum_{m=0}^\infty K^t(z)(m)\, \overline{c}_m , \quad z \in \mathbb{C} ,$$

where $\{c_m\}_{m=0}^\infty \in \ell^2(\mathbb{N}_0)$, can be recovered through the Lagrange-type interpolation series

$$f(z) = \sum_{i=0}^\infty f(z_i^t) \frac{G_t(z)}{(z - z_i^t) G_t'(z_i^t)}, \quad z \in \mathbb{C} .$$

The convergence of the above series is absolute and uniform on compact subsets of $\mathbb{C}$.

Finally, it is worth to mention that more can be said about the kernel K^t and the sampling points $\{z_i^t\}_{i=0}^\infty$. Indeed, it is known that, associated with the self-adjoint extension $\mathcal{S}_t$ of $\mathcal{A}$, there exists a positive measure μ_t solution of the indeterminate Hamburger moment problem $s_n = \int_{-\infty}^\infty x^n d\mu_t(x)$, $n \geq 0$, for which the polynomials $\{P_n\}_{n=0}^\infty$ are dense in $L^2(\mu_t)$ (an extremal measure). Equivalently, the Hamburger moment problem is indeterminate if and only if the discrete Sturm-Liouville problem (17) or (18) belongs to the limit-circle case. Taking into account the components $A(z)$, $B(z)$, $C(z)$ and $D(z)$ of the Nevalinna matrix of the indeterminate Hamburger moment problem (see [21, p. 124]) we have that [21, p. 126]

$$m_t(z) := \frac{A(z) + tC(z)}{B(z) + tD(z)} = \int_{-\infty}^\infty \frac{d\mu_t(x)}{z - x}, \quad z \in \mathbb{C} \setminus \mathbb{R} .$$

The poles of the meromorphic function $m_t(z)$ (which coincide with the zeros of the entire function $B(z) + tD(z)$ if $t \in \mathbb{R}$ or the zeros of $D(z)$ if $t = \infty$) are precisely the eigenvalues of $\mathcal{S}_t$, that is, the sampling points $\{z_i^t\}_{i=0}^\infty$ (see [21, p. 127]). Concerning the kernel K^t, for each $z \in \mathbb{C}$, we have that

$$K^t(z)(m) = G_t(z) \left[Q_m(z) + m_t(z) P_m(z)\right] , \quad m \in \mathbb{N}_0 \, .$$

Since we are dealing with an indeterminate Hamburger moment problem, note that, for each $z \in \mathbb{C}$, the sequences $\{P_m(z)\}_{m=0}^\infty$ and $\{Q_m(z)\}_{m=0}^\infty$ belong to $\ell^2(\mathbb{N}_0)$. See [13] and [21] for the details.

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