Filter Banks on Discrete Abelian Groups

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Abstract

In this work we provide polyphase, modulation, and frame theoretical analyses of a filter bank on a discrete abelian group. Thus, multidimensional or cyclic filter banks as well as filter banks for signals in $\ell^2(\mathbb{Z}^d \times \mathbb{Z}_s)$ or $\ell^2(\mathbb{Z}_r \times \mathbb{Z}_s)$ spaces are studied in a unified way. We obtain perfect reconstruction conditions and the corresponding frame bounds.

1 Introduction

The aim of this paper is to provide a filter bank theory for processing signals in the space $\ell^2(G)$ where G denotes a discrete abelian group. Working in this general setting allows the study, at the same time, of unidimensional (setting $\ell^2(G) = \ell^2(\mathbb{Z})$), multidimensional ($\ell^2(G) = \ell^2(\mathbb{Z}^d)$), cyclic filter banks ($\ell^2(G) = \ell^2(\mathbb{Z}_s)$), as well as filter banks processing signals in the spaces $\ell^2(\mathbb{Z}^d \times \mathbb{Z}_s)$, $\ell^2(\mathbb{Z}_r \times \mathbb{Z}_s)$, $\ell^2(\mathbb{Z}_s^d)$ or $\ell^2(\mathbb{Z}_r \times \mathbb{Z}_s \times \mathbb{Z}_v)$.

Thus, the proposed abstract group approach is not just a unified way of dealing with classical discrete groups $\mathbb{Z}, \mathbb{Z}^d$ or $\mathbb{Z}_s$; it also allows us to deal with products of these groups. As a consequence, the availability of an abstract filter bank theory becomes a useful tool to handle these problems at the same time and it could also be applied to some applications in the future. Moreover, the notation is easier, especially compared to this of multidimensional setting.

Filter banks have proven to be very useful in digital signal processing or in wavelet theory (see, for instance, [15, 18, 21] and references therein). The original theory for filter banks for signals in $\ell^2(\mathbb{Z})$ was extended to multidimensional filter banks (see, for instance, [7, 18, 22]), as well as for cyclic filter banks [19, 20]. Associated to an unidimensional analysis filter bank there is a sequence of shifts $\{f_k(\cdot - n)\}_{k=1,\dots,K;\, n\in\mathbb{Z}}$ of K elements f_k in $\ell^2(\mathbb{Z})$. The frame properties of this sequence give information about the corresponding filter bank: its dual frames provide the synthesis filter banks, and its frame bounds provide information on the filter bank stability. A frame analysis of unidimensional filter banks has been carried out in Refs. [2, 9]; the frame analysis on cyclic filter banks was also done in Refs. [4, 5, 6, 10]. All these results can be recovered from our unified approach. It is also worth to mention the related Refs. [1, 13, 14].

In the present paper we carry out polyphase, modulation, and frame theoretical analyses for filter banks associated with signals on a general discrete abelian group. Thus, we obtain necessary and sufficient perfect reconstruction (PR) conditions. We also obtain necessary and sufficient frame conditions, yielding the optimal frame bounds. In particular, we extend the

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frame analysis given in Refs. [2, 4, 6, 9, 10] to multidimensional filter banks. Our study, frame analysis included, is done in the polyphase domain, which has proven to be more useful and efficient in filter banks design. For the sake of completeness, we also include filter banks representation in the modulation domain, as well as the relationship between polyphase and modulation matrices.

The paper is organized as follows: Section II introduces the properties of Fourier transform for discrete abelian group used along the article. Section III deals with filter banks on discrete abelian groups. In Section IV we apply the obtained results in Section III to a wide variety of examples. An appendix section includes the proofs of the involved results.

Finally, let us mention that the reader interested only in multidimensional or cyclic filter banks can go directly to Section 4; the underlying meaning of the terms appearing there can be taken directly from Fig. 1 and Eqs. (2) and (9).

2 Harmonic analysis on Discrete Abelian Groups

Most of the results included here are borrowed from [11]; they can also be found in [12] and [16]; for an introduction to groups theory and symmetries in signal processing, see [17].

2.1 Convolutions

Let G be a abelian discrete countable group with the operation group denoted by $+$. For, $1 \le p < \infty$, $\ell^p(G)$ denotes the set of functions $x : G \mapsto \mathbb{C}$ such that $\|x\|_p^p = \sum_{n\in G} |x(n)|^p < \infty$.

For $x, y \in \ell^2(G)$ we define its convolution as

$$(x * y)(m) = \sum_{n\in G} x(n)y(m-n)\,, \quad m \in G\,.$$

The series above converges absolutely for any $m \in G$ [11, Proposition 2.40]. According to [11, Proposition 2.39], if $y \in \ell^1(G)$ then $x * y \in \ell^2(G)$ and

$$\|x * y\|_2 \le \|x\|_2\, \|y\|_1. \tag{1}$$

2.2 The Fourier transform

Let

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

be the torus. We said that $\xi : G \mapsto \mathbb{T}$ is a character of G if $\xi(n+m) = \xi(n)\xi(m)$ for all $n, m \in G$. We denote $\xi(n) = \langle n, \xi\rangle$. Defining $(\xi + \gamma)(n) = \xi(n)\gamma(n)$, the set of characters $\widehat{G}$ with the operation $+$ is a group, called the dual group of G.

For $x \in \ell^1(G)$ we define its Fourier transform as

$$X(\xi) = \widehat{x}(\xi) = \sum_{n\in G} x(n)\overline{\langle n, \xi\rangle}, \quad \xi \in \widehat{G}.$$

It is known [11, Theorem 4.5] that

$$\widehat{\mathbb{Z}} \cong \mathbb{T}, \quad \text{with } \langle n, z\rangle = z^n,$$
$$\widehat{\mathbb{Z}}_s \cong \mathbb{Z}_s = \mathbb{Z}/s\mathbb{Z}, \quad \text{with } \langle n, m\rangle = W_s^{nm},$$

where $W_s = e^{2\pi i/s}$. Thus, the Fourier transform on $\mathbb{Z}$ is the z-transform,

$$X(z) = \sum_{n\in\mathbb{Z}} x(n) z^{-n}$$

and the Fourier transform on $\mathbb{Z}_s$ is the s-point DFT,

$$X(m) = \sum_{n\in\mathbb{Z}_s} x(n) W_s^{-nm}.$$

There exists a unique measure, called the Haar measure, μ on $\widehat{G}$ satisfying $\mu(\xi + E) = \mu(E)$, for every Borel set $E \subset \widehat{G}$ [11, Section 2.2], and $\mu(\widehat{G}) = 1$. We denote $\int_{\widehat{G}} X(\xi)d\xi = \int_{\widehat{G}} X(\xi)d\mu(\xi)$. If $G = \mathbb{Z}$,

$$\int_{\widehat{G}} X(\xi)d\xi = \int_{\mathbb{T}} X(z)dz = \frac{1}{2\pi}\int_0^{2\pi} X(e^{iw})dw\,,$$

and if $G = \mathbb{Z}_s$,

$$\int_{\widehat{G}} X(\xi)d\xi = \int_{\mathbb{Z}_s} X(n)dn = \frac{1}{s}\sum_{n\in\mathbb{Z}_s} X(n)\,.$$

For $1 \le p < \infty$, $L^p(\widehat{G})$ denotes the set of functions $X : \widehat{G} \mapsto \mathbb{C}$ such that $\|X\|_p^p = \int_{\widehat{G}} |X(\xi)|^p d\xi < \infty$. The Fourier transform on $\ell^1(G)\cap\ell^2(G)$ is an isometry on a dense subspace of $L^2(\widehat{G})$. Thus, it can be extended in a unique manner to a surjective isometry of $\ell^2(G)$ onto $L^2(\widehat{G})$ [11, p. 99].

The Fourier transform satisfies that if $x \in \ell^1(G)$ and $X \in L^1(\widehat{G})$ then

$$x(n) = \int_{\widehat{G}} X(\xi)\langle n, \xi\rangle d\xi, \quad n \in G \qquad \text{(Inversion Theorem [11, Theorem 4.32])}.$$

and if $x \in \ell^2(G)$ and $h \in \ell^1(G)$ then

$$(x * h)^\wedge(\xi) = X(\xi)H(\xi), \quad \text{a.e. } \xi \in \widehat{G} \qquad \text{[12, Theorem 31.27]}.$$

If $G_1, \dots G_d$ are abelian discrete groups then the dual group of the product group is

$$\big(G_1 \times \dots \times G_d\big)^\wedge \cong \widehat{G}_1 \times \dots \times \widehat{G}_n$$

[11, Proposition 4.6] with

$$\big\langle\, (x_1, x_2, \dots, x_d)\,,\, (\xi_1, \xi_2 \dots, \xi_d)\,\big\rangle = \langle x_1, \xi_1\rangle\langle x_2, \xi_2\rangle \cdots \langle x_d, \xi_d\rangle.$$

Hence, $\widehat{\mathbb{Z}^d} \cong \mathbb{T}^d$ and the corresponding Fourier transform is

$$X(z) = \sum_{n\in\mathbb{Z}^d} x(n) z^{-n}, \quad z = (z_1, \dots, z_d) \in \mathbb{T}^d,$$

where $z^n = z_1^{n_1} \dots z_d^{n_d}$. Besides, $\widehat{\mathbb{Z}_s \times \mathbb{Z}_r} \cong \mathbb{Z}_s \times \mathbb{Z}_r$ and the corresponding Fourier transform is

$$X(m) = \sum_{n\in\mathbb{Z}_s\times\mathbb{Z}_r} x(n) W_s^{-n_1 m_1} W_r^{-n_2 m_2}, \quad m \in \mathbb{Z}_s \times \mathbb{Z}_r.$$

2.3 The lattice M

Throughout the article, we assume that M is a subgroup of G with finite index L; we fix a set of representatives of the cosets of M, $\mathcal{L} = \{\ell_0, \ldots, \ell_{L-1}\}$, i.e., the group G can be decomposed as

$$G = (\ell_0 + M) \cup (\ell_1 + M) \cup \ldots \cup (\ell_{L-1} + M)$$

with $(\ell_r + M) \cap (\ell_{r'} + M) = \varnothing$ for $r \neq r'$ ($\mathcal{L}$ is also called a transversal or a section of M).

For instance, for $G = \mathbb{Z}$ and $M = m\mathbb{Z}$ we can take $\mathcal{L} = \{0, 1, \ldots, m-1\}$ since

$$\mathbb{Z} = m\mathbb{Z} \cup (1 + m\mathbb{Z}) \cup \cdots \cup (m - 1 + m\mathbb{Z}).$$

We denote by $*_M$ the convolution with respect to the subgroup M, i.e.,

$$\big(c *_M d\big)(n) = \sum_{m \in M} c(m)\, d(n-m)\,, \quad n \in M\,.$$

2.4 The M-Fourier transform

The annihilator of M is a subgroup of $\widehat{G}$ defined by $M^\perp = \big\{\xi \in \widehat{G} : \langle m, \xi \rangle = 1 \text{ for all } m \in M\big\}$. $M^\perp$ has L elements [11, Section 4.3]. We have that

$$\widehat{M} \cong \widehat{G}/M^\perp \quad \text{with} \quad \big\langle m, \xi + M^\perp \big\rangle = \langle m, \xi \rangle \qquad \text{[11, Theorem 4.39]}\,.$$

We denote by $C(\xi + M^\perp)$ or $\widehat{c}(\xi + M^\perp)$ the Fourier transform of a function c in the group M, i.e.

$$C(\xi + M^\perp) = \sum_{m \in M} c(m) \overline{\langle m, \xi + M^\perp \rangle} = \sum_{m \in M} c(m) \overline{\langle m, \xi \rangle}.$$

The polyphase analysis in Section 3 relies on this transform. Thus, in many occasions to simplify the notation we denote the characters of $\widehat{M}$ by γ instead of $\xi + M^\perp$. To prevent confusions, we call $C(\gamma)$ the M-Fourier transform of c.

3 Filter banks

For a complex function x with domain in G, we denote its *restriction* to the subgroup M as

$$(\downarrow_M x)(m) = x(m), \quad m \in M.$$

For a complex function x with domain M we define the *expander* to G as

$$(\uparrow_M x)(n) = \begin{cases} x(n), & n \in M \\ 0, & n \notin M. \end{cases}$$

Notice that when $G = \mathbb{Z}$ and $M = m\mathbb{Z}$, the functions $\downarrow_M x$ and $\uparrow_M x$ are similar to the decimation and the expander defined as $(\downarrow_m x)(n) = x(nm)$, $(\uparrow_m x)(n) = 1_M(n) x(n/m)$, but are not the same. Nevertheless, $\uparrow_m \downarrow_m x = \uparrow_M \downarrow_M x$.

We consider the filter bank represented in Fig. 1, i.e.,

$$c_k = \big(\downarrow_M (x * h_k)\big) \quad \text{and} \quad y = \sum_{k \in \mathcal{K}} (\uparrow_M c_k) * g_k\,, \quad k \in \mathcal{K} := \{1, 2, \ldots, K\}\,,$$

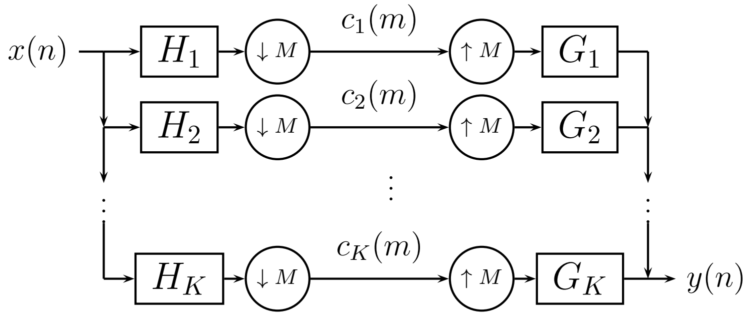

Figure 1: K-Channel Filter Bank

or equivalently

$$y(n) = \sum_{k\in\mathcal{K}} \sum_{m\in M} (x * h_k)(m)\, g_k(n-m), \quad n \in G\,. \tag{2}$$

Throughout the article, we assume that the filters $h_k, g_k \in \ell^1(G)$, for $k \in \mathcal{K}$. This assumption guarantees the convergence of series involved in (2) for any $x \in \ell^2(G)$. Indeed, from (1) we have that $x * h_k \in \ell^2(G)$ and then, from (1) again, we have that the series in (2) converges absolutely, and $y \in \ell^2(G)$.

3.1 Polyphase analysis

For $k \in \mathcal{K}$ and $\ell \in \mathcal{L}$ we define the polyphase components

$$\begin{aligned} &x_\ell(m) = x(m+\ell), \quad y_\ell(m) = y(m+\ell), \\ &h_{k,\ell}(m) = h_k(m-\ell), \quad g_{\ell,k}(m) = g_k(m+\ell), \quad m \in M, \end{aligned}$$

and we denote their M-Fourier transforms by $X_\ell(\gamma), Y_\ell(\gamma), H_{k,\ell}(\gamma), G_{\ell,k}(\gamma)$ respectively.

For any $x \in \ell^2(G)$ and $m \in M$, we have

$$\begin{aligned} c_k(m) = (x * h_k)(m) &= \sum_{n\in G} x(n)\, h_k(m-n) = \sum_{\ell\in\mathcal{L}} \sum_{n\in M} x(n+\ell)\, h_k(m-n-\ell) \\ &= \sum_{\ell\in\mathcal{L}} \sum_{n\in M} x_\ell(n)\, h_{k,\ell}(m-n) = \sum_{\ell\in\mathcal{L}} (x_\ell *_M h_{k,\ell})(m). \end{aligned}$$

All the series above converge absolutely since we have assumed that $h_k \in \ell^1(G)$. Moreover, $\mathrm{c}_k \in \ell^2(M)$ since $x * h_k \in \ell^2(G)$. Taking the M-Fourier transform, we obtain

$$C_k(\gamma) = \sum_{\ell\in\mathcal{L}} H_{k,\ell}(\gamma) X_\ell(\gamma). \tag{3}$$

Thus, we have the matrix expression

$$\mathbf{C}(\gamma) = \mathbf{H}(\gamma)\mathbf{X}(\gamma) \quad \text{a.e. } \gamma \in \widehat{M}, \tag{4}$$

where

$$\mathbf{C} = [C_k]_{k\in\mathcal{K}}\,, \quad \mathbf{X} = [X_\ell]_{\ell\in\mathcal{L}}\,, \quad \mathbf{H} = \big[H_{k,\ell}\big]_{k\in\mathcal{K},\,\ell\in\mathcal{L}}\,. \tag{5}$$

Above, $\mathbf{C}$ and $\mathbf{X}$ denote column vectors, i.e., $\mathbf{C} = [C_1, \ldots, C_K]^\top$ and $\mathbf{X} = [X_{\ell_0}, \ldots, X_{\ell_{L-1}}]^\top$.

The polyphase components of the output y are

$$\begin{aligned} y_\ell(m) = y(m+\ell) &= \sum_{k\in\mathcal{K}}\sum_{n\in M} c_k(n) g_k(m+\ell-n) \\ &= \sum_{k\in\mathcal{K}}\sum_{n\in M} c_k(n)\, g_{\ell,k}(m-n) = \sum_{k\in\mathcal{K}} (c_k *_{_M} g_{\ell,k})(m). \end{aligned}$$

Taking the M-Fourier transform, we obtain

$$Y_\ell(\gamma) = \sum_{k\in\mathcal{K}} G_{\ell,k}(\gamma) C_k(\gamma) \qquad \text{a.e. } \gamma \in \widehat{M},$$

which can be written as

$$\mathbf{Y}(\gamma) = \mathbf{G}(\gamma)\mathbf{C}(\gamma) \quad \text{a.e. } \gamma \in \widehat{M}, \tag{6}$$

where

$$\mathbf{Y} = [Y_\ell]_{\ell\in\mathcal{L}}\,, \quad \mathbf{G} = \big[G_{\ell,k}\big]_{\ell\in\mathcal{L},\, k\in\mathcal{K}}\,. \tag{7}$$

Thus, from (4) and (6), we have

$$\mathbf{Y}(\gamma) \;=\; \mathbf{G}(\gamma)\,\mathbf{H}(\gamma)\,\mathbf{X}(\gamma) \quad \text{a.e. } \gamma \in \widehat{M}. \tag{8}$$

On the other hand, we consider in the following Proposition a generalization to discrete groups of the polyphase transform (see [3]).

Proposition 1 *The transformation $\mathcal{P}: \ell^2(G) \to L^2(\widehat{M}) \times \cdots \times L^2(\widehat{M})$ (L times) defined by $\mathcal{P}(x) = \mathbf{X} = [X_\ell]_{\ell\in\mathcal{L}}$, is a surjective isometry.*

From (8) and by using Proposition 1, we easily deduce

Theorem 1 *The filter bank in Fig. 1 satisfies $y = x$ for all $x \in \ell^2(G)$ (Perfect Reconstruction) if and only if $\mathbf{G}(\gamma)\,\mathbf{H}(\gamma) = \mathbf{I}_L$ for all $\gamma \in \widehat{M}$.*

Since we have assume that the filters h_k, g_k are in $\ell^1(G)$, the entries of matrices $\mathbf{G}(\gamma)$ and $\mathbf{H}(\gamma)$ are continuous (see proof of Theorem 1). That explains why in the PR characterization appears for all $\gamma \in \widehat{M}$ instead of a.e. $\gamma \in \widehat{M}$.

It is easy to check that between the polyphase transform and the Fourier transform, there exists the relationship

$$X(\xi) = \mathbf{p}^\top(\xi)\mathbf{X}(\xi + M^\perp),\; \xi \in \widehat{G}, \quad \text{where} \quad \mathbf{p}(\xi) = \big[\,\overline{\langle \ell, \xi\rangle}\,\big]_{\ell\in\mathcal{L}}\,.$$

Then, from (8) the Fourier transform of the output y is expressed as

$$Y(\xi) = \mathbf{p}^\top(\xi)\,\mathbf{G}(\xi + M^\perp)\mathbf{H}(\xi + M^\perp)\;\mathbf{X}(\xi + M^\perp) \quad \text{a.e. } \xi \in \widehat{G}.$$

3.2 Frame analysis

We denote $(T_m f)(n) = f(n-m)$ and the involution of the analysis filter by $f_k(n) = \overline{h_k(-n)}$. Then

$$c_k(m) = (x * h_k)(m) = \sum_{n \in G} x(n)\, h_k(m-n) = \langle x, T_m f_k \rangle_{\ell^2(G)}, \quad m \in G,\, k \in \mathcal{K}\,, \tag{9}$$

and expansion (2) can be written as

$$y = \sum_{k \in \mathcal{K}} \sum_{m \in M} \langle x, T_m f_k \rangle_{\ell^2(G)}\ T_m g_k.$$

Thus, the filter bank in Fig. 1 is related to the sequences $\{T_m f_k\}_{k \in \mathcal{K},\, m \in M}$ and $\{T_m g_k\}_{k \in \mathcal{K},\, m \in M}$. The following theorems provide frame properties of these sequences. In Ref. [8] the reader can find the main properties of frames and Riesz bases. Recall, that we have assumed that $h_k \in \ell^1(G)$ which is equivalent to assume that $f_k \in \ell^1(G)$.

Theorem 2 *The sequences $\{T_m f_k\}_{k \in \mathcal{K},\, m \in M}$ and $\{T_m g_k\}_{k \in \mathcal{K},\, m \in M}$ are dual frames for $\ell^2(G)$ if and only if $\mathbf{G}(\gamma)\,\mathbf{H}(\gamma) = \mathbf{I}_L$ for all $\gamma \in \widehat{M}$.*

Let $\mathbf{H}^*$ denote the transpose conjugate of the matrix $\mathbf{H}$.

Theorem 3 *The sequence $\{T_m f_k\}_{k \in \mathcal{K},\, m \in M}$ is a frame for $\ell^2(G)$ if and only if Rank $\mathbf{H}(\gamma) = L$ for all $\gamma \in \widehat{M}$. In this case, the optimal frame bounds are*

$$A = Min_{\gamma \in \widehat{M}} \lambda_{min}(\gamma) \quad and \quad B = Max_{\gamma \in \widehat{M}} \lambda_{max}(\gamma)$$

where $\lambda_{min}(\gamma)$ and $\lambda_{max}(\gamma)$ are the smallest and the largest eigenvalue of the matrix $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)$, and the canonical dual frame is $\{T_m \widetilde{f}_k\}_{k \in \mathcal{K},\, m \in M}$ where $\mathcal{P}\widetilde{f}_k = (\mathbf{H}^*\mathbf{H})^{-1}\mathcal{P} f_k,\ k \in \mathcal{K}$.

The synthesis matrix $\mathbf{G}(\gamma)$ corresponding to the canonical dual frame is $[\mathbf{H}(\gamma)^*\mathbf{H}(\gamma)]^{-1}\mathbf{H}^*(\gamma)$, which is the Moore-Penrose pseudoinverse $\mathbf{H}^\dagger(\gamma)$ of the analysis matrix $\mathbf{H}(\gamma)$.

Analogously, the optimal frame bounds of the dual frame $\{T_m g_k\}_{k \in \mathcal{K},\, m \in M}$ are given by $A_g = \text{Min}_{\gamma \in \widehat{M}} \mu_{\min}(\gamma)$ and $B_g = \text{Max}_{\gamma \in \widehat{M}} \mu_{\max}(\gamma)$, where $\mu_{\min}(\gamma)$ and $\mu_{\max}(\gamma)$ are the smallest and the largest eigenvalue of the matrix $\mathbf{G}(\gamma)\mathbf{G}^*(\gamma)$. For the canonical dual frame $g_k = \widetilde{f}_k$, we have that $A_g = 1/B$ and $B_g = 1/A$ [8, Lemma 5.1.1].

The frame bounds give information about the stability of the filter bank. Notice that, by its definition, the optimal frames bounds of $\{T_m f_k\}_{k \in \mathcal{K}, m \in M}$ are the tightest numbers $0 < A \le B$ such that

$$A\|x\|_2^2 \le \sum_{k \in \mathcal{K}} \sum_{m \in M} |c_k(m)|^2 = \sum_{k \in \mathcal{K}} \sum_{m \in M} |(x * h_k)(m)|^2 \le B\|x\|_2^2\,, \quad x \in \ell^2(G)\,.$$

Thus B gives a measure of how an error in the input x of the analysis filter bank affects to subband signals c_k. For the synthesis, we have that B_g is the tightest number such that [8, Theorem 3.2.3]

$$\|y\|^2 = \Big\| \sum_{k \in \mathcal{K}} \sum_{m \in M} c_k(m) g_k(\cdot - m) \Big\|^2 \le B_g \sum_{k \in \mathcal{K}} \sum_{m \in M} |c_k(m)|^2\,.$$

Thus B_g gives a measure of how an error in the subband signals c_k affects to the recovered signal y. The smallest possible value for B_g is $1/A$, which correspond to take the canonical dual frame. One can find a sensitivity analysis based on frame bounds in Ref. [2]; see also [8, p. 118].

Having in mind that $A = B$ if and only if $\mathbf{H}^*(\gamma)\,\mathbf{H}(\gamma) = A\,\mathbf{I}_L$ for all $\gamma \in \widehat{M}$, we deduce

Corollary 1 *The sequence $\{T_m f_k\}_{k\in\mathcal{K},\, m\in M}$ is a tight frame for $\ell^2(G)$ if and only if there exists $A > 0$ such that $\mathbf{H}^*(\gamma)\,\mathbf{H}(\gamma) = A\,\mathbf{I}_L$ for all $\gamma \in \widehat{M}$. In this case, the frame bound is A.*

For maximally decimated filter banks, we have the following result

Theorem 4 *Assume that $L = K$. The sequence $\{T_m f_k\}_{k\in\mathcal{K},\, m\in M}$ is Riesz basis for $\ell^2(G)$ if and only if $\det \mathbf{H}(\gamma) \neq 0$ for all $\gamma \in \widehat{M}$. In this case, the optimal Riesz bounds are the constants A and B defined in Theorem 3.*

3.3 Modulation Analysis

Recall that $M^\perp$, the annihilator of M, is a subgroup of $\widehat{G}$ with L elements.

Proposition 2 *For any $x \in \ell^2(G)$, the M-Fourier transform of $(\downarrow_M x)$ is*

$$(\downarrow_M x)^\wedge(\xi + M^\perp) = \frac{1}{L} \sum_{\eta\in M^\perp} X(\xi + \eta), \quad \text{a.e. } \xi \in \widehat{G}.$$

Hence, the M-Fourier transform of $c_k = \downarrow_M (x * h_k)$ is $C_k(\xi + M^\perp) = \frac{1}{L}\sum_{\eta\in M^\perp} X(\xi+\eta) H_k(\xi+\eta)$. Hence, denoting $\mathbf{C} = \big[C_k\big]_{k\in\mathcal{K}}$, we have

$$\mathbf{C}(\xi + M^\perp)^\top = \frac{1}{L}\mathbf{H}_{\mathrm{mod}}(\xi)\,\mathbf{x}_{\mathrm{mod}}(\xi), \tag{10}$$

where

$$\mathbf{x}_{\mathrm{mod}}(\xi) = \big[X(\xi+\eta)\big]_{\eta\in M^\perp}, \quad \mathbf{H}_{\mathrm{mod}}(\xi) = \big[H_k(\xi+\eta)\big]_{k\in\mathcal{K},\,\eta\in M^\perp}. \tag{11}$$

For any $c \in \ell^2(M)$, the Fourier transform of $(\uparrow_M c)$ is $M^\perp$-periodic; specifically, for any $\eta \in M^\perp$,

$$(\uparrow_M c)^\wedge(\xi+\eta) = \sum_{n\in G} [\uparrow_M c](n)\overline{\langle n, \xi+\eta\rangle} = \sum_{m\in M} c(m)\overline{\langle m, \xi + M^\perp\rangle} = C(\xi + M^\perp).$$

Then the Fourier transform of

$$y(n) = \sum_{k\in\mathcal{K}}\sum_{m\in M} c_k(m) g_k(n-m) = \sum_{k\in\mathcal{K}}\sum_{l\in G} (\uparrow_M c_k)(l) g_k(n-l) = \sum_{k\in\mathcal{K}} \big((\uparrow_M c_k) * g_k\big)(n)$$

is $Y(\xi) = \sum_{k\in\mathcal{K}} C_k(\xi + M^\perp)^\top G_k(\xi)$. Then, from (10), the Fourier transform of the output y to the filter bank in Fig. 1 is

$$Y(\xi) = \frac{1}{L}\big[G_1(\xi), G_2(\xi), \cdots, G_K(\xi)\big]\,\mathbf{H}_{\mathrm{mod}}(\xi)\,\mathbf{x}_{\mathrm{mod}}(\xi), \quad \xi \in \widehat{G}.$$

This modulation representation of the output to the filter bank was obtained in [1].

Proposition 3 *The $K \times L$ matrices $\mathbf{H}_{mod}(\xi)$ and $\mathbf{H}(\xi)$, defined in (11) and (5) respectively, are related by*

$$\mathbf{H}_{mod}(\xi) = \mathbf{H}(\xi + M^{\perp})\, \mathbf{D}(\xi)\, \mathbf{W} \quad \text{for all } \xi \in \widehat{G},$$

where $\mathbf{W} = \big[\langle \ell_i, \eta\rangle\big]_{i=0,1,\dots,L-1,\, \eta\in M^{\perp}}$ *and* $\mathbf{D}(\xi) = diag\big(\langle \ell_0, \xi\rangle, \langle \ell_1, \xi\rangle, \dots, \langle \ell_{L-1}, \xi\rangle\big)$.

It is worth note that $\mathbf{W}\mathbf{W}^* = L\,\mathbf{I}_L$ (see (12)) and then $\mathbf{H}(\xi+M^{\top}) = (1/L)\mathbf{H}_{\text{mod}}(\xi)\mathbf{W}^*\overline{\mathbf{D}(\xi)}$.

4 Examples

In this section we particularize the above general theory on filter banks in four different contexts:

4.1 The case $G = \mathbb{Z}^d$ and $M = \{\mathbf{M}n : n \in \mathbb{Z}^d\}$

Let $\mathbf{M}$ be a $d \times d$ matrix with integer entries and positive determinant. For the case $G = \mathbb{Z}^d$ and $M = \{\mathbf{M}n : n \in \mathbb{Z}^d\}$, we could take $\mathcal{L} = -\mathcal{N}(\mathbf{M})$ where $\mathcal{N}(\mathbf{M}) = \mathbf{M}[0,1)^n \cap \mathbb{Z}^d$ which has $\det \mathbf{M}$ elements (see [18]), $\mathcal{L} = \mathcal{N}(\mathbf{M}^{\top})$ (see [7]), or even other possibilities (see [22]). In the following corollary we write some of the results of Section 3 in terms of the polyphase matrices usually used in this context [7, 18, 22]:

$$\mathbf{E}(z) = \Big[\sum_{n\in\mathbb{Z}^d} h_k(\mathbf{M}n - \ell) z^{-n}\Big]_{k\in\mathcal{K},\, \ell\in\mathcal{L}}, \quad \mathbf{R}(z) = \Big[\sum_{n\in\mathbb{Z}^d} g_k(\mathbf{M}n + \ell) z^{-n}\Big]_{\ell\in\mathcal{L}, k\in\mathcal{K}}, \quad z \in \mathbb{T}^d.$$

Corollary 2 *Let $\lambda_{min}(z)$ and $\lambda_{max}(z)$ be the smallest and the largest eigenvalue of the* $\det \mathbf{M} \times \det \mathbf{M}$ *matrix $\mathbf{E}^*(z)\mathbf{E}(z)$. Then, the sequence $\{T_m f_k\}_{m\in M,\, k\in\mathcal{K}}$ is a frame for $\ell^2(\mathbb{Z}^d)$ if and only if Rank $\mathbf{E}(z) = \det \mathbf{M}$ for all $z \in \mathbb{T}^d$. In this case, the optimal frame bounds are*

$$A = Min_{z\in\mathbb{T}^d} \lambda_{min}(z) \quad and \quad B = Max_{z\in\mathbb{T}^d} \lambda_{max}(z).$$

It is tight frame if and only if $\mathbf{E}^*(z)\mathbf{E}(z) = A\,\mathbf{I}_{\det \mathbf{M}}$, $z \in \mathbb{T}^d$. The sequences $\{T_m f_k\}_{m\in M,\, k\in\mathcal{K}}$ and $\{T_m g_k\}_{m\in M,\, k\in\mathcal{K}}$ are dual frames if and only if $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}_{\det \mathbf{M}}$ for all $z \in \mathbb{T}^d$. Whenever $\det \mathbf{M} = K$*, the sequence $\{T_m f_k\}_{k\in\mathcal{K},\, m\in M}$ is a Riesz basis for $\ell^2(\mathbb{Z}^d)$ if and only if* $\det \mathbf{E}(z) \neq 0$ *for all $z \in \mathbb{Z}^d$. In this case, the optimal Riesz bounds are A and B.*

This corollary generalizes, to the multidimensional case, the results obtained in [2] and [9] in the unidimensional case.

4.2 The case $G = \mathbb{Z}_s$ and $M = L\,\mathbb{Z}_s$

Assume that $s = L\,N$, with $L, N \in \mathbb{N}$. Whenever $G = \mathbb{Z}_s$ and $M = L\mathbb{Z}_s$ we could take $\mathcal{L} = \{0, -1, \dots, -(L-1)\}$ (mod s) (see [19, 20]). In the following corollary we write the results in terms of the polyphase matrices defined in [19, 20]:

$$\mathbf{E}(n) = \Big[\sum_{m=0}^{N-1} h_k(Lm - \ell) W_N^{-mn}\Big]_{k\in\mathcal{K},\, \ell\in\mathcal{L}}, \quad \mathbf{R}(n) = \Big[\sum_{m=0}^{N-1} g_k(Lm + \ell) W_N^{-mn}\Big]_{\ell\in\mathcal{L}, k\in\mathcal{K}}, \quad n \in \mathbb{Z}_N.$$

Note that the N-point DFT appears since $\widehat{M} \cong \mathbb{Z}_s/M^{\perp} \cong \mathbb{Z}_s/(N\mathbb{Z}_s) \cong \mathbb{Z}_N$.

Corollary 3 *Let $\lambda_{min}(n)$ and $\lambda_{max}(n)$ be the smallest and the largest eigenvalue of the $L \times L$ matrix $\mathbf{E}^*(n)\mathbf{E}(n)$. The sequence $\{T_m f_k\}_{m\in M,\, k\in\mathcal{K}}$ is a frame for $\ell^2(\mathbb{Z}_s)$ if and only if Rank $\mathbf{E}(n) = L$ for all $n \in \mathbb{Z}_N$. In this case, the optimal frame bounds are*

$$A = Min_{n\in\mathbb{Z}_N}\lambda_{min}(n) \quad \text{and} \quad B = Max_{n\in\mathbb{Z}_N}\lambda_{max}(n).$$

It is tight frame if and only if $\mathbf{E}^*(n)\mathbf{E}(n) = A\mathbf{I}_L$ for all $n \in \mathbb{Z}_N$. The sequences $\{T_m f_k\}_{m\in M,\, k\in\mathcal{K}}$ and $\{T_m g_k\}_{m\in M,\, k\in\mathcal{K}}$ are dual frames if and only if $\mathbf{R}(n)\mathbf{E}(n) = \mathbf{I}_L$ for all $n \in \mathbb{Z}_N$. Whenever $L = K$, the sequence $\{T_m f_k\}_{k\in\mathcal{K},\, m\in M}$ is a Riesz basis for $\ell^2(\mathbb{Z}_s)$ if and only if $\det \mathbf{E}(n) \neq 0$ for all $n \in \mathbb{Z}_N$. In this case, the optimal Riesz bounds are A and B.

Some of these results can be found in [4, 5, 6, 10].

4.3 The case $G = \mathbb{Z}^d \times \mathbb{Z}_s$ and $M = \mathbf{M}\,\mathbb{Z}^d \times L\,\mathbb{Z}_s$

Consider now the tensor product of previous two examples, i.e., $G = \mathbb{Z}^d \times \mathbb{Z}_s$ and $M = \mathbf{M}\mathbb{Z}^d \times L\mathbb{Z}_s$, where $\mathbf{M}$ is a matrix with integer entries, $\det \mathbf{M} > 0$ and $s = LN$. We could take $\mathcal{L} = \mathcal{N}(\mathbf{M}) \times \{0, 1, \ldots, (L-1)\}$. Set the matrices

$$\mathbf{E}(z,n) = \Big[\sum_{m=0}^{N-1}\sum_{u\in\mathbb{Z}^d} h_k\big([\mathbf{M}u, Lm] - \ell\big) z^{-u} W_N^{-mn}\Big]_{k\in\mathcal{K},\, \ell\in\mathcal{L}}$$

$$\mathbf{R}(z,n) = \Big[\sum_{m=0}^{N-1}\sum_{u\in\mathbb{Z}^d} g_k\big([\mathbf{M}u, Lm] + \ell\big) z^{-u} W_N^{-mn}\Big]_{\ell\in\mathcal{L},\, k\in\mathcal{K}}$$

Corollary 4 *The filter bank described in Fig. 1 satisfies the perfect reconstruction property if and only if $\mathbf{R}(z,n)\mathbf{E}(z,n) = \mathbf{I}_{L\det\mathbf{M}}$ for all $z \in \mathbb{T}^d$ and $n \in \mathbb{Z}_N$. Let $\lambda_{min}(z,n)$ and $\lambda_{max}(z,n)$ be the smallest and the largest eigenvalue of the $L\det\mathbf{M} \times L\det\mathbf{M}$ matrix $\mathbf{E}^*(z,n)\mathbf{E}(z,n)$. The sequence $\{T_m f_k\}_{m\in M,\, k\in\mathcal{K}}$ is a frame for $\ell^2(\mathbb{Z}^d \times \mathbb{Z}_s)$ if and only if Rank $\mathbf{E}(z,n) = L\det\mathbf{M}$ for all $z \in \mathbb{T}^d$ and $n \in \mathbb{Z}_N$. In this case, the optimal frame bounds are*

$$A = Min_{z\in\mathbb{T}^d, n\in\mathbb{Z}_N}\lambda_{min}(z,n) \quad \text{and} \quad B = Max_{z\in\mathbb{T}^d, n\in\mathbb{Z}_N}\lambda_{max}(z,n).$$

Whenever $K = L\det\mathbf{M}$, the sequence $\{T_m f_k\}_{k\in\mathcal{K},\, m\in M}$ is a Riesz basis for $L^2(\mathbb{Z}^d \times \mathbb{Z}_s)$ if and only if $\det \mathbf{E}(z,n) \neq 0$ for all $z \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_N$. In this case, the optimal Riesz bounds are A and B.

4.4 The case $G = \mathbb{Z}_{2P} \times \mathbb{Z}_{2Q}$ and M the Quincunx

The Quincunx M consists of the elements (n, m) in $\mathbb{Z}_{2P} \times \mathbb{Z}_{2Q}$ such that n and m are both even or both odd; it is a subgroup of $\mathbb{Z}_{2P} \times \mathbb{Z}_{2Q}$. In this case we could take $\mathcal{L} = \{(0,0), (1,0)\}$. Consider the $[P, Q]$-Points DFT transform

$$\big[\mathrm{DFT}\, x\big](n,m) = \sum_{u=0}^{P-1}\sum_{v=0}^{Q-1} x(u,v) W_P^{-un}\, W_Q^{-vm}\,,$$

and the transform

$$[\Lambda x](n,m) = \big[\mathrm{DFT}\, x_0\big](n,m) + W_{2P}^{-n}\, W_{2Q}^{-m}\big[\mathrm{DFT}\, x_1\big](n,m),$$

where $x_0(n,m) = x(2n,2m)$ and $x_1(n,m) = x(2n+1,2m+1)$. Set the matrices

$$\mathbf{E}(n,m) = \Big[\Lambda h_{k,\ell}(n,m)\Big]_{k\in\mathcal{K},\,\ell\in\mathcal{L}} \quad \text{and} \quad \mathbf{R}(n,m) = \Big[\Lambda g_{\ell,k}(n,m)\Big]_{\ell\in\mathcal{L},k\in\mathcal{K}}.$$

Corollary 5 *The filter bank described in Fig. 1 has the perfect reconstruction property if and only if* $\mathbf{R}(n,m)\mathbf{E}(n,m) = \mathbf{I}_2$ *for all* $(n,m) \in \mathbb{Z}_{2P}\times\mathbb{Z}_Q$.

Note that $\widehat{M} \cong (\mathbb{Z}_{2P}\times\mathbb{Z}_{2Q})/M^\perp \cong (\mathbb{Z}_{2P}\times\mathbb{Z}_{2Q})/\{(0,0),(P,Q)\} \cong \mathbb{Z}_{2P}\times\mathbb{Z}_Q$.

Appendix

In this appendix we include the proofs of the results in Sections III and IV.

Proof of Proposition 1: For $x, y \in \ell^2(G)$ we have

$$\begin{aligned}\langle x,y\rangle_{\ell^2(G)} &= \sum_{\ell\in\mathcal{L}}\sum_{m\in M} x(m+\ell)\overline{y(m+\ell)} = \sum_{\ell\in\mathcal{L}}\sum_{m\in M} x_\ell(m)\overline{y_\ell(m)} = \sum_{\ell\in\mathcal{L}}\langle x_\ell, y_\ell\rangle_{\ell^2(M)} \\ &= \sum_{\ell\in\mathcal{L}}\langle X_\ell, Y_\ell\rangle_{L^2(\widehat{M})} = \langle \mathbf{X},\mathbf{Y}\rangle_{L^2(\widehat{M})\times\cdots\times L^2(\widehat{M})}.\end{aligned}$$

Then $\mathcal{P}$ is a isometry. Besides, for any $\mathbf{X} \in L^2(\widehat{M})\times\cdots\times L^2(\widehat{M})$, since the M-Fourier transform is a surjective isometric between $\ell^2(M)$ and $L^2(\widehat{M})$, there exists a function x such that its polyphase components $[X_\ell]_{\ell\in\mathcal{L}} = \mathbf{X}$. Hence, $\mathcal{P}$ is surjective. □

Proof of Theorem 1: Having in mind Proposition 1 and (8) the filter bank satisfies the Perfect Reconstruction property if and only if $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{M}$. Since we have assume that h_k and g_k belong to $\ell^1(G)$, their polyphase components, $h_{k,\ell}$ and $g_{\ell,k}$ belong to $L^1(M)$. Then their M-Fourier transform are continuous [11, Proposition 4.13]. Hence, the entries of $\mathbf{G}(\gamma)\mathbf{H}(\gamma)$ are continuous. Therefore, $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{M}$ if and only if $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ for all $\gamma \in \widehat{M}$. □

Proof of Theorem 2: By using (9) and (1), we obtain that for each $k \in \mathcal{K}$,

$$\begin{aligned}&\sum_{m\in M}|\langle x, T_m f_k\rangle|^2 \le \sum_{n\in G}|\langle x, T_n f_k\rangle|^2 = \sum_{n\in G}|x * h_k(n)|^2 \\ &= \|x*h_k\|_2^2 \le \|x\|_2^2\|h_k\|_1^2\,, \text{ for all } x\in\ell^2(G)\,.\end{aligned}$$

Hence, $\big\{T_m f_k\big\}_{k\in\mathcal{K},m\in M}$ is a Bessel sequence for $\ell^2(G)$. Analogously one proves that the sequence $\big\{T_m g_k\big\}_{k\in\mathcal{K},m\in M}$ is a Bessel sequence for $\ell^2(G)$. Having in mind Lemma 5.6.2 in [8], the theorem is now a consequence of Theorem 1. □

Proof of Theorem 3: First, notice that $\lambda_{\min}(\gamma)$ and $\lambda_{\max}(\gamma)$ have a minimum and a maximum value over $\widehat{M}$. Indeed, since $h_k \in \ell^1(G)$, the entries of $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)$ are continuous functions [11, Proposition 4.13] and then $\lambda_{\min}(\gamma)$ and $\lambda_{\max}(\gamma)$ are real continuous functions (see [23]). Besides, since M is discrete, $\widehat{M}$ is compact [11, Proposition 4.4].

In the proof of Theorem 2 we have showed that $\{T_m f_k\}_{k\in\mathcal{K},m\in M}$ is a Bessel sequence. Now, we obtain a representation in the polyphase domain for its frame operator

$$Sx = \sum_{k\in\mathcal{K}} \sum_{m\in M} \langle x, T_m f_k\rangle \, T_m f_k, \quad x \in \ell^2(G)\,.$$

Indeed, when $g_k(n) = f_k(n) = \overline{h_k(-n)}$, then $\mathbf{G} = \mathbf{H}^*$, and the representation (8) reads

$$\big[\mathcal{P}Sx\big](\gamma) = \mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\mathbf{X}(\gamma).$$

By using Proposition 1, we get

$$\begin{aligned}\sum_{k\in\mathcal{K}} \sum_{m\in M} \big|\langle x, T_m f_k\rangle\big|^2 &= \langle Sx, x\rangle_{\ell^2(G)} = \langle \mathcal{P}Sx, \mathcal{P}x\rangle_{L^2(\widehat{M})\times\ldots\times L^2(\widehat{M})} \\ &= \int_{\widehat{M}} \mathbf{X}^*(\gamma)\big[\mathcal{P}Sx\big](\gamma)d\gamma = \int_{\widehat{M}} \mathbf{X}^*(\gamma)\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\mathbf{X}(\gamma)\, d\gamma.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k\in\mathcal{K}} \sum_{m\in M} \big|\langle x, T_m f_k\rangle\big|^2 &\ge \int_{\widehat{M}} \lambda_{\min}(\gamma)|\mathbf{X}(\gamma)|^2 d\gamma \ge A \int_{\widehat{M}} |\mathbf{X}(\gamma)|^2 d\gamma \\ &= A\|\mathbf{X}\|^2_{L^2(\widehat{M})\times\ldots\times L^2(\widehat{M})} = A\|x\|_2^2.\end{aligned}$$

Let $J > A$. Then, there exist a subset $\Omega \subset \widehat{M}$ with positive measure such that $\lambda_{\min}(\gamma) < J$ for $\gamma \in \Omega$. Let $\mathbf{X}(\gamma)$ equal to 0 when $\gamma \notin \Omega$ and equal to unitary eigenvector of $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)$ corresponding to $\lambda_{\min}(\gamma)$ when $\gamma \in \Omega$. Notice that $\mathbf{X} \in L^2(\widehat{M})\times\ldots\times L^2(\widehat{M})$ since $\|\mathbb{F}\|^2_{L^2(\widehat{M})\times\ldots\times L^2(\widehat{M})} =$ measure$(\Omega) \le 1$. The function $x = \mathcal{P}^{-1}\mathbf{X}$ satisfies

$$\sum_{k\in\mathcal{K}} \sum_{m\in M} \big|\langle x, T_m f_k\rangle\big|^2 = \int_{\Omega} \mathbf{X}^*(\gamma)\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\mathbf{X}(\gamma)d\gamma = \int_{\Omega} \lambda_{\min}(\gamma)\mathbf{X}^*(\gamma)\mathbf{X}(\gamma)d\gamma \le J\|x\|^2$$

Therefore, the sequence $\{T_m f_k\}_{k\in\mathcal{K},m\in M}$ is a frame for $\ell^2(G)$ if and only if $A > 0$, and in this case the lower optimal bound is A. In the same way it is proved that the optimal Bessel bound is B. Since $\lambda_{\min}(\gamma)$ is a continuous function, $A > 0$ if and only if $\lambda_{\min}(\gamma) > 0$ for all $\gamma \in \widehat{M}$ which is equivalent to be the rank of $\mathbf{H}(\gamma)$ equal to L for all $\gamma \in \widehat{M}$.

It is easy to check that $ST_m x = T_m Sx$. The canonical dual frame is given by (see [8, Lemma 5.1.1])

$$S^{-1}T_m f_k = T_m S^{-1} f_k = T_m \mathcal{P}^{-1}\mathcal{P}S^{-1} f_k = T_m \mathcal{P}^{-1}(\mathbf{H}^*\mathbf{H})^{-1}\mathcal{P} f_k. \qquad \square$$

Proof of Theorem 4: If the sequence $\{T_m f_k\}_{k\in\mathcal{K},m\in M}$ is a Riesz basis then it is a frame. Then, by Theorem 3, Rank $\mathbf{H}(\gamma) = L = K$, and thus $\det \mathbf{H}(\gamma) \neq 0$, for all $\gamma \in \widehat{M}$.

To prove the reciprocal, assume that $\det \mathbf{H}(\gamma) \neq 0$, for all $\gamma \in \widehat{M}$. Then Rank $\mathbf{H}(\gamma) = L$ and, from Theorem 3, $\{T_m f_k\}_{m\in M,k=1,\ldots,K}$ is a frame. Thus, to prove that it is a Riesz basis it only remains prove that it has a biorthogonal sequence [8, Theorem 6.1.1]. Notice that since $|\det \mathbf{H}(\gamma)|$ is continuous on the compact $\widehat{M}$, and $|\det \mathbf{H}(\gamma)| > 0$ for all $\gamma \in \widehat{M}$, then there exists $J > 0$ such that $|\det \mathbf{H}(\gamma)| > J$ for all $\gamma \in \widehat{M}$. Then the rows of $\mathbf{H}^{-1}(\gamma)$ belong to $L^2(\widehat{M}) \times \ldots \times L^2(\widehat{M})$. We denote by $g_1, \ldots, g_k$ the inverse polyphase transform (see Proposition 1) of these rows. Thus $\mathbf{G}(\gamma)$ defined by (7) is $\mathbf{G}(\gamma) = \mathbf{H}(\gamma)^{-1}$. From (3), we obtain that the M-Fourier transform of $c_{k,k'} = \downarrow_M (g_{k'} * h_k)$ is

$$C_{k,k'}(\gamma) = \sum_{\ell \in \mathcal{L}} H_{k,\ell}(\gamma) G_{\ell,k'}(\gamma).$$

Since $\mathbf{G}(\gamma) = \mathbf{H}^{-1}(\gamma)$ we obtain that $C_{k,k'}(\gamma) = \delta_{k,k'}$ Then, having in mind that the inverse M-Fourier transform of $C_{k,k} = 1$ is δ, by using (9) we obtain

$$\langle T_{m'} g_{k'}, T_m f_k \rangle = \langle g_{k'}, T_{m-m'} f_k \rangle = \big(g_{k'} * h_k\big)(m - m') = c_{k,k'}(m - m') = \delta_{k,k'} \delta_{m,m'}$$

which proves that the sequence $\{T_m f_k\}_{m \in M, k \in \mathcal{K}}$ is a Riesz basis for $\ell^2(G)$. The optimal Riesz bounds are the optimal frame bounds [8, Theorem 5.4.1], and then, from Theorem 3, they are A and B. □

Proof of Proposition 2: If $n \notin M$ we have that there exist $\eta_r \in M^\perp$ such that $\langle n, \eta_r \rangle \neq 1$ [11, Proposition 4.38]. Since $M^\perp$ is a group,

$$\sum_{\eta \in M^\perp} \langle n, \eta \rangle = \sum_{\eta \in M^\perp} \langle n, \eta + \eta_r \rangle = \langle n, \eta_r \rangle \sum_{\eta \in M^\perp} \langle n, \eta \rangle.$$

Therefore

$$\sum_{\eta \in M^\perp} \langle n, \eta \rangle = \begin{cases} L & n \in M \\ 0 & n \notin M. \end{cases} \tag{12}$$

By using this relation, we obtain

$$\begin{aligned} (\downarrow_M x)^\wedge(\xi + M^\perp) &= \sum_{m \in M} x(m) \overline{\langle m, \xi \rangle} = \frac{1}{L} \sum_{n \in G} \sum_{\eta \in M^\perp} \overline{\langle n, \eta \rangle} x(n) \overline{\langle n, \xi \rangle} \\ &= \frac{1}{L} \sum_{\eta \in M^\perp} \sum_{n \in G} x(n) \overline{\langle n, \xi + \eta \rangle} = \frac{1}{L} \sum_{\eta \in M^\perp} \widehat{x}(\xi + \eta). \end{aligned}$$ □

Proof of Proposition 3: We have

$$\begin{aligned} H_k(\xi) &= \sum_{n \in G} h_k(n) \overline{\langle n, \xi \rangle} = \sum_{\ell \in \mathcal{L}} \sum_{m \in M} h_k(m - \ell) \overline{\langle m - \ell, \xi \rangle} \\ &= \sum_{\ell \in \mathcal{L}} \langle \ell, \xi \rangle \sum_{m \in M} h_k(m - \ell) \overline{\langle m, \xi \rangle} = \sum_{\ell \in \mathcal{L}} \langle \ell, \xi \rangle H_{k,\ell}(\xi + M^\perp). \end{aligned}$$

Then $H_k(\xi + \eta) = \sum_{\ell \in \mathcal{L}} \langle \ell, \xi \rangle H_{k,\ell}(\xi + M^\perp) \langle \ell, \eta \rangle$ for all $\xi \in \widehat{G}$, $\eta \in M^\perp$. □

Proof of Corollary 2: For a matrix with integer entries $\mathbf{A}$ we define $z^{\mathbf{A}}$ as the vector whose k-component is $z_1^{\mathbf{A}_{1,k}} z_2^{\mathbf{A}_{2,k}} \ldots z_d^{\mathbf{A}_{d,k}}$. It can be verified that $[z^{\mathbf{A}}]^{\mathbf{B}} = z^{\mathbf{AB}}$ (see [18, pp. 581-582]).

Then

$$H_{k,\ell}(z+M^\perp)=\sum_{m\in M} h_k(m-\ell)z^{-m}=\sum_{n\in\mathbb{Z}^d} h_k(\mathbf{M}n-\ell)z^{-\mathbf{M}n}$$
$$=\sum_{n\in\mathbb{Z}^d} h_k(\mathbf{M}n-\ell)[z^{\mathbf{M}}]^{-n}=E_{k,\ell}(z^{\mathbf{M}}).$$

($z+M^\perp$ denotes an element of $\mathbb{T}^d/M^\perp$) and analogously $G_{\ell,k}(z+M^\perp)=R_{\ell,k}(z^{\mathbf{M}})$. Then

$$\mathbf{H}(z+M^\perp)=\mathbf{E}(z^{\mathbf{M}}),\quad \mathbf{G}(z+M^\perp)=\mathbf{R}(z^{\mathbf{M}}).$$

Besides, for any $z\in\mathbb{T}^d$ there exist $s\in\mathbb{T}^d$ such $s^{\mathbf{M}}=z$. Indeed, there exists $r\in\mathbb{T}^d$ such that $r_j^{\det\mathbf{M}}=z_j$ and then $[r^{\mathrm{adj}\mathbf{M}}]^{\mathbf{M}}=r^{(\mathrm{adj}\mathbf{M})\mathbf{M}}=r^{\mathbf{I}\det\mathbf{M}}=z$. By using these two facts, the corollary is a consequence of Theorems 2, 3 and 4 and Corollary 1. □

Proof of Corollary 3: We have $\widehat{M}\cong\widehat{G}/M^\perp\cong\mathbb{Z}_s/M^\perp$ with $\langle Lm,n+M^\perp\rangle=W_s^{mLn}=W_N^{mn}$. Then

$$H_{k,\ell}(n+M^\perp)=\sum_{m=0}^{N-1} h_k(Lm-\ell)\overline{\langle Lm,n\rangle}=\sum_{m=0}^{N-1} h_k(Lm-\ell)W_N^{-mn}=E_{k,\ell}(n)$$

and analogously $G_{\ell,k}(n)=R_{\ell,k}(n)$. Hence,the corollary is a consequence of Theorems 2, 3 and 4 and Corollary 1, having in mind that $\mathbf{E}(n)$ and $\mathbf{R}(n)$ are N-periodic. □

Proof of Corollary 4: We have

$$H_{k,\ell}\big((z,n)+M^\perp\big)=\sum_{u\in\mathbb{Z}^d}\sum_{m=0}^{N-1} h_k\big((\mathbf{M}u,mL)-\ell\big)\overline{\langle(\mathbf{M}u,mL),(z,n)\rangle}$$
$$=\sum_{u\in\mathbb{Z}^d}\sum_{m=0}^{N-1} h_k\big((\mathbf{M}u,mL)-\ell\big)z^{-\mathbf{M}u}W_s^{-mLn}$$
$$=\sum_{u\in\mathbb{Z}^d}\sum_{m=0}^{N-1} h_k\big((\mathbf{M}u,mL)-\ell\big)[z^{\mathbf{M}}]^{-u}W_N^{-mn}=E_{k,\ell}(z^{\mathbf{M}},n)$$

and analogously $G_{\ell,k}\big((z,n)+M^\perp\big)=R_{\ell,k}(z^{\mathbf{M}},n)$. Besides (it was proved in previous proof) for any $z\in\mathbb{T}^d$ there exist $s\in\mathbb{T}^d$ such $s^{\mathbf{M}}=z$. Thus, having in mind the N-periodicity, the corollary is a consequence of Theorems 1, 3 and 4. □

Proof of Corollary 5: We have $\widehat{M}\cong\widehat{G}/M^\perp\cong(\mathbb{Z}_{2P}\times\mathbb{Z}_{2Q})/M^\perp$ with

$$\big\langle(2u,2v),(n,m)+M^\perp\big\rangle=W_{2P}^{2un}\,W_{2Q}^{2vm}=W_P^{un}\,W_Q^{vm}$$
$$\big\langle(2u+1,2v+1),(n,m)+M^\perp\big\rangle=W_{2P}^{(2u+1)n}\,W_{2Q}^{(2v+1)m}=W_{2P}^{n}\,W_{2Q}^{m}\,W_P^{un}\,W_Q^{vm}.$$

Then, the M-Fourier transform of a function h is given by

$$H\big((n,m)+M^\perp\big)=\sum_{u=0}^{P-1}\sum_{v=0}^{Q-1}\big[h_0(u,v)W_P^{-un}\,W_Q^{-vm}+h_1(u,v)W_P^{-un}\,W_Q^{-vm}\,W_{2P}^{-n}\,W_{2Q}^{-m}\big].$$

Hence, from Theorem 1, the filter bank satisfies the PR property if and only if $\mathbf{R}(n,m)\,\mathbf{E}(n,m)=\mathbf{I}_2$ for all $(n,m)\in\mathbb{Z}_{2P}\times\mathbb{Z}_{2Q}$. Since $\Lambda(n+P,m+Q)=\Lambda(n,m)$ and $\Lambda(n,m+Q)=\Lambda(n+P,m)$, it suffices to consider $(n,m)\in\mathbb{Z}_{2P}\times\mathbb{Z}_Q$. □

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