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A distributional study of discrete classical orthogonal polynomials

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Abstract

For the sequences of discrete classical orthogonal polynomials (Charlier, Meixner, Hahn) we can find a functional u, which satisfies the difference distributional equation $\Delta(\phi u) = \psi u$ where ϕ and ψ are polynomials of degrees $\leqslant 2$ and 1 respectively. From this it follows that these polynomials are solutions of a second-order difference equation; also, they can be represented by a Rodrigues-type formula. The sequence of difference polynomials derived from them constitutes an orthogonal polynomial sequence. Their weight functions satisfy a Pearson-type difference equation. A structure relation ($\phi \Delta P_{n+1} = a_n P_{n+2} + b_n P_{n+1} + c_n P_n$) also holds.

Keywords: Orthogonal polynomials; Difference equations; Moment functionals

1. Introduction

In this paper we study the properties of discrete classical orthogonal polynomials. The properties of continuous case orthogonal polynomials (Hermite, Laguerre, Jacobi, Bessel) are well known (see [1] for instance):

(1) They satisfy a Sturm–Liouville second-order differential equation.
(2) Their derivatives constitute an orthogonal polynomial sequence (OPS).
(3) They can be expressed by a Rodrigues formula.
(4) Their associated weight functions satisfy a Pearson differential equation.
(5) They verify a structure relation.

These properties, among others, can be derived from the distributional differential equation $\mathrm{D}(\phi u) = \psi u$ [9], where u is the functional associated with the OPS. In this paper the derivative operator D is substituted by a difference operator Δ. We define the set $\mathscr{D}$ of all regular functional moments u, satisfying a distributional difference equation $\Delta(\phi u) = \psi u$, where $\phi(x)$ and $\psi(x)$ are

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polynomials with $\deg(\phi) \leqslant 2$, $\deg(\psi) = 1$. We study the following properties that the (P_n) OPS corresponding to $u \in \mathscr{D}$ satisfy:

(1) (ΔP_n) is an OPS.
(2) A structure relation holds.
(3) They satisfy a Sturm–Liouville linear second-order difference equation.
(4) Their weight functions are solutions of a Pearson-type difference equation.
(5) They can be expressed by a discrete Rodrigues formula.

Let $\mathscr{P}$ be the linear space of complex polynomials, $\mathscr{P}_N$ be those of degree less than or equal to N and u a complex linear functional on $\mathscr{P}$ (moment functional).

Definition 1.1. A set of polynomials (P_n) is said to be an orthogonal polynomial sequence (OPS) *associated with* the functional u if for each n, $\deg(P_n) = n$ and

$$\langle u, P_n P_m \rangle = K_n \delta_{nm} \quad (K_n \neq 0)$$

for $n, m = 0, 1, 2 \ldots$.

Definition 1.2. For every polynomial ϕ the *left product* of u by ϕ is defined as

$$\langle \phi u, p \rangle = \langle u, \phi p \rangle \quad \forall p \in \mathscr{P}.$$

Definition 1.3. The *forward distributional difference* for u is given by

$$\langle \Delta u, p \rangle = -\langle u, \Delta p \rangle \quad \forall p \in \mathscr{P}$$

and, similarly, the *backward distributional difference* is

$$\langle \nabla u, p \rangle = -\langle u, \nabla p \rangle \quad \forall p \in \mathscr{P}.$$

Lemma 1.4. $\forall p, q \in \mathscr{P}$:

(1) $\Delta[p(x)q(x)] = q(x)\Delta p(x) + p(x+1)\Delta q(x)$.
(2) $\nabla[p(x)q(x)] = q(x)\nabla p(x) + p(x-1)\nabla q(x)$.

From this we can obtain the Leibniz-type formulas:

(3) $\Delta^n[p(x)q(x)] = q(x)\Delta^n p(x) + n\Delta q(x)\Delta^{n-1}p(x+1) + \binom{n}{2}\Delta^2 q(x)\Delta^{n-2}p(x+2) + \cdots$.
(4) $\nabla^n[p(x)q(x)] = q(x)\nabla^n p(x) + n\nabla q(x)\nabla^{n-1}p(x-1) + \binom{n}{2}\nabla^2 q(x)\nabla^{n-2}p(x-2) + \cdots$.

Lemma 1.5. *If* $\phi \in \mathscr{P}$ *and* u *is a moment functional then*

$$\Delta(\phi u) = \phi(x-1)\Delta u + \Delta\phi(x-1)u.$$

Proof.

$$\begin{aligned}\langle \Delta(\phi u), p \rangle &= -\langle u, \phi \Delta p \rangle \\ &= -\langle u, \Delta(\phi(x-1)p) - p\Delta\phi(x-1) \rangle\end{aligned}$$

$$= \langle \Delta u, \phi(x-1)p \rangle + \langle \Delta\phi(x-1)u, p \rangle$$
$$= \langle \phi(x-1)\Delta u + \Delta\phi(x-1)u, p \rangle \quad \forall p \in \mathscr{P}. \qquad \square$$

By induction we can obtain the following Leibniz-type formula.

Lemma 1.6.

$$\Delta^n(\phi u) = \binom{n}{0}\phi(x-n)\Delta^n u + \binom{n}{1}\Delta\phi(x-n)\Delta^{n-1}u + \cdots + \binom{n}{n-1}\Delta^{n-1}\phi(x-n)\Delta u + \binom{n}{n}\Delta^n\phi(x-n)u.$$

Let (P_n) be a monic orthogonal polynomial sequence (MOPS) and u its associated functional. We consider the dual basis (α_n) associated with (P_n) in $\mathscr{P}^*$. If $v \in P^*$ and if $\langle v, P_i \rangle = 0$, $\forall i \geqslant l$, then

$$v = \sum_{i=0}^{l-1} \lambda_i \alpha_i;$$

in particular

$$u = \langle u, 1 \rangle \alpha_0.$$

Lemma 1.7. *Each element of the dual basis α_n can be written as*

$$\alpha_i = \frac{P_i}{\langle u, P_i^2 \rangle} u, \quad i = 0, 1, 2, \ldots. \tag{1}$$

Lemma 1.8. *Let (P_n) and (Q_n) be such that*

$$Q_n = \frac{\Delta P_{n+1}}{n+1}.$$

If $(\alpha_n), (\beta_n)$ are their corresponding dual bases in $\mathscr{P}^*$ then

$$\Delta\beta_n = -(n+1)\alpha_{n+1}. \tag{2}$$

Proof.

$$\langle \Delta\beta_n, P_{m+1} \rangle = -\langle \beta_n, \Delta P_{m+1} \rangle = -(m+1)\langle \beta_n, Q_m \rangle = -(m+1)\delta_{n,m}. \qquad \square$$

Definition 1.9. We define the *nth factorial power* of x ($n \in \mathbb{N}$) as

$$x_h^{(n)} = x(x-h)(x-2h)\cdots(x-(n-1)h), \quad n \geqslant 1,$$
$$x_h^{(0)} = 1.$$

Lemma 1.10.

$$\Delta_h x_h^{(n)} = nhx_h^{(n-1)}, \tag{3}$$

$$x_h^{(n)} = \sum_{i=1}^{n} S_i^n h^{n-i} x^i, \tag{4}$$

where S_i^n are the Stirling numbers of the first kind.

In the following, $h = 1$, if not indicated otherwise.

2. Properties of the set $\mathscr{D}$

Referring back to the definition of the set $\mathscr{D}$,

$$\mathscr{D} = \{u\colon u \text{ regular};\ \Delta(\phi u) = \psi u;\ \deg(\phi) \leqslant 2;\ \deg(\psi) = 1\}.$$

If (P_n) is an MOPS, it is well known that (R_n) and $(\hat{R}_n)$ defined by

$$R_n(x) = a^n P_n\left(\frac{x}{a}\right), \quad \hat{R}_n(x) = P_n(x - b) \quad (a, b \in \mathscr{C})$$

are MOPS. Their associated functionals v and $\hat{v}$ are given by

$$\langle v, x^n \rangle = \langle u, (ax)^n \rangle, \quad \langle \hat{v}, x^n \rangle = \langle u, (x + b)^n \rangle.$$

See [2, Ex. 2.4, p. 10].

Proposition 2.1. *If $u \in \mathscr{D}$, then v and $\hat{v} \in \mathscr{D}$.*

Proof. First we prove that $v \in \mathscr{D}$,

$$\begin{aligned}\langle \Delta(\phi_1 v), x^{(n)} \rangle &= -\, n\langle \phi_1 v, x^{(n-1)} \rangle \\ &= -\, n\langle \phi_1 v, \sum_{i=1}^{n-1} S_i^{n-1} x^i \rangle = -\, n\langle u, \phi_1(ax) \sum_{i=1}^{n-1} S_i^{n-1} (ax)^i \rangle.\end{aligned}$$

Let $\phi(x) = \phi_1(ax)$,

$$\begin{aligned}\langle \Delta(\phi_1 v), x^{(n)} \rangle &= -\, n\langle \phi u, a^{n-1} \sum_{i=1}^{n-1} S_i^{n-1} (1/a)^{n-i-1} x^i \rangle \\ &= -\, n\langle \phi u, a^{n-1} x_{1/a}^{(n-1)} \rangle = -\, a^n \langle \phi u, \Delta x_{1/a}^{(n)} \rangle.\end{aligned}$$

Therefore

$$\langle \Delta(\phi_1 v), x^{(n)} \rangle = \langle \psi u, a^n x_{1/a}^{(n)} \rangle,$$

$$\langle u, \psi a^n \sum_{i=1}^{n} S_i^n (1/a)^{n-i} x^i \rangle = \langle u, \psi \sum_{i=1}^{n} S_i^n (ax)^i \rangle,$$

$$\langle v, \psi(x/a) \sum_{i=1}^{n} S_i^n x^i \rangle = \langle \psi_1 v, x^{(n)} \rangle,$$

if $\psi_1(x) = \psi(x/a)$ then

$$\Delta(\phi_1 v) = \psi_1 v$$

and $\deg(\phi_1(x)) \leqslant 2$ and $\deg(\psi_1(x)) = 1$.

It could be proved in a similar way that $\hat{v} \in \mathscr{D}$. □

Corollary 2.2. *Let (P_n) be an MOPS associated with u. The sequence of polynomials (R_n) given by*

$$R_n(x) = a^n P_n\left(\frac{x-b}{a}\right) \quad (a, b \in \mathscr{C},\ a \neq 0)$$

is orthogonal with respect to the functional v and

$$\langle v, x^n \rangle = \langle u, (ax+b)^n \rangle.$$

If $u \in \mathscr{D}$, then $v \in \mathscr{D}$.

Given a functional u, we define its *factorial moments* as

$$\langle u, x^{(n)} \rangle = \hat{\eta}_n, \quad n \geqslant 0.$$

Let $u \in \mathscr{D}$ and $\phi(x) = ax^2 + bx + c$, $\psi(x) = px + q$ and $p \neq 0$.

Proposition 2.3. *$u \in \mathscr{D}$ iff the factorial moment sequence $(\hat{\eta}_n)$ satisfies the difference equation*

$$(an+p)\hat{\eta}_{n+1} + (an(2n-1) + (b+p)n + q)\hat{\eta}_n + (an(n-1)^2 + bn(n-1) + cn)\hat{\eta}_{n-1} = 0 \qquad (5)$$

if $n \geqslant 0$ with the initial conditions $\hat{\eta}_{-1} = 0$, $\hat{\eta}_0 = 1$.

Proof. We use formulas

$$x.x^{(n)} = nx^{(n)} + x^{(n+1)},$$

$$x^2.x^{(n)} = x^{(n+2)} + (2n+1)x^{(n+1)} + n^2 x^{(n)}.$$

If $u \in \mathscr{D}$ then

$$\langle \Delta(\phi u), x^{(n)} \rangle = \langle \psi u, x^{(n)} \rangle$$

$$\Leftrightarrow -\langle u, nax^2 x^{(n-1)} + nbxx^{(n-1)} + cnx^{(n-1)} \rangle = \langle u, pxx^{(n)} + qx^{(n)} \rangle$$

$$\Leftrightarrow -na(\hat{\eta}_{n+1} + (2(n-1)+1)\hat{\eta}_n + (n-1)^2\hat{\eta}_{n-1}) - nb((n-1)\hat{\eta}_{n-1} + \hat{\eta}_n) - cn\hat{\eta}_{n-1}$$

$$= pn\hat{\eta}_n + p\hat{\eta}_{n+1} + q\hat{\eta}_n.$$

As a result of these computations we obtain (5), for every $n = 0, 1, \ldots$. □

Proposition 2.4. *If (P_n) is an MOPS associated with the functional u, and the sequence (Q_n), defined by*

$$Q_n = \frac{\Delta P_{n+1}}{n+1},$$

is an MOPS, then $u \in \mathscr{D}$.

Proof. Let (Q_n) be associated with the functional v and $(\alpha_n), (\beta_n)$ be, respectively, the dual bases of $(P_n), (Q_n)$. By (1) and (2) we have

$$\Delta v = \frac{P_1(x)}{\langle u, P_1^2 \rangle} u = \psi u.$$

As $P_n(x) = \Delta(xP_n(x)) - x\Delta P_n(x) - \Delta P_n(x)$,

$$\begin{aligned}\langle v, P_n \rangle &= \langle v, \Delta(xP_n(x)) \rangle - \langle v, x\Delta P_n(x) \rangle - \langle v, \Delta P_n(x) \rangle \\ &= -\left\langle \frac{P_1 u}{\langle u, P_1^2 \rangle}, xP_n \right\rangle - \langle xv, nQ_{n-1} \rangle - \langle v, nQ_{n-1} \rangle = 0\end{aligned}$$

if $n \geqslant 3$. Then $v = \bar{a}\alpha_0 + \bar{b}\alpha_1 + \bar{c}\alpha_2$.

The action of v on P_0, P_1 and P_2 gives us $\bar{a} = 1$, $\bar{b} = \langle v, P_1 \rangle$, $\bar{c} = \langle v, P_2 \rangle$. Then

$$v = u + \frac{\langle v, P_1 \rangle}{\langle u, P_1^2 \rangle} P_1(x)u + \frac{\langle v, P_2 \rangle}{\langle u, P_2^2 \rangle} P_2(x)u = \phi u$$

with $\deg(\phi) \leqslant 2$, $\deg(\psi) = 1$ and $\Delta(\phi u) = \psi u$. □

Proposition 2.5. *If (P_n) is an MOPS associated with the functional $u \in \mathscr{D}$, then the sequence of polynomials (Q_n) defined by*

$$Q_n(x) = \frac{\Delta P_{n+1}}{n+1}$$

constitutes an MOPS.

Proof. Let $\phi(x) = ax^2 + bx + c$, $\psi(x) = px + q$ and $p \neq 0$. If $v = \phi u$ then for all $m \leqslant n$ we have

$$\begin{aligned}\langle v, x^{(m)}Q_n \rangle &= (n+1)^{-1}\langle v, x^{(m)}\Delta P_{n+1}(x) \rangle \\ &= (n+1)^{-1}\langle \phi u, \Delta(x^{(m)}P_{n+1}(x)) - mx^{(m-1)}P_{n+1}(x+1) \rangle \\ &= -(n+1)^{-1}\langle \psi u, x^{(m)}P_{n+1} \rangle - (n+1)^{-1}\langle \phi u, mx^{(m-1)}(\Delta P_{n+1}(x) + P_{n+1}(x)) \rangle,\end{aligned}$$

where each term of this sum is either

$$\langle \psi u, x^{(m)}P_{n+1} \rangle = \langle u, x^{(m)}P_{n+1}\psi \rangle = pK_{n+1}\delta_{nm} \quad \text{and} \quad K_{n+1} \neq 0$$

or

$$\begin{aligned}\langle \phi u, mx^{(m-1)}(\Delta P_{n+1} + P_{n+1}) \rangle &= m\langle \phi u, x^{(m-1)}\Delta P_{n+1} \rangle + m\langle \phi u, x^{(m-1)}P_{n+1} \rangle \\ &= m\langle \phi u, x^{(m-1)}\Delta P_{n+1} \rangle + naK_{n+1}\delta_{nm},\end{aligned}$$

$$\begin{aligned}m\langle \phi u, x^{(m-1)}\Delta P_{n+1} \rangle &= m\langle \phi u, \Delta(x^{(m-1)}P_{n+1}) - (m-1)x^{(m-2)}P_{n+1}(x+1) \rangle \\ &= -m\langle \psi u, (x^{(m-1)}P_{n+1}) \rangle - m(m-1) \\ &\quad \times \langle \phi u, x^{(m-2)}(\Delta P_{n+1} + P_{n+1}) \rangle.\end{aligned}$$

The first term of the equation is

$$-m\langle \psi u, x^{(m-1)}P_{n+1}\rangle = -n\langle u, \psi x^{(m-1)}P_{n+1}\rangle = 0$$

because $m < n + 1$, and the second one is

$$-m(m-1)\{\langle \phi u, x^{(m-2)}\Delta P_{n+1}\rangle + \langle \phi u, x^{(m-2)}P_{n+1}\rangle\} = -m(m-1)\langle \phi u, x^{(m-2)}\Delta P_{n+1}\rangle.$$

Repeating the process until the factorial power vanishes one obtains the following result:

$$\langle \phi u, \Delta P_{n+1}\rangle = -\langle \psi u, P_{n+1}\rangle = -\langle u, \psi P_{n+1}\rangle \quad \text{if } n \geqslant 1.$$

Summarizing the results

$$\langle v, x^{(m)}Q_n\rangle = -(n+1)^{-1}K_{n+1}(p+na)\delta_{nm},$$
$$\forall n \geqslant 1, \quad m \leqslant n.$$

For $n = 0$, $\langle v, 1\rangle = \langle u, \phi\rangle \neq 0$.

Then the sequence (Q_n) is an MOPS iff $p + na \neq 0$ $\forall n \in \mathbb{N}$. Otherwise we obtain a finite orthogonal polynomial family until $n \in \mathbb{N}$ for which $p + na = 0$. □

Proposition 2.6. *If (P_n) is an MOPS associated with the functional u, and satisfies a structure relation*

$$\phi(x)\Delta P_{n+1}(x) = a_n P_{n+2}(x) + b_n P_{n+1}(x) + c_n P_n(x), \quad n \geqslant 0, \quad c_n \neq 0,$$

then $u \in \mathscr{D}$.

Conversely, if $u \in \mathscr{D}$ then the polynomials (P_n) associated with u satisfy the previous relation.

Proof. Using the functional u in the relation above,

$$\langle u, \phi\Delta P_{n+1}\rangle = a_n\langle u, P_{n+2}\rangle + b_n\langle u, P_{n+1}\rangle + c_n\langle u, P_n\rangle = 0 \quad \forall n \geqslant 1,$$

then

$$\langle u, \phi\Delta P_{n+1}\rangle = -\langle \Delta(\phi u), P_{n+1}\rangle = 0 \quad \text{if } n \geqslant 1.$$

Therefore

$$\Delta(\phi u) = \bar{a}\alpha_0 + \bar{b}\alpha_1$$

and (α_n) is the dual basis of (P_n).

Acting with $\Delta(\phi u)$ on 1 and x one obtains $\bar{a} = 0$ and $\bar{b} = -\langle u, \phi\rangle \neq 0$, therefore

$$\Delta(\phi u) = \frac{-\langle u, \phi\rangle}{\langle u, P_1^2\rangle} P_1 u = \psi u$$

and $\deg(\psi) = 1$. Conversely, as $\Delta(\phi u) = \psi u$ with $\deg(\phi) \leqslant 2$, and $\deg(\phi\Delta P_{n+1}) \leqslant n + 2$, we have that

$$\phi(x)\Delta P_{n+1}(x) = \sum_{j=0}^{n+2} c_{n,j}P_j(x)$$

with

$$c_{n,j} = \frac{\langle u, \phi P_j \Delta P_{n+1} \rangle}{\langle u, P_j^2 \rangle}.$$

On the other hand

$$\langle u, \phi P_j \Delta P_{n+1} \rangle = (n+1) \langle \phi u, P_j Q_n \rangle$$
$$= (n+1) \langle v, P_j Q_n \rangle = 0 \quad \text{if } j < n$$

using the orthogonality of the sequence (Q_n) associated with the functional v. Therefore

$$\phi(x) \Delta P_{n+1}(x) = c_{n,n} P_n(x) + c_{n,n+1} P_{n+1}(x) + c_{n,n+2} P_{n+2}(x).$$

We shall see that $c_{n,n} \neq 0$,

$$\langle u, P_n^2 \rangle c_n = - \langle u, \psi P_n P_{n+1} \rangle - \langle u, \phi (\Delta P_n) P_{n+1} \rangle$$

as $\langle v, \phi(\Delta P_n)(n+1) Q_n \rangle = 0$. If $\phi(x) = ax^2 + bx + c$ and $\psi(x) = px + q$, then

$$\langle u, P_n^2 \rangle c_{n,n} = - \langle u, \{(px+q)(x^n + \cdots) + (ax^2 + \cdots)(nx^{n-1} + \cdots)\} P_{n+1} \rangle$$
$$= - \langle u, (p + na) x^{n+1} P_{n+1} \rangle \neq 0$$

except if $p + na = 0$, for some $n \in \mathbb{N}$. □

Remark. If (P_n) is an MOPS associated with the functional $u \in \mathscr{D}$, from the above it follows that the sequence (Q_n) defined by

$$Q_n = \frac{\Delta P_{n+1}}{n+1}$$

is an MOPS associated with the functional $v = \phi u$.

It is easy to prove that $v \in \mathscr{D}$:

$$\langle \Delta(\phi(x+1)v), p \rangle = - \langle v, \phi(x+1) \Delta p \rangle = - \langle \phi u, \Delta(p\phi) - p \Delta \phi \rangle$$
$$= \langle \Delta(\phi u), p\phi \rangle + \langle \phi u, p \Delta \phi \rangle = \langle \psi u, p\phi \rangle + \langle (\Delta \phi) v, p \rangle,$$

then $\Delta(\phi(x+1)v) = (\psi + \Delta\phi)v$, and $v \in \mathscr{D}$.

Iterating the above procedure we deduce that the sequence $(Q_{n,2})$ defined as

$$Q_{n,2}(x) = \frac{\Delta Q_{n+1}}{n+1} = \frac{\Delta^2 P_{n+2}(x)}{(n+1)(n+2)}$$

is an MOPS associated with the functional $w \in \mathscr{D}$ since it satisfies

$$\Delta(\phi(x+2)w) = (\psi(x) + \Delta(\phi(x) + \phi(x+1)))w.$$

Then we have the following result.

Proposition 2.7. *Let (P_n) be an MOPS associated with $u \in \mathcal{D}$ and $\Delta(\phi u) = \psi u$. Then the sequence $(Q_{n,k})$ defined by*

$$Q_{n,k} = \frac{\Delta^k P_{n+k}}{(n+1)(n+2)\cdots(n+k)}$$

constitutes an MOPS associated with the functional

$$v_k = \phi_{(k)} u$$

where $\phi_{(k)}(x) = \phi(x)\phi(x+1)\cdots\phi(x+k-1)$ and $v_k \in \mathcal{D}$ satisfy a difference distributional equation

$$\Delta(\phi(x+k)v_k) = (\psi(x) + \Delta(\phi(x) + \phi(x+1) + \cdots + \phi(x+k-1)))v_k = \psi_k(x)v_k. \tag{6}$$

Lemma 2.8. *The distributional equations $\Delta(\phi u) = \psi u$ and $\nabla(\phi_1 u) = \psi_1 u$, where $\phi_1(x+1) = \phi(x) - \psi(x)$ and $\psi_1(x+1) = \psi(x)$ are equivalent in the following sense: The MOPS associated with the regular solutions of each one are the same up to a linear change of variables.*

Proof. Developing the expression

$$\langle \Delta(\phi_1 u), p\rangle = \langle \psi_1 u, p\rangle, \quad p \in \mathcal{P},$$

we obtain

$$\langle \Delta(\phi(x-1)u), p(x-1)\rangle = \langle \psi(x-1)u, p(x-1)\rangle.$$

Therefore, $\Delta(\phi(x-1)u) = \psi(x-1)u$. The result follows from Proposition 2.1. □

Proposition 2.9. *Let (P_n) be an MOPS associated with the functional u, then the two following statements are equivalent:*

(a) *(P_n) satisfy the difference equation*

$$\sigma_1(x)\Delta\nabla y(x) + \tau_1(x)\Delta y(x) + \lambda_n y(x) = 0 \quad \forall n \geqslant 0,$$

where $(\lambda_n) \subset \mathbb{C}$, and $\sigma_1, \tau_1 \in \mathcal{P}$ with $\deg(\sigma_1) \leqslant 2$ *and* $\deg(\tau_1) = 1$.

(b) *$u \in \mathcal{D}$.*

Proof. (a ⇒ b) Consider $v = \nabla(\sigma_1 u)$ and the basis

$$Q_n = \frac{\Delta P_{n+1}}{n+1} \in \mathcal{P}.$$

We have

$$\langle v, Q_n\rangle = \frac{1}{n+1}\langle \nabla(\sigma_1 u), \Delta P_{n+1}\rangle$$

$$= -\frac{1}{n+1}\langle u, -\tau_1 \Delta P_{n+1} - \lambda_n P_{n+1}\rangle = \langle \tau_1 u, Q_n\rangle$$

although $\langle u, P_{n+1}\rangle = 0\ \forall n \geqslant 0$. Therefore $v = \tau_1 u$ and $u \in \mathscr{D}$ by the previous lemma.

(b $\Rightarrow$ a) Using a method similar to the one of Proposition 2.5

$$\left(\frac{\nabla P_n(x)}{n}\right)$$

constitutes an MOPS associated with the functional $\phi_1 u$ with $\phi_1(x+1) = \phi(x) - \psi(x)$. Therefore

$$\langle \phi_1 u, \nabla P_n \nabla x^{(m)}\rangle = \begin{cases} 0 & \text{if } m < n,\\ K_n & \text{if } m = n,\end{cases}$$

$$K_n \neq 0$$

and

$$\langle \phi_1 u, \nabla P_n \nabla x^{(m)}\rangle = \langle \phi_1 u, \Delta P_n(x-1) \nabla x^{(m)}\rangle.$$

Because $(\nabla x^{(m)} \Delta P_n) = x^{(m)} \nabla \Delta P_n + \Delta P_n(x-1)\nabla x^{(m)}$,

$$\begin{aligned}\langle \phi_1 u, \nabla P_n \nabla x^{(m)}\rangle &= -\langle \nabla(\phi_1 u), x^{(m)}\Delta P_n\rangle - \langle u, x^{(m)}\phi_1 \nabla\Delta P_n\rangle\\ &= -\langle u, x^{(m)}(\psi_1 \Delta P_n + \phi_1 \nabla \Delta P_n)\rangle.\end{aligned}$$

From the uniqueness (up to multiplicative constants) of the OPS associated with u, there exists $(c_n) \subset \mathbb{C}$ such that

$$\phi_1 \nabla\Delta P_n + \psi_1 \Delta P_n = c_n P_n, \quad n \geqslant 0,$$

where $\phi_1(x+1) = \phi(x) - \psi(x)$ and $\psi_1(x+1) = \psi(x)$. □

Proposition 2.10. *Let (P_n) be an MOPS associated with the functional u. Then, $u \in \mathscr{D}$ if and only if*

$$P_n = Q_n + a_n Q_{n-1} + b_n Q_{n-2}, \quad n \geqslant 2,$$

where Q_n is defined in Proposition 2.5.

Proof. If $u \in \mathscr{D}$, u satisfies a distributional equation $\Delta(\phi u) = \psi u$. We know that Q_n is an MOPS associated with $v = \phi u$. For $n \geqslant 2$ we can write

$$P_n = Q_n + \sum_{j=0}^{n-1} \lambda_{n,j} Q_j,$$

where

$$\lambda_{n,j} = \frac{\langle v, P_n Q_j\rangle}{\langle v, Q_j^2\rangle} = \frac{\langle u, \phi Q_j P_n\rangle}{\langle v, Q_j^2\rangle} = 0 \quad \text{for } j+2 \leqslant n,$$

and then

$$P_n = Q_n + \lambda_{n,n-1} Q_{n-1} + \lambda_{n,n-2} Q_{n-2}.$$

Conversely, suppose that P_n satisfies the above relation. Let (α_n) and (β_n) be the dual bases associated with (P_n) and (Q_n), respectively. Then,

$$\langle \beta_0, P_n \rangle = \langle \beta_0, Q_n \rangle + a_n \langle \beta_0, Q_{n-1} \rangle + b_n \langle \beta_0, Q_{n-2} \rangle = 0$$

for $n \geqslant 3$. From Lemma 1.7 it follows that $\beta_0 = \phi u$, where ϕ is a polynomial with $\deg \phi \leqslant 2$.

On the other hand, taking into account Lemmas 1.7 and 1.8,

$$\Delta \beta_0 = -\alpha_1 = -\frac{P_1}{\langle u, P_1^2 \rangle} u = \psi u.$$

Therefore, $\Delta(\phi u) = \psi u$ with $\deg \phi \leqslant 2$, $\deg \psi = 1$ and $u \in \mathscr{D}$. □

Proposition 2.11 (distributional Rodrigues formula). *Let (P_n) be an MOPS associated with the functional u. Then $u \in \mathscr{D}$ iff there exists $(K_n) \subset \mathbb{C}$, $K_n \neq 0$ satisfying*

$$K_n^{-1} P_n(x) u = \Delta^n(\phi_{(n)}(x) u) \quad \forall n \geqslant 0$$

and $\phi \in \mathscr{P}_2$.

Proof. ($\Leftarrow$) If $n = 1$ as $K_1^{-1} P_1(x) u = \Delta(\phi(x) u)$ then $u \in \mathscr{D}$.

($\Rightarrow$) (P_i) is a basis in $\mathscr{P}$. For a fixed $n \in N$, if $i \leqslant n$

$$\begin{aligned}\langle \Delta^n(\phi_{(n)} u), P_i \rangle &= (-1)^n \langle \phi_{(n)} u, \Delta^n P_i \rangle \\ &= (-1)^n n! \langle u, \phi_{(n)} \rangle \delta_{in}.\end{aligned}$$

If $i > n$ let $i = m + n$,

$$\begin{aligned}\langle \Delta^n(\phi_{(n)} u), P_i \rangle &= \langle \Delta^n(\phi_{(n)} u), P_{m+n} \rangle \\ &= (-1)^n \langle \phi_{(n)} u, \Delta^n P_{m+n} \rangle \\ &= (-1)^n (m+1)(m+2) \cdots (m+n) \langle \phi_{(n)} u, Q_{m+n} \rangle = 0.\end{aligned}$$

Taking

$$K_n^{-1} = \frac{(-1)^n n! \langle u, \phi_{(n)} \rangle}{\langle u, P_n^2 \rangle},$$

then

$$\begin{aligned}\langle K_n^{-1} P_n u, P_i \rangle &= K_n^{-1} \langle u, P_n P_i \rangle \\ &= K_n^{-1} \langle u, P_n^2 \rangle \delta_{ni} = \langle \Delta^n(\phi_{(n)} u), P_i \rangle,\end{aligned}$$

therefore

$$\Delta^n(\phi_{(n)} u) = K_n^{-1} P_n u. \qquad \square$$

3. A derivation of the Rodrigues formula

Let (P_n) be an MOPS associated with $u \in \mathscr{D}$ and $\Delta(\phi u) = \psi u$. From Proposition 2.9 it follows that

$$\Delta(\phi_{(n+1)}(x)u) = \psi_n(x)\phi_{(n)}u, \quad n \geqslant 0,$$

where

$$\phi_{(n)}(x) = \phi(x)\phi(x+1) \cdots \phi(x+n-1),$$
$$\phi_{(0)}(x) = 1,$$
$$\psi_n(x) = \psi(x) + \Delta\phi(x) + \Delta\phi(x+1) + \cdots + \Delta\phi(x+n-1),$$
$$\psi_0(x) = \psi(x).$$

Lemma 3.1. *If* $2 \leqslant m \leqslant n+1$ *then*

$$\Delta^m(\phi_{(n+1)}u) = \psi_n(x-m+1)\Delta^{m-1}(\phi_{(n)}u) + (m-1)\Delta\psi_n(x-m+1)\Delta^{m-2}(\phi_{(n)}u). \tag{7}$$

Proof.

$$\Delta^m(\phi_{(n+1)}u) = \Delta^{m-1}(\Delta\phi_{(n+1)}u) = \Delta^{m-1}(\psi_n\phi_{(n)}u).$$

By using Lemma 1.6, and because $\Delta^i\psi_n(x-m+1) = 0$, for $i \geqslant 2$, we have (7). □

Lemma 3.2. *Let* $\omega(x)$ *be a positive function which is a solution of the difference equation:*

$$\Delta(\phi(x-1)\omega(x)) = \psi(x)\omega(x+1).$$

Then:

(1) $\Delta(\phi_{(n+1)}(x-1)\omega(x)) = \psi_n(x)\phi_{(n)}(x)\omega(x+1), n \geqslant 0.$

(2) *If* $2 \leqslant m \leqslant n+1$,

$$\nabla^m(\phi_{(n+1)}(x)\omega(x+1)) = \psi_n(x-m+1)\nabla^{m-1}(\phi_{(n)}(x)\omega(x+1)) + (m-1)\nabla\psi_n(x-m+1)\nabla^{m-2}(\phi_{(n)}(x)\omega(x+1)).$$

Proof. Lemma 3.2(1) will be proved by induction. For $n = 0$ it is straightforward. Assuming that the formula holds up to $n-1$ then

$$\begin{aligned}\Delta(\phi_{(n+1)}(x-1)\omega(x)) &= \Delta(\phi(x-1)\phi(x) \cdots \phi(x+n-2)\omega(x)\phi(x+n-1)) \\ &= \Delta\phi(x+n-1)\phi_{(n)}(x)\omega(x+1) + \phi(x+n-1)\Delta(\phi_{(n)}(x-1)\omega(x)),\end{aligned}$$

then by using the hypothesis of induction on the second term

$$\begin{aligned}\Delta(\phi_{(n+1)}(x-1) &= \Delta\phi(x+n-1)\phi_{(n)}(x)\omega(x+1) \\ &\quad + \phi(x+n-1)\phi_{(n-1)}(x)\psi_{n-1}(x)\omega(x+1) \\ &= (\Delta\phi(x+n-1) + \psi_{n-1}(x))\phi_{(n)}(x)\omega(x+1) \\ &= \psi_n(x)\phi_{(n)}(x)\omega(x+1).\end{aligned}$$

To prove Lemma 3.2(2), we can use the formula deduced previously and write

$$\nabla(\phi_{(n+1)}(x)\omega(x+1)) = \psi_n(x)\phi_{(n)}(x)\omega(x+1),$$

$$\nabla^m(\phi_{(n+1)}(x)\omega(x+1)) = \nabla^{m-1}(\psi_n(x)\phi_{(n)}(x)\omega(x+1)).$$

Using the results of Lemma 1.4

$$\nabla^m(\phi_{(n+1)}(x)\omega(x+1))$$
$$= \psi_n(x-m+1)\nabla^{m-1}(\phi_{(n)}(x)\omega(x+1)) + (m-1)\nabla\psi_n(x-m)\nabla^{m-2}(\phi_{(n)}(x)\omega(x+1))$$

as $\deg(\psi_n) = 1$ and $\nabla\psi_n(x-m) = \nabla\psi_n(x-m+1) = \Delta\psi_n(x-m+1)$. □

Lemma 3.3. *The following results hold*:

(1) *If*

$$\frac{\nabla^{m-j}(\phi_{(n)}(x)\omega(x+1))}{\omega(x+1)}u = \Delta^{m-j}(\phi_{(n)}u), \quad j = 1, 2,$$

then

$$\frac{\nabla^m(\phi_{(n+1)}(x)\omega(x+1))}{\omega(x+1)}u = \Delta^m(\phi_{(n+1)}u).$$

(2) *If*

$$\Delta^i(\phi_{(n)}u) = \frac{\nabla^i(\phi_{(n)}(x)\omega(x+1))}{\omega(x+1)}u, \quad i = 0, 1, \ldots, n,$$

then

$$\Delta^j(\phi_{(n+1)}u) = \frac{\nabla^j(\phi_{n+1)}(x)\omega(x+1))}{\omega(x+1)}u, \quad j = 0, 1, \ldots, n+1.$$

(3)

$$\frac{\nabla^n(\phi_{(n)}(x)\omega(x+1))}{\omega(x+1)}u = \Delta^n(\phi_{(n)}u).$$

Proof. Lemma 3.3(1) can be deduced by comparing the expression obtained to $\Delta^m(\phi_{(n+1)}u)$ and $\nabla^m(\phi_{(n+1)}\omega(x+1))$. Lemma 3.3(2) can be deduced from Lemma 3.3(1).

Lemma 3.3(3) is obtained by induction in Lemma 3.3(2) and making $j = n+1$. Therefore it is enough to prove that Lemma 3.3(2) holds for $n = 1$. If $i = 0$ it is immediate that

$$\phi u = \frac{\phi(x)\omega(x+1)}{\omega(x+1)}u.$$

If $i = 1$

$$\Delta(\phi u) = \psi u = \frac{\nabla[\phi(x)\omega(x+1)]}{\omega(x+1)}u. \qquad □$$

Theorem 3.4. *Let (P_n) be an MOPS with respect to a functional $u \in \mathscr{D}$. Then, the P_n polynomials satisfy the Rodrigues formula:*

$$P_n(x) = \frac{B_n}{\omega(x+1)} \nabla^n [\phi_{(n)}(x)\omega(x+1)], \quad n \geqslant 0,$$

where (B_n) are normalizing constants.

Proof. We define the polynomial of degree n:

$$S_n(x) = \frac{\nabla^n [\phi_{(n)}(x)\omega(x+1)]}{\omega(x+1)}.$$

Using Proposition 2.10 and Lemma 2.8, there exist no null constants (K_n) such that

$$K_n^{-1} P_n u = \Delta[\phi_{(n)} u] = S_n u.$$

Then, (S_n) is an OPS associated with the functional u. By the uniqueness, the Rodrigues formula is obtained. □

4. Consequences

Using the known properties of the classical orthogonal polynomials, it is clear that their functionals belong to $\mathscr{D}$. From Section 3 we can deduce that they are the only ones. Therefore the functionals corresponding to the classical discrete polynomials are characterized (up to a linear change of variable) by the difference distributional equation:

$$\Delta(\phi u) = \psi u,$$

where ϕ and ψ are polynomials with $\deg(\phi) \leqslant 2$ and $\deg(\psi) = 1$.

According to Lesky's [8] classification, we can give Table 1 as an illustrative example, where $a, b, c, d \in \mathbb{R}$ and $N \in \mathbb{N}$.

We can obtain ϕ and ψ from Nikiforov's table [11]. In this case,

$$\phi(x) = \sigma(x+1) + \tau(x+1) \quad \text{and} \quad \psi(x) = \tau(x+1),$$

where σ and τ are coefficients of the equation

$$\sigma \nabla \Delta y + \tau \Delta y + \lambda y = 0.$$

Table 1

P_n	ϕ	ψ
Charlier	1	$-x+a$
Meixner	x	$-x+b$
Krawtchouk	x	$2x+c$
Hahn	x^2+x+1	$-2Nx+d$

For instance, for the Charlier polynomials $(C_n^{(\mu)}(x))$,

$$\sigma(x) = x \quad \text{and} \quad \tau(x) = \mu - x,$$

therefore

$$\phi(x) = \mu \quad \text{and} \quad \psi(x) = -x + \mu - 1.$$

Another important consequence is that the discrete classical OPSs are completely characterized by the properties given in the Introduction and proved in Section 2.

5. Remarks

(1) As we note in Proposition 2.11, it is possible to define $\mathscr{D}$ as

$$\mathscr{D} = \{u\colon u \text{ regular}; \nabla\phi_1 u = \psi_1 u; \deg(\phi_1(x)) \leqslant 2; \deg(\psi_1(x)) = 1\}$$

and obtain similar properties to those obtained in Section 2, using a backward difference ∇.

(2) It is easy to justify the choice of $\omega(x)$ as a positive solution to the difference equation

$$\Delta[\phi(x-1)\omega(x)] = \psi(x)\omega(x+1).$$

By Proposition 2.11 we know that polynomials (P_n) associated with the functional $u \in \mathscr{D}$ are eigenfunctions of the equation

$$\sigma(x)\Delta\nabla y + \tau(x)\Delta y + \lambda y = 0 \tag{8}$$

with $\phi(x) = \sigma(x+1) + \tau(x+1)$, and $\psi(x) = \tau(x+1)$.

It can be proved that

$$\Delta(\sigma\omega) = \tau\omega \Leftrightarrow \Delta[\phi(x-1)\omega(x)] = \psi(x)\omega(x+1)$$

and therefore the integrating factor $\omega(x)$ transforms (8) into a self-adjoint equation.

(3) The following expression

$$\frac{\nabla^k[\phi_{(n)}(x)\omega(x+1)]}{\omega(x+1)},$$

is a polynomial of degree $\leqslant 2n-k$, where $\omega(x)$ is a positive solution of

$$\Delta[\phi(x-1)\omega(x)] = \psi(x)\omega(x+1) \quad [4].$$

If $k = n$, the degree is exactly n.

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