JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **214,** 587–598 (1997)
ARTICLE NO. AY975598

Sampling Theorem Associated with a Dirac Operator and the Hartley Transform

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Submitted by S. M. Meerkov

Received June 10, 1996

Kramer's sampling theorem indicates that, under certain conditions, sampling theorems associated with boundary-value problems involving nth order self-adjoint differential operators can be obtained. In this paper, we extend this result and derive a sampling theorem associated with boundary-value problems involving a one-dimensional system of Dirac operators. As a special case, we obtain a sampling theorem for the Hartley transform of a bandlimited function.

1. INTRODUCTION

The Whittaker–Shannon–Kotel'nikov (WSK) sampling theorem states that if $f(t)$ is a bandlimited function with bandwidth σ, then it is completely determined by its values at the points $t_n = n\pi/\sigma$, $n \in \mathbb{Z}$, and can be reconstructed by means of the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\sin \sigma(t - t_n)}{\sigma(t - t_n)}, \tag{1}$$

where the series is absolutely and uniformly convergent on compact sets [13]. By a bandlimited function we mean a function $f(t)$ that can be

0022-247X/97 $25.00

written in the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(w) e^{-itw} dw, \tag{2}$$

for some $F \in L^2(-\sigma, \sigma)$. The famous Paley–Wiener theorem asserts that f is an entire function of exponential type σ that belongs to $L^2(\mathbb{R})$ when restricted to the real axis, i.e.,

$$|f(z)| \le Ae^{\sigma|z|}, \qquad \text{for some } A > 0,$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty, \qquad \text{where } x = \operatorname{Re} z.$$

It should be noted that the series (1) can be written in the form

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{(t - t_n) G'(t_n)}, \tag{3}$$

where $G(t) = \sin \sigma t / \sigma$.

Formula (3) resembles the well-known Lagrange interpolation formula, and for this reason, we call it a Lagrange-type interpolation formula. The WSK theorem is considered an important tool in communication engineering because it enables engineers to reconstruct analogue bandlimited signals by sampling them at certain instants.

The connection between the WSK sampling theorem and boundary-value problems was first observed in 1957 by H. P. Kramer [5], who derived the following generalization of the WSK sampling theorem.

Let there exist a function $K(x, t)$ continuous in t such that $K(x, t) \in L^2(I)$ for every real number t. Assume that there exists a sequence of real numbers $\{t_n\}_{n \in \mathbb{Z}}$ such that $\{K(x, t_n)\}_{n \in \mathbb{Z}}$ is a complete orthogonal family in $L^2(I)$ for some finite interval $I = [a, b]$. Then for any function of the form

$$f(t) = \int_a^b F(x) K(x, t) dx, \tag{4}$$

with $F \in L^2(I)$, we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) S_n^*(t), \tag{5}$$

where

$$S_n^*(t) = \frac{\int_a^b K(x, t) K(x, t_n) dx}{\int_a^b |K(x, t_n)|^2 dx}. \tag{6}$$

In the case where $I = [-\sigma, \sigma]$, $K(x, t) = e^{ixt}$, $t_n = n\pi/\sigma$, it is easy to see that

$$S_n^*(t) = \frac{\sin \sigma(t - t_n)}{\sigma(t - t_n)},$$

and Eqs. (4) and (5) reduce to (2) and (1).

Kramer noted that the function $K(x, t)$ and the sampling points $\{t_n\}_{n \in \mathbb{Z}}$ can be found from certain boundary-value problems. More precisely, let

$$L \equiv p_0(x) \frac{d^n}{dx^n} + \cdots + p_{n-1}(x) \frac{d}{dx} + p_n(x), \qquad x \in I$$

be a self-adjoint differential operator, where $p_k(x)$ is a complex-valued function with $n - k$ continuous derivatives, $k = 0, 1, \ldots, n$, and $p_0(x) \neq 0$ for any $x \in I$. Let $U_j(y) = 0$, $j = 1, \ldots, n$, be linearly independent homogeneous self-adjoint boundary conditions. If the boundary-value problem

$$Ly = ty, \qquad x \in I,$$
$$U_j(y) = 0, \qquad j = 1, \ldots, n,$$

possesses a function $\varphi(x, t)$ that generates the eigenfunctions of the problem $\{\varphi_n(x)\}$ when the parameter t is replaced by the eigenvalues $\{t_n\}$, i.e., $\varphi(x, t_n) = \varphi_n(x)$, then one can take the sampling points to be $\{t_n\}$ and the function $K(x, t)$ to be $\varphi(x, t)$.

In recent years, this connection between sampling theorems and boundary-value problems has been the focus of many research papers. In [8], sampling theorems associated with regular Sturm–Liouville problems were investigated. It was shown, among other things, that for such boundary-value problems Kramer's sampling series (5) and (6) is nothing more than a Lagrange-type interpolation formula of the form (3). These results were refined and extended to singular Sturm–Liouville boundary-value problems in [9]. Sampling theorems associated with differential equations [4, 7], and more general types of boundary-value problems have been investigated in [2, 10, 11], and in [12], where the Green's function was used to generate the interpolation functions. Sampling theorems associated with integral equations have also been reported [1].

In this paper, we derive a sampling theorem associated with boundary-value problems involving one-dimensional Dirac operator. We show that the sampling series in this case, as in the Sturm–Liouville case, is nothing more than a Lagrange-type interpolation series. As a special case of our main result, we obtain a sampling series for the Hartley transform of a bandlimited function. The Hartley transform of a function $f(t)$ is defined as

$$F(x) = \int_0^{\infty} [\cos xt + \sin xt] f(t)\, dt.$$

It was introduced in [3] by an electric engineer, Hartley, as a way to overcome what he considered a drawback of the Fourier transform, i.e.,

$$F(x) = \int_{\infty}^{-\infty} [\cos xt + i \sin xt] f(t)\, dt,$$

namely, representing a real-valued signal $f(t)$ by a complex-valued one, $F(x)$. For more details, see [14, Chap. 13].

2. PRELIMINARIES

Consider the one-dimensional Dirac system of differential equations

$$\begin{aligned} y_2' + p_{11}(x) y_1 + p_{12}(x) y_2 &= \lambda y_1 \\ -y_1' + p_{21}(x) y_1 + p_{22}(x) y_2 &= \lambda y_2, \end{aligned}$$

where $p_{ij}(x), i, j = 1, 2$, are real-valued functions defined on $[0, \pi]$ and λ is a parameter. Using matrix notation, we can write this system in the form

$$\mathbf{B}\frac{d\mathbf{y}}{dx} + \mathbf{P}(x)\mathbf{y} = \lambda \mathbf{y},$$

where

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{P}(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

By using a smooth and orthogonal transformation of the form

$$\mathbf{H}(x) = \begin{pmatrix} \cos \phi(x) & -\sin \phi(x) \\ \sin \phi(x) & \cos \phi(x) \end{pmatrix},$$

with $\phi(x) = (1/2)\tan^{-1}[2p_{12}(x)/(p_{11}(x) - p_{22}(x))]$, and setting $\mathbf{y} = \mathbf{H}(x)\mathbf{z}$, we can transform the system into the canonical form [6, p. 186],

$$\mathbf{B}\frac{d\mathbf{z}}{dx} + \mathbf{Q}(x)\mathbf{z} = \lambda \mathbf{z}, \tag{7}$$

where

$$\mathbf{Q}(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix},$$

for some appropriate functions $p(x)$ and $r(x)$. Therefore, from now on, we shall confine our attention to Dirac systems in their canonical form (7).

Let us consider the following boundary-value problem for Eq. (7),

$$y_2' - p(x)y_1 = \lambda y_1, \qquad y_1' + r(x)y_2 = -\lambda y_2, \tag{8}$$

$$y_1(0)\sin\alpha + y_2(0)\cos\alpha = 0 \tag{9}$$

$$y_1(\pi)\sin\beta + y_2(\pi)\cos\beta = 0, \tag{10}$$

where $p(x)$ and $r(x)$ are continuous on $[0, \pi]$.

It is known [6] that the problem (8)–(10) has a countable number of eigenvalues $\{\lambda_n\}_{n=-\infty}^{\infty}$, which are all real and simple, and to every eigenvalue λ_n, there corresponds a vector-valued eigenfunction $\mathbf{y}_\mathbf{n}^T(x, \lambda_n) = (y_{n,1}(x, \lambda_n), y_{n,2}(x, \lambda_n))$, where T stands for the transpose. Moreover, vector-valued eigenfunctions belonging to different eigenvalues are orthogonal, i.e.,

$$\int_0^\pi \mathbf{y}_\mathbf{n}^T(x, \lambda_n)\mathbf{y}_\mathbf{m}(x, \lambda_m)\,dx$$

$$= \int_0^\pi \left[y_{n,1}(x, \lambda_n) y_{m,1}(x, \lambda_m) + y_{n,2}(x, \lambda_n) y_{m,2}(x, \lambda_m) \right] dx = 0.$$

Let $\mathbf{y}(x, \lambda)$ and $\mathbf{z}(x, \lambda')$ be solutions of (8); hence

$$y_2' - p(x)y_1 = \lambda y_1, \qquad y_1' + r(x)y_2 = -\lambda y_2 \tag{11}$$

$$z_2' - p(x)z_1 = \lambda' z_1, \qquad z_1' + r(x)z_2 = -\lambda' z_2. \tag{12}$$

Multiplying (11) by z_1 and $-z_2$, and (12) by $-y_1$ and y_2 respectively, and adding them together, we obtain

$$\frac{d}{dx}\left\{ z_1(x, \lambda')y_2(x, \lambda) - z_2(x, \lambda')y_1(x, \lambda) \right\}$$

$$= (\lambda - \lambda')\left\{ y_1(x, \lambda)z_1(x, \lambda') + y_2(x, \lambda)z_2(x, \lambda') \right\}. \tag{13}$$

It is also known that if $\mathbf{f}^T(x) = (f_1(x), f_2(x))$ is a vector-valued function which has a continuous derivative and satisfies the boundary conditions (9) and (10), then it can be expanded into an absolutely and uniformly convergent series of the vector-valued eigenfunctions of problem (8)–(10), viz

$$\mathbf{f}(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{\mathbf{\Phi_n}(x)}{\|\mathbf{\Phi_n}\|^2}, \tag{14}$$

where

$$\hat{f}_n = \int_0^\pi \mathbf{f}^T(x)\mathbf{\Phi_n}(x)\,dx = \int_0^\pi \left[f_1(x)\Phi_{n,1}(x) + f_2(x)\Phi_{n,2}(x)\right]dx,$$

$\mathbf{\Phi_n}^T(x) = (\Phi_{n,1}(x), \Phi_{n,2}(x))$ is the vector-valued eigenfunction corresponding to the eigenvalue λ_n, and

$$\|\mathbf{\Phi_n}\|^2 = \int_0^\pi \mathbf{\Phi_n}^T(x)\mathbf{\Phi_n}(x)\,dx = \int_0^\pi \left[\Phi_{n,1}^2(x) + \Phi_{n,2}^2(x)\right]dx.$$

If **f** is merely a square-integrable function, then the series (14) converges to **f** in the mean.

3. THE MAIN RESULT

Now we can derive a sampling theorem associated with the boundary-value problem (8)–(10).

THEOREM 3.1. *Let* $\mathbf{f}(\mathbf{x})$ *be a square-integrable vector-valued function on* $[0, \pi]$. *Set*

$$F(\lambda) = \int_0^\pi \mathbf{f}^T(x)\mathbf{\Phi}(x, \lambda)\,dx, \tag{15}$$

where $\mathbf{\Phi}(\mathbf{x}, \lambda)$ *is a solution of the differential equation* (8), *together with* $\Phi_1(0, \lambda) = \cos\alpha$ *and* $\Phi_2(0, \lambda) = -\sin\alpha$. *Then* $F(\lambda)$ *is an entire function of exponential type at most* π *that can be reconstructed using its values at the points* $\{\lambda_n\}_{n=-\infty}^{\infty}$ *by means of the formula*

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n)\frac{G(\lambda)}{(\lambda - \lambda_n)G'(\lambda_n)}, \tag{16}$$

where the series converges uniformly on compact sets of $\mathbb{C}$, *and* $G(\lambda)$ *is an entire function whose zeros are exactly the eigenvalues of the problem* (8)–(10) *and which can be written in the form*

$$G(\lambda) = (\lambda - \lambda_0)\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right)\left(1 - \frac{\lambda}{\lambda_{-n}}\right). \tag{17}$$

Proof. First, the infinite product given by (17) defines an entire function because it is known that the eigenvalues of the problem (8)–(10) satisfy the asymptotic formula $\lambda_n = a_0 + n + a_1/n + O(1/n^2)$ as $|n| \to \infty$. See [6, p. 193]. That $F(\lambda)$ is an entire function of exponential type at most π follows from the relation

$$|F(\lambda)| \leq \int_0^{\pi} |f_1(x)|\,|\Phi_1(x,\lambda)|\,dx + \int_0^{\pi} |f_2(x)|\,|\Phi_2(x\,\lambda)|\,dx$$

and the fact that $\Phi_1(x,\lambda)$ and $\Phi_2(x,\lambda)$ are entire functions of exponential type at most π; see Lemma 7.2.1 in [6].

Since $\mathbf{\Phi}(x,\lambda)$ is in $L^2(0,\pi)$ for any λ, we have

$$\mathbf{\Phi}(x,\lambda) = \sum_{n=-\infty}^{\infty} \hat{\Phi}_n \frac{\mathbf{\Phi}_{\mathbf{n}}(x)}{\|\Phi_n\|^2}, \tag{18}$$

where

$$\begin{aligned} \hat{\Phi}_n &= \int_0^{\pi} \mathbf{\Phi}^T(x,\lambda)\mathbf{\Phi}_{\mathbf{n}}(x)\,dx \\ &= \int_0^{\pi} [\Phi_1(x,\lambda)\Phi_{n,1}(x) + \Phi_2(x,\lambda)\Phi_{n,2}(x)]\,dx, \end{aligned} \tag{19}$$

$\mathbf{\Phi}^{\mathbf{T}}(x,\lambda) = (\Phi_1(x,\lambda), \Phi_2(x,\lambda))$ and $\mathbf{\Phi}_{\mathbf{n}}^{\mathbf{T}}(x) = (\Phi_{n,1}(x), \Phi_{n,2}(x))$.

Since $\mathbf{f}$ is in $L^2[0,\pi]$, it has an expansion of the form (14). In view of Parseval's relation and definition (15), we obtain

$$F(\lambda) = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{\hat{\mathbf{\Phi}}_n}{\|\mathbf{\Phi}_{\mathbf{n}}\|^2}.$$

But

$$F(\lambda_n) = \int_0^{\pi} \mathbf{f}^T(x)\mathbf{\Phi}(x,\lambda_n)\,dx = \int_0^{\pi} \mathbf{f}^T(x)\mathbf{\Phi}_{\mathbf{n}}(x)\,dx = \hat{f}_n;$$

thus

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n) \frac{\hat{\mathbf{\Phi}}_n}{\|\mathbf{\Phi}_{\mathbf{n}}\|^2}. \tag{20}$$

From relation (13), we have

$$(\lambda - \lambda')\int_0^{\pi} \mathbf{y}^{\mathbf{T}}(x,\lambda)\mathbf{z}(x,\lambda')\,dx = \mathbf{W}(\mathbf{z},\mathbf{y})\big|_{x=\pi} - \mathbf{W}(\mathbf{z},\mathbf{y})\big|_{x=0}, \tag{21}$$

where

$$\mathbf{W}(\mathbf{y},\mathbf{z}) = \begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ z_1(x,\lambda') & z_2(x,\lambda') \end{vmatrix}.$$

Let $\boldsymbol{\Psi}(x,\lambda)$ be the solution of the differential equation (8) with $\Psi_1(\pi,\lambda) = \cos\beta$, and $\Psi_2(\pi,\lambda) = -\sin\beta$. Set

$$\begin{aligned} G(\lambda) &= \mathbf{W}(\boldsymbol{\Phi}(\pi,\lambda), \boldsymbol{\Psi}(\pi,\lambda)) \\ &= \Phi_1(\pi,\lambda)\Psi_2(\pi,\lambda) - \Phi_2(\pi,\lambda)\Psi_1(\pi,\lambda) \\ &= -\sin\beta\Phi_1(\pi,\lambda) - \cos\beta\Phi_2(\pi,\lambda). \end{aligned} \tag{22}$$

Similarly,

$$G(\lambda') = -\sin\beta\Phi_1(\pi,\lambda') - \cos\beta\Phi_2(\pi,\lambda'). \tag{23}$$

Multiplying (22) by $\Phi_1(\pi,\lambda')$ and (23) by $\Phi_1(\pi,\lambda)$ and subtracting the resulting equations, we have

$$\begin{aligned} &G(\lambda)\Phi_1(\pi,\lambda') - G(\lambda')\Phi_1(\pi,\lambda) \\ &\quad = \cos\beta\left[\Phi_2(\pi,\lambda')\Phi_1(\pi,\lambda) - \Phi_1(\pi,\lambda')\Phi_2(\pi,\lambda)\right]. \end{aligned} \tag{24}$$

If we put $\mathbf{y}(x,\lambda) = \boldsymbol{\Phi}(x,\lambda)$, and $\mathbf{z}(x,\lambda') = \boldsymbol{\Phi}(x,\lambda')$ in (21), and observe that $\mathbf{W}(\boldsymbol{\Phi}(0,\lambda), \boldsymbol{\Phi}(0,\lambda')) = 0$ since both $\boldsymbol{\Phi}(0,\lambda)$ and $\boldsymbol{\Phi}(0,\lambda')$ satisfy the same boundary condition (9) at 0, it then follows that

$$\begin{aligned} &(\lambda - \lambda')\int_0^\pi \boldsymbol{\Phi}^T(x,\lambda)\boldsymbol{\Phi}(x,\lambda')\,dx \\ &\quad = \Phi_1(\pi,\lambda')\Phi_2(\pi,\lambda) - \Phi_2(\pi,\lambda')\Phi_1(\pi,\lambda) \end{aligned} \tag{25}$$

which, in view of (24), can be reduced to

$$(\lambda - \lambda')\int_0^\pi \boldsymbol{\Phi}^T(x,\lambda)\boldsymbol{\Phi}(x,\lambda')\,dx = \frac{G(\lambda')\Phi_1(\pi,\lambda) - G(\lambda)\Phi_1(\pi,\lambda')}{\cos\beta},$$

provided that $\cos\beta \neq 0$. By taking the limit as $\lambda' \to \lambda_n$, we obtain

$$(\lambda - \lambda_n)\int_0^\pi \boldsymbol{\Phi}^T(x,\lambda)\boldsymbol{\Phi}_{\mathbf{n}}(x)\,dx = \frac{-G(\lambda)\Phi_{n,1}(\pi)}{\cos\beta} \tag{26}$$

since $G(\lambda_n) = 0$ by (22) and (10). Similarly, we can show that

$$(\lambda - \lambda_n)\int_0^\pi \boldsymbol{\Phi}^{\mathbf{T}}(x,\lambda)\boldsymbol{\Phi}_{\mathbf{n}}(x)\,dx = \frac{-G(\lambda)\Phi_{n,2}(\pi)}{\sin\beta}, \tag{27}$$

provided that $\sin\beta \neq 0$. Differentiating with respect to λ and taking the limit as $\lambda \to \lambda_n$, we obtain

$$\|\mathbf{\Phi_n}\|^2 = \int_0^\pi \mathbf{\Phi_n}^T(x)\mathbf{\Phi_n}(x)\,dx = \frac{-G'(\lambda_n)\Phi_{n,1}(\pi)}{\cos\beta} \tag{28}$$

$$= \frac{-G'(\lambda_n)\Phi_{n,2}(\pi)}{\sin\beta}. \tag{29}$$

From (19), (26), and (28), we have for $\beta \neq \pi/2$

$$\frac{\hat{\mathbf{\Phi}}_n}{\|\mathbf{\Phi_n}\|^2} = \frac{G(\lambda)}{(\lambda - \lambda_n)G'(\lambda_n)}, \tag{30}$$

and if $\beta = \pi/2$, we use (19), (27), and (29) to obtain the same result. Now the substitution of Eq. (30) into (20) yields (16).

To show the uniform convergence of the series (16) on a compact set K of the complex plane, let

$$\mathbf{S_N}(x) = \sum_{n=-N}^{N} F(\lambda_n)\frac{\mathbf{\Phi_n}(x)}{\|\mathbf{\Phi_n}\|^2}.$$

For $\beta \neq \pi/2$, we have by the Cauchy–Schwarz inequality, Eqs. (20), (26) (or (27) if $\beta = \pi/2$), and (28),

$$\begin{aligned}
&\left| F(\lambda) - \sum_{n=-N}^{N} F(\lambda_n)\frac{G(\lambda)}{(\lambda-\lambda_n)G'(\lambda_n)}\right|^2 \\
&= \left| F(\lambda) - \sum_{n=-N}^{N} F(\lambda_n)\frac{\hat{\mathbf{\Phi}}_n}{\|\mathbf{\Phi_n}\|^2}\right|^2 \\
&= \left| \int_0^\pi [\mathbf{f}(x) - \mathbf{S_N}(x)]^T \mathbf{\Phi}(x,\lambda)\,dx \right|^2 \\
&\leq \left(\int_0^\pi \mathbf{\Phi}^T(x,\lambda)\mathbf{\Phi}(x,\lambda)\,dx \right) \|\mathbf{f} - \mathbf{S_N}\|^2 \\
&\leq C(K)\|\mathbf{f} - \mathbf{S_N}\|^2,
\end{aligned}$$

but the last term goes to zero as $N \to \infty$, where

$$C(K) = \max_{\lambda \in K}\left(\int_0^\pi \mathbf{\Phi}^T(x,\lambda)\mathbf{\Phi}(x,\lambda)\,dx \right) < \infty.$$

To complete the proof, we must show that the function $G(\lambda)$ in (22) is essentially the infinite product in (17). Of course, the function defined in (22) is an entire function whose only zeros are the eigenvalues of the Dirac problem. If we denote the infinite product in (17) temporarily by $\hat{G}(\lambda)$, we can write $G(\lambda) = g(\lambda)\hat{G}(\lambda)$, where $g(\lambda)$ is an entire function with no zeros. Thus,

$$\frac{G(\lambda)}{G'(\lambda_n)} = \frac{g(\lambda)\hat{G}(\lambda)}{g(\lambda_n)\hat{G}'(\lambda_n)},$$

and the conclusion of the theorem remains valid for the function $F(\lambda)/g(\lambda)$. ▮

4. APPLICATIONS

Consider the boundary-value problem (8)–(10) in which $p(x) = 0 = r(x)$:

$$y_2' = \lambda y_1, \qquad y_1' = -\lambda y_2, \tag{31}$$

$$y_1(0)\sin\alpha + y_2(0)\cos\alpha = 0 \tag{32}$$

$$y_1(\pi)\sin\beta + y_2(\pi)\cos\beta = 0. \tag{33}$$

It is easy to see that a solution of (31) and (32) is given by $\mathbf{\Phi}^T(x,\lambda) = (\cos(\lambda x - \alpha), \sin(\lambda x - \alpha))$. By substituting this solution in (33), we obtain $\cos(\lambda\pi - \alpha)\sin\beta + \sin(\lambda\pi - \alpha)\cos\beta = \sin(\lambda\pi + \beta - \alpha) = 0$, hence, the eigenvalues are $\lambda_n = n - \gamma/\pi$, where $\gamma = \beta - \alpha$.

Therefore, the function $G(\lambda)$ defined in Eq. (17) is given by

$$G(\lambda) = \prod_{n=-\infty}^{\infty}\left(1 - \frac{\lambda}{n-\delta}\right) = \left(1 + \frac{\lambda}{\delta}\right)\prod_{n=0}^{\infty}\left(1 - \frac{\lambda^2}{n^2 - \delta^2}\right)$$
$$= \frac{\sin\pi(\lambda+\delta)}{\sin\pi\delta},$$

where $\delta = \gamma/\pi$, and $\gamma \neq 0$. If $\gamma = 0$, then $G(\lambda) = \lambda\prod_{n=1}^{\infty}(1 - \lambda^2/n^2) = \sin\pi\lambda/\pi$.

Therefore, Theorem 3.1 now takes on the form: If

$$F(\lambda) = \int_0^{\pi}\{f_1(x)\cos(\lambda x - \alpha) + f_2(x)\sin(\lambda x - \alpha)\}dx,$$

for some f_1 and $f_2 \in L^2[0,\pi]$, then

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(n-\delta)\frac{\sin\pi(\lambda - n + \delta)}{\pi(\lambda - n + \delta)},$$

for all δ.

In particular, if $f_1(x) = f_2(x) = f(x)$, we have a generalization of the Hartley transform $F(\lambda)$ of a bandlimited function $f(x)$

$$F(\lambda) = \int_0^{\pi} \{\cos(\lambda x - \alpha) + \sin(\lambda x - \alpha)\} f(x)\, dx$$

which can be reconstructed via

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(n - \delta) \frac{\sin \pi(\lambda - n + \delta)}{\pi(\lambda - n + \delta)},$$

where $\delta = c - \alpha/\pi$, for some arbitrary constant c. When $\alpha = 0$ we obtain the Hartley transform

$$F(\lambda) = \int_0^{\pi} \{\cos \lambda x + \sin \lambda x\} f(x)\, dx,$$

and

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(n - c) \frac{\sin \pi(\lambda - n + c)}{\pi(\lambda - n + c)}.$$

ACKNOWLEDGMENT

This work was done while the first author was visiting the "Universidad Carlos III de Madrid," Spain and he takes this opportunity to thank the Mathematics Department for its hospitality and stimulating atmosphere.

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