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A General Sampling Theorem Associated with Differential Operators

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In this paper we prove a general sampling theorem associated with differential operators with compact resolvent. Thus, we are able to recover, through a Lagrange-type interpolatory series, functions defined by means of a linear integral transform. The kernel of this transform is related with the resolvent of the differential operator. Most of the well-known sampling theorems associated with differential operators are shown to be nothing but limit cases of this result.

KEY WORDS: Kramer's sampling theorem; symmetric and self-adjoint operators; compact resolvents; Hilbert–Schmidt operators; Lagrange-type interpolatory series.

1. INTRODUCTION

The classical Whittaker–Shanonn–Kotel'nikov (WSK) sampling theorem given by the formula

$$f(\lambda)=\sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(\lambda-n)}{\pi(\lambda-n)} \tag{1}$$

holds for functions in $L^2(\mathbb{R})$ whose Fourier transform has support in $[-\pi, \pi]$, i.e.,

$$f(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} F(\omega)\, e^{i\lambda\omega}\, d\omega,$$

for some $F\in L^2[-\pi, \pi]$. The sampling series (1) converges absolutely on $\mathbb{C}$, and uniformly on horizontal strips of $\mathbb{C}$ (in particular, on $\mathbb{R}$).

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One direction in which this therorem has been generalized is replacing the kernel function, $e^{i\lambda\omega}$, by a more general kernel $K(\omega, \lambda)$, leading to the following generalization by Kramer [6]: Let $K(\omega, \lambda)$ be a function, continuous in λ such that, as a function of ω, $K(\omega, \lambda) \in L^2(I)$ for every real number λ, where I is an interval of the real line. Assume that there exists a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that $\{K(\omega, \lambda_n)\}_{n\in\mathbb{Z}}$ is a complete orthogonal sequence of functions of $L^2(I)$. Then for any f of the form

$$f(\lambda) = \int_I F(\omega)K(\omega, \lambda)\, d\omega,$$

where $F \in L^2(I)$, we have

$$f(\lambda) = \sum_{n=-\infty}^{\infty} f(\lambda_n) S_n(\lambda), \tag{2}$$

with

$$S_n(\lambda) = \frac{\int_I K(\omega, \lambda)\overline{K(\omega, \lambda_n)}\, d\omega}{\int_I |K(\omega, \lambda_n)|^2\, d\omega}. \tag{3}$$

The series (2) converges absolutely and uniformly wherever $\|K(\cdot, \lambda)\|_{L^2(I)}$ is bounded. In particular, if $I = [-\pi, \pi]$, $K(\omega, \lambda) = e^{i\lambda\omega}$ and $\{\lambda_n = n\}_{n\in\mathbb{Z}}$, we get the WSK sampling theorem.

One way to generate the kernel $K(\omega, \lambda)$ and the sampling points $\{\lambda_n\}_{n\in\mathbb{Z}}$ is to consider Sturm–Liouville boundary-value problems [10,13]. Another possibility is to use the Green functions method described in Ref. [12]. For many self-adjoint boundary-value problems, the Green function can be written in the form

$$G_\lambda(x, y) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda - \lambda_n},$$

where $\{\lambda_n\}_{n=0}^{\infty}$ are the eigenvalues and $\{\phi_n\}_{n=0}^{\infty}$ the corresponding eigenfunctions. The Green function method can also be used to derive sampling theorems associated with Fredholm integral operators [1], and, as we will see in this paper, to derive sampling theorems associated with a densely defined self-adjoint operator whose resolvent is a Hilbert–Schmidt operator.

A general way to generate the kernel $K(x, \lambda)$ is to consider a symmetric, densely defined, differential operator $A: \mathscr{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ with compact resolvent R_λ for λ in the resolvent set. In this case, we have a sequence of eigenvalues and an orthornormal basis of eigenfunctions (see [4], [5] for the general theory). Thus, we can derive, at least in the cases where all the eigenspaces are one-dimensional, a kernel verifying all the requirements in Kramer's theorem. Namely, let

$$K_g(x,\lambda) = P(\lambda)[R_\lambda g](x), \tag{4}$$

where Ω is a domain of $\mathbb{R}^n$, $g \in L^2(\Omega)$ and $P(\lambda)$ is the canonical product of the eigenvalues, provided this product exists. Multiplication by $P(\lambda)$ removes the singularities in the resolvent and, therefore, $K_g(x,\lambda)$ can be defined on $\Omega \times \mathbb{C}$. The hypothesis about the existence of $P(\lambda)$ will allow us to write the sampling functions in (3) as Lagrange-type interpolation functions.

Defining kernels as in (4) allows one degree of freedom: The choice of function g. Through particular choices we will obtain that most of the well-known sampling theorems associated with differential or integral operators [1,10,12] are limit cases of the main result of this paper.

2. A GENERAL SAMPLING THEOREM

Let $A: \mathscr{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ be a symmetric (formally self-adjoint) linear operator, densely defined on $L^2(\Omega)$. Assume that there exists an inverse operator $T = A^{-1}$, compact and defined on $L^2(\Omega)$. We know from the spectral theorem for symmetric compact operators defined on a Hilbert space [7] that T has discrete spectrum. Moreover, if $\{\mu_n\}_{n=0}^{\infty}$ is the sequence of eigenvalues of T, then $\lim_{n\to\infty} |\mu_n| = 0$. We may assume

$$|\mu_0| \geq |\mu_1| \geq \cdots \geq |\mu_n| \geq \cdots$$

Moreover, the eigenspace associated with each eigenvalue μ_n is finite dimensional. Set $k_n = \dim \operatorname{Ker}(\mu_n I - T) < \infty$. Note that 0 is not an eigenvalue of T, so the sequence $\{\phi_n\}_{n=0}^{\infty}$ of eigenfunctions of T is a complete orthonormal system (applying the Gram–Schmidt method in each eigenspace) of $L^2(\Omega)$. The sequences $\{\lambda_n = \mu_n^{-1}\}_{n=0}^{\infty}$ and $\{\phi_n\}_{n=0}^{\infty}$ are, respectively, the sequence of eigenvalues and the sequence of associated eigenfunctions of the operator A. Since $\lim_{n\to\infty} |\mu_n| = 0$, we have

$$0 < |\lambda_0| \leq |\lambda_1| \leq \cdots \leq |\lambda_n| \leq \cdots$$

and $\lim_{n\to\infty} |\lambda_n| = \infty$.

We assume that the exponent of convergence of the sequence $\{\lambda_n\}_{n=0}^{\infty}$ is finite, i.e.,

$$\eta = \inf\left\{\alpha > 0 \,\middle|\, \sum_{k=0}^{\infty} \frac{1}{|\lambda_k|^\alpha} < +\infty\right\} < \infty.$$

Hence, we can define the canonical product, $P(\lambda)$, associated with the sequence of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ as

$$P(\lambda) = \begin{cases} \prod_{n=0}^{\infty} (1-\lambda/\lambda_n)\exp(\sum_{i=0}^{p}(1/i)(\lambda/\lambda_n)^i) & \text{if } p \geq 1 \\ \prod_{n=0}^{\infty}(1-\lambda/\lambda_n) & \text{if } p=0 \end{cases} \tag{5}$$

where p is the smallest non-negative integer larger than $\eta - 1$. $P(\lambda)$ is an entire function whose zeros are $\{\lambda_n\}_{n=0}^{\infty}$.

The following lemma, which appears in [11], will be needed.

Lemma 1. Let $\mathscr{K}$ be a compact subset of $\mathbb{C}$. Then, there exists a constant $C_{\mathscr{K}}$ such that

$$\sup_{\lambda\in\mathscr{K}} \left|\frac{P(\lambda)}{\lambda-\lambda_n}\right| \leq C_{\mathscr{K}}, \qquad \text{for any } n\in\mathbb{N}_0 = \mathbb{N}\cup\{0\}.$$

Now let us fix a function $g\in L^2(\Omega)$. We define the kernel

$$\Phi_g(x,\lambda) \doteq P(\lambda)[R_\lambda g](x) \qquad x\in\Omega, \qquad \lambda\in\mathbb{C},$$

where $R_\lambda = (\lambda I - A)^{-1}$ is the resolvent operator of A. Under these conditions the following sampling theorem holds

Theorem 1. Let f be defined as

$$f(\lambda) = \int_\Omega F(x)\Phi_g(x,\lambda)\,dx,$$

where $F\in L^2(\Omega)$. Then, f is an entire function which can be recovered from its values on the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of A through the Lagrange-type interpolation series

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}, \tag{6}$$

where $P(\lambda)$ is given by (5). The convergence of the series in (6) is absolute and uniform on compact subsets of $\mathbb{C}$.

Proof. Since $\dim \mathrm{Ker}(\lambda_n Id - A) = k_n$, we can arrange the sequence of eigenfunctions of A as $\{\phi_{n,i}\}$, where $n\in\mathbb{N}_0$ and $i=1,2,\ldots,k_n$. Since $[R_\lambda g]$ can be expanded in $L^2(\Omega)$ [7] as

$$[R_\lambda g](x) = \sum_{n=0}^{\infty}\left(\frac{1}{\lambda-\lambda_n}\sum_{i=1}^{k_n}\langle g,\phi_{n,i}\rangle_{L^2(\Omega)}\phi_{n,i}(x)\right), \tag{7}$$

the kernel $\Phi_g(x,\lambda)$ is a function of $L^2(\Omega)$ in $x\in\Omega$ and an entire function in

$\lambda \in \mathbb{C}$. Applying the Cauchy–Schwarz inequality, we conclude that $f(\lambda)$ is well defined for any $\lambda \in \mathbb{C}$.

If we expand F and $\Phi_g(\cdot, \lambda)$ respect to the basis $\{\phi_{n,i}\}$, we obtain

$$F(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{k_n} \langle F, \phi_{n,i} \rangle_{L^2(\Omega)} \phi_{n,i}(x)$$

and

$$\Phi_g(x, \lambda) = \sum_{n=0}^{\infty} \frac{P(\lambda)}{\lambda - \lambda_n} \sum_{i=1}^{k_n} \langle g, \phi_{n,i} \rangle_{L^2(\Omega)} \phi_{n,i}(x).$$

Now, using Parseval's identity, we have

$$f(\lambda) = \langle F, \overline{\Phi_g(\cdot, \lambda)} \rangle_{L^2(\Omega)} = \sum_{n=0}^{\infty} \frac{P(\lambda)}{\lambda - \lambda_n} \sum_{i=1}^{k_n} \langle g, \phi_{n,i} \rangle_{L^2(\Omega)} \langle F, \phi_{n,i} \rangle_{L^2(\Omega)}. \tag{8}$$

On the other hand

$$f(\lambda_n) = \lim_{\lambda \to \lambda_n} \int_{\Omega} F(x) \Phi_g(x, \lambda)\, dx = \int_{\Omega} F(x) \lim_{\lambda \to \lambda_n} \Phi_g(x, \lambda)\, dx,$$

and since

$$\begin{aligned} \lim_{\lambda \to \lambda_n} \Phi_g(x, \lambda) &= \lim_{\lambda \to \lambda_n} \left[\frac{P(\lambda)}{\lambda - \lambda_n} \sum_{i=1}^{k_n} \langle g, \phi_{n,i} \rangle_{L^2(\Omega)} \phi_{n,i}(x) \right. \\ &\qquad \left. + P(\lambda) \sum_{m \neq n}^{\infty} \frac{1}{\lambda - \lambda_m} \sum_{i=1}^{k_m} \langle g, \phi_{m,i} \rangle_{L^2(\Omega)} \phi_{m,i}(x) \right] \\ &= P'(\lambda_n) \sum_{i=1}^{k_n} \langle g, \phi_{n,i} \rangle_{L^2(\Omega)} \phi_{n,i}(x), \end{aligned}$$

we obtain

$$\begin{aligned} f(\lambda_n) &= P'(\lambda_n) \sum_{i=1}^{k_n} \langle g, \phi_{n,i} \rangle_{L^2(\Omega)} \int_{\Omega} F(x) \phi_{n,i}(x)\, dx \\ &= P'(\lambda_n) \sum_{i=1}^{k_n} \langle g, \phi_{n,i} \rangle_{L^2(\Omega)} \langle F, \phi_{n,i} \rangle_{L^2(\Omega)}. \end{aligned}$$

Hence, we can write (8) as

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{P(\lambda)}{(\lambda - \lambda_n) P'(\lambda_n)}. \tag{9}$$

Let $\mathscr{K}\subset\mathbb{C}$ be a compact subset of $\mathbb{C}$. We will prove that the series in (9) converges absolutely and uniformly on $\mathscr{K}$. Indeed,

$$\begin{aligned}
&\left|f(\lambda)-\sum_{n=0}^{N} f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\right| \\
&=\left|\sum_{n=N+1}^{\infty} f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\right| \\
&\le \sum_{n=N+1}^{\infty}\left|f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\right| \\
&\le \sum_{n=N+1}^{\infty}\left(\left|\frac{P(\lambda)}{\lambda-\lambda_n}\right|\sum_{i=1}^{k_n}|\langle g,\phi_{n,i}\rangle_{L^2(\Omega)}||\langle F,\phi_{n,i}\rangle_{L^2(\Omega)}|\right) \\
&\le C_{\mathscr{K}}\left(\sum_{n=N+1}^{\infty}\sum_{i=1}^{k_n}|\langle F,\phi_{n,i}\rangle|^2\right)^{1/2}\left(\sum_{n=N+1}^{\infty}\sum_{i=1}^{k_n}|\langle g,\phi_{n,i}\rangle|^2\right)^{1/2}.
\end{aligned}\tag{10}$$

The last inequality follows from the Cauchy–Schwarz inequality and Lemma 1. Since $F, g\in L^2(\Omega)$, the last term in (10) goes to 0 as $N\to\infty$, regardless of $\lambda\in\mathscr{K}$. The uniform convergence of the series in (9) on compact subsets of $\mathbb{C}$ implies that $f(\lambda)$ is an entire function. □

3. SAMPLING THEOREMS AS LIMIT CASES

As we pointed out in the Introduction, in this section we will prove that most of the well-known sampling theorems are indeed limit cases of Theorem 1. We concentrate on two cases.

3.1. Sampling Theorems Associated with Sturm–Liouville Problems: The Singular Case on the Halfline

In [10] Zayed extends the Weiss–Kramer sampling theorem [6,9] to a singular Sturm–Liouville problem on the halfline $[0,\infty)$.

Consider the following singular Sturm–Liouville boundary-value problem:

$$-y''+q(x)y=\lambda y,\qquad 0\le x<\infty,\tag{11}$$

$$y(0)\cos\alpha+y'(0)\sin\alpha=0,\tag{12}$$

where $q(x)$ is a continuous function on $[0, \infty)$. Let $\phi(x, \lambda)$, $\theta(x, \lambda)$ be the solutions of (11) such that

$$\phi(0) = \sin\alpha, \qquad \phi'(0) = -\cos\alpha,$$
$$\theta(0) = \cos\alpha, \qquad \theta'(0) = \sin\alpha.$$

From the Weyl–Titchmarsh theory [8] it follows that there exists a complex valued function $m_\infty(\lambda)$ such that for every nonreal λ, Eq. (11) has a solution $\psi_\infty(x, \lambda) = \theta(x, \lambda) + m_\infty(\lambda)\phi(x, \lambda)$ belonging to $L^2(0, \infty)$. In the so-called *limit-point case* $m_\infty(\lambda)$ is unique, while in the *limit-circle case* there are uncountably many such functions (see [8] for details).

The function $m_\infty(\lambda)$ is analytic in $\mathbb{C} - \mathbb{R}$ and, if it has poles on the real axis, they are all simple. Although, in general, the function $m_\infty(\lambda)$ cannot be extended to a meromorphic function defined on $\mathbb{C}$, we assume in what follows that $m_\infty(\lambda)$ is a meromorphic function, real valued on the real axis and whose singularities are simple poles on the real axis. For $m_\infty(\lambda)$ to satisfy these assumptions, it is sufficient [8] that $\lim_{x\to\infty} q(x) = +\infty$. The poles of $m_\infty(\lambda)$ are the eigenvalues of problem (11–12). These form an increasing sequence of real numbers, $\{\lambda_n\}_{n=0}^{\infty}$, whose only accumulation point is ∞. Without loss of generality we may assume that all the eigenvalues are different from zero. The eigenfunction corresponding to the eigenvalue λ_n is given by $\phi_n(x) = \phi(x, \lambda_n)$.

Theorem 2. (see [10]. Consider the singular Sturm–Liouville problem

$$-y'' + q(x)y = \lambda y, \qquad 0 \le x < \infty,$$
$$y(0)\cos\alpha + y'(0)\sin\alpha = 0,$$

where $q(x)$ is a continuous function on $[0, \infty)$. Assume that $m_\infty(\lambda)$ is a meromorphic function, real valued on the real axis and whose only singularities are simple poles $\{\lambda_n\}_{n=0}^{\infty}$ on the positive real axis. Further, we suppose that its exponent of convergence is finite.

Let f be defined as

$$f(\lambda) = \int_0^\infty F(s)\Phi(s, \lambda)\, ds,$$

where $F \in L^2(0, \infty)$, $\Phi(x, \lambda) = P(\lambda)\psi_\infty(x, \lambda)$, and $P(\lambda)$ is the canonical product (5) associated with $\{\lambda_n\}_{n=0}^{\infty}$. Then, f is an entire function that can be recovered through the Lagrange-type interpolation series

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{P(\lambda)}{(\lambda - \lambda_n)P'(\lambda_n)}.$$

The convergence of the series is absolute and uniform on compact subsets of $\mathbb{C}$.

In order to obtain Theorem 2 as a limit case of Theorem 1, we have to rewrite the problem (11–12) in the language of the differential operator theory. First, we are reminded how to define a self-adjoint operator $A: \mathscr{D}(A) \subset L^2(0,\infty) \to L^2(0,\infty)$ associated with the problem (11–12). Let $T(\tau) = \tau|_{\mathscr{D}(\tau)}$, where $\tau f = -f'' + qf$ and $\mathscr{D}(\tau) = \{y, y' \in AC_{\text{loc}}[0,\infty) | \tau y \in L^2(0,\infty)\}$. Following the general theory in Edmund and Evans [5], we can construct a self-adjoint extension $A: \mathscr{D}(A) \subset L^2(0,\infty) \to L^2(0,\infty)$ of $T_0(\tau)$, the closure of the restriction of $T(\tau)$ to

$$\mathscr{D}_0'(\tau) = \{y \in \mathscr{D}(\tau) | y(a) = y'(a) = 0, \text{ and } y = 0 \text{ outside a compact subset of } [0,\infty)\}.$$

In the limit–point case, the maximal operator $T(\tau)$ associated with the problem (11–12) is self-adjoint, i.e., $A = T(\tau)$ and $\mathscr{D}(A) = \mathscr{D}(\tau)$. On the other hand, in the limit–circle case, we need to add a suitable boundary condition at the infinity to the operator $T(\tau)$ in order to obtain a densely defined and self-adjoint operator in $L^2(0,\infty) A: \mathscr{D}(A) \subset L^2(0,\infty) \to L^2(0,\infty)$ (see [5], [8] for details).

Under the hypotheses of Theorem 2, the resolvent operator of the self-adjoint operator A associated with the problem (11–12) is compact. Indeed, the spectrum is discrete and the sequence of eigenfunctions constitutes an orthogonal basis of $L^2(0,\infty)$ [5].

Now, we show that Zayed's theorem is a consequence of Theorem 1. Consider the singular Sturm–Liouville problem (11–12) and assume the hypotheses in Theorem 2 hold. Let $A: \mathscr{D}(A) \subset L^2(0,\infty) \to L^2(0,\infty)$ be a self-adjoint operator associated with the problem and let R_λ be its compact resolvent operator. Since $\mathscr{D}(\mathscr{A})$ is a dense subset of $L^2(0,\infty)$, there exists a sequence $\{g_n\}_{n=0}^{\infty} \subset L^2(0,\infty)$ such that

$$R_\lambda g_n \to \psi_\infty(\cdot, \lambda) \qquad \text{for any } \lambda \in \mathbb{C} - \{\lambda_n\}_{n=0}^{\infty} \tag{13}$$

in $L^2(0,\infty)$. We note that $\psi_\infty(x,\cdot)$ can be extended to $\mathbb{C} - \{\lambda_n\}_{n=0}^{\infty}$ by means of the formula

$$\psi_\infty(x,\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n} \frac{\phi_n(x)}{\|\phi_n\|^2_{L^2(0,\infty)}} \qquad \text{for any } \lambda \in \mathbb{C} - \{\lambda_n\}_{n=0}^{\infty}. \tag{14}$$

Let $\Phi(x,\lambda) = P(\lambda)\psi_\infty(x,\lambda)$ and $f(\lambda) = \int_0^\infty \Phi(x,\lambda) F(x)\,dx$, where $F \in L^2(0,\infty)$. If we consider $f_n(\lambda) = \int_0^\infty \Phi_n(x,\lambda) g(x)\,dx$, where $\Phi_n(x,\lambda) = P(\lambda) R_\lambda g_n(x)$, using Theorem 1, for each $n \in \mathbb{N}_0$ we have

$$f_n(\lambda) = \sum_{m=0}^{\infty} f_n(\lambda_m) \frac{P(\lambda)}{(\lambda - \lambda_m) P'(\lambda_m)} \qquad \text{for any } \lambda \in \mathbb{C}.$$

Taking limits as $n \to \infty$, we will prove that $\lim_{n\to\infty} f_n(\lambda) = f(\lambda)$ and

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} f_n(\lambda_m) \frac{P(\lambda)}{(\lambda-\lambda_m)P'(\lambda_m)} = \sum_{m=0}^{\infty} f(\lambda_m) \frac{P(\lambda_m)}{(\lambda-\lambda_m)P'(\lambda_m)}, \tag{15}$$

and therefore we obtain the sampling expansion in Theorem 2.

In fact, from (13) we have that $\Phi_n(\cdot, \lambda) \to \Phi(\cdot, \lambda)$ in $L^2(0, \infty)$ for $\lambda \in \mathbb{C} - \{\lambda_n\}_{n=0}^{\infty}$. Using the Cauchy–Schwarz inequality, we obtain that

$$\lim_{n\to\infty} f_n(\lambda) = f(\lambda) \qquad \text{for any } \lambda \in \mathbb{C} - \{\lambda_n\}_{n=0}^{\infty}.$$

Now, we prove that $\lim_{n\to\infty} f_n(\lambda_m) = f(\lambda_m)$ for any eigenvalue λ_m. Taking into account that for any m

$$R_\lambda g_m(x) = \sum_{n=0}^{\infty} \frac{\langle g_m, \phi_n \rangle}{\lambda - \lambda_n} \frac{\phi_n(x)}{\|\phi_n\|^2_{L^2(0,\infty)}} \qquad \text{for any } \lambda \in \mathbb{C} - \{\lambda_n\}_{n=0}^{\infty}, \tag{16}$$

we can choose the sequence $\{g_n\}_{n=0}^{\infty} \subset L^2(0, \infty)$ verifying (13) and the condition

$$\langle g_n, \phi_m \rangle = \begin{cases} 0, & \text{if } m > n \\ 1, & \text{if } m \le n. \end{cases}$$

Now, from (14) and (16) we have that

$$f(\lambda_m) = \lim_{\lambda \to \lambda_m} \int_0^\infty \Phi(x, \lambda) F(x)\, dx = P'(\lambda_m) \langle F, \phi_m \rangle \frac{1}{\|\phi_m\|^2}$$

and

$$f_n(\lambda_m) = \lim_{\lambda \to \lambda_m} \int_0^\infty \Phi_n(x, \lambda) F(x)\, dx = P'(\lambda_m) \langle g_n, \phi_m \rangle \langle F, \phi_m \rangle \frac{1}{\|\phi_m\|^2}.$$

Since $\lim_{n\to\infty} \langle g_n, \phi_m \rangle = 1$ for any m, we have that $\lim_{n\to\infty} f_n(\lambda_m) = f(\lambda_m)$ for any m.

The interchange of the limit and the series in (15), is a consequence of the Moore–Smith Theorem, the proof of which can be found in [3].

Theorem 3. Let M be a complete metric space with metric ρ, and let $\{x_{n,m}\}$, $n, m \in \mathbb{N}_0$, be given. Assume there are sequences $\{y_n\}$, $\{z_m\}$ in M such that (1) $\lim_{n\to\infty} \rho(x_{n,m}, z_m) = 0$ uniformly in m, and (2) for each $n \in \mathbb{N}_0$, $\lim_{m\to\infty} \rho(x_{n,m}, y_n) = 0$. Then there is $x \in M$ such that

$$\lim_{m\to\infty} \lim_{n\to\infty} \rho(x_{n,m}, x) = \lim_{n\to\infty} \lim_{m\to\infty} \rho(x_{n,m}, x) = \lim_{m,n\to\infty} \rho(x_{n,m}, x) = 0.$$

Indeed, if we denote

$$x_{n,m}(\lambda) = \sum_{i=0}^{n} f_m(\lambda_i) \frac{P(\lambda)}{(\lambda - \lambda_i)P'(\lambda_i)},$$

where $\lambda \in \mathcal{K}$, $\mathcal{K}$ a fixed compact of $\mathbb{C}$, then, $x_{n,m}(\lambda) \to z_m(\lambda) = f_m(\lambda)$ as $n \to \infty$ and

$$x_{n,m}(\lambda) \to y_n(\lambda) = \sum_{i=0}^{n} f(\lambda_i) \frac{P(\lambda)}{(\lambda - \lambda_i)P'(\lambda_i)},$$

as $m \to \infty$. The first limit above is uniform in m. Indeed, using (10), (13), and (16) we conclude that

$$|x_{n,m}(\lambda) - z_m(\lambda)| \leq M \|\Phi(\cdot, \lambda)\|_{L^2(0,\infty)} \left(\sum_{i=n+1}^{\infty} \frac{|\langle F, \phi_i \rangle|^2}{\|\phi_i\|^2} \right)^{1/2}, \tag{17}$$

where $\Phi(x, \lambda) = P(\lambda)\psi_\infty(x, \lambda)$ and M is a suitable constant. We have that (see [11, p. 120])

$$\|\Phi(\cdot, \lambda)\|_{L^2(0,\infty)} = \left(\sum_{i=0}^{\infty} \frac{|P(\lambda)|^2}{|\lambda - \lambda_i|^2} \right)^{1/2} \leq C_{\mathcal{K}},$$

where $C_{\mathcal{K}}$ is a constant depending only of $\mathcal{K}$, therefore, the right-hand term of (17) goes to zero as $n \to \infty$ regardless of m, and of $\lambda \in \mathcal{K}$.

3.2. Operators Whose Resolvent is of Hilbert–Schmidt Type

When the resolvent operator R_λ of a symmetric, densely defined, operator A: $\mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ is of Hilbert–Schmidt type, it has a symmetric kernel $G_\lambda \in L^2(\Omega \times \Omega)$ [2], such that

$$[R_\lambda f](x) = \int_\Omega G_\lambda(x, y) f(y)\, dy \tag{18}$$

for any $f \in L^2(\Omega)$. In this case, it is possible to obtain a sampling theorem where the resolvent kernel will determine the class of functions that can be recovered through a sampling expansion.

Fixing $y_0 \in \Omega$, we define the kernel

$$\Psi(x, \lambda) = P(\lambda) G_\lambda(x, y_0) \qquad x \in \Omega, \qquad \lambda \in \mathbb{C},$$

where $P(\lambda)$ is the canonical product associated with the sequence of eigenvalues, $\{\lambda_n\}_{n=0}^{\infty}$, of A. Since the resolvent operator is of Hilbert–Schmidt type, we have that $\sum_{n=0}^{\infty} 1/|\lambda_n|^2 < \infty$. Therefore, the canonical product $P(\lambda)$ will be

$$P(\lambda) = \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \exp\left(\frac{\lambda}{\lambda_n}\right)$$

if $\sum_{n=0}^{\infty} |\lambda_n|^{-1} = \infty$, or

$$P(\lambda) = \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)$$

if $\sum_{n=0}^{\infty} |\lambda_n|^{-1} < \infty$.

Theorem 4. Any function f defined as

$$f(\lambda) = \int_{\Omega} F(x)\Psi(x, \lambda)\, dx,$$

where $F \in L^2(\Omega)$, is an entire function, which can be recovered through a Lagrange interpolation-type series as

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{P(\lambda)}{(\lambda - \lambda_n) P'(\lambda_n)}. \tag{19}$$

The convergence of the series in (19) is absolute and uniform on compact subsets of $\mathbb{C}$.

Proof. It is known [2] that the kernel G_λ can be written as

$$G_\lambda(x, y) = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n} \sum_{i=1}^{k_n} \phi_{n,i}(x)\phi_{n,i}(y), \tag{20}$$

where $\{\phi_{n,i}\}_{i=1}^{k_n}$ are the eigenfunctions associated with the eigenvalue λ_n and k_n is the dimension of the eigenspace associated with λ_n. Without loss of generality, we suppose that $\{\phi_{n,i}\}$ is an orthonormal basis of $L^2(\Omega)$.

From (20), we have that $\Psi(x, \lambda)$ is a function of $L^2(\Omega)$ in x and an entire function in λ. Using the Cauchy–Schwarz inequality, we conclude that f is well defined.

Expanding F and $\Psi(\cdot, \lambda)$ with respect to the basis of eigenfunctions, we obtain

$$F(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{k_n} \langle F, \phi_{n,i} \rangle_{L^2(\Omega)} \phi_{n,i}(x)$$

and

$$\Psi(x, \lambda) = \sum_{n=0}^{\infty} \frac{P(\lambda)}{\lambda - \lambda_n} \sum_{i=1}^{k_n} \phi_{n,i}(y_0)\phi_{n,i}(x).$$

Applying Parseval's identity, we have that

$$f(\lambda) = \langle F, \overline{\Psi(\cdot, \lambda)}\rangle_{L^2(\Omega)} = \sum_{n=0}^{\infty} \frac{P(\lambda)}{\lambda - \lambda_n} \sum_{i=1}^{k_n} \langle F, \phi_{n,i}\rangle_{L^2(\Omega)}\phi_{n,i}(y_0). \tag{21}$$

From this we obtain $f(\lambda_n)$ on the eigenvalues as in Theorem 1. Indeed,

$$\begin{aligned} f(\lambda_n) = \lim_{\lambda\to\lambda_n} f(\lambda) &= \int_\Omega F(x) \lim_{\lambda\to\lambda_n} \Psi(x, \lambda)\, dx \\ &= P'(\lambda_n) \sum_{i=1}^{k_n} \langle F, \phi_{n,i}\rangle_{L^2(\Omega)}\phi_{n,i}(y_0). \end{aligned}$$

Therefore

$$\frac{f(\lambda_n)}{P'(\lambda_n)} = \sum_{i=1}^{k_n} \langle F, \phi_{n,i}\rangle_{L^2(\Omega)}\phi_{n,i}(y_0).$$

Substituting in (21) we obtain that

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{P(\lambda)}{(\lambda - \lambda_n)P'(\lambda_n)}.$$

Let $\mathscr{K} \subset \mathbb{C}$ be a compact subset of $\mathbb{C}$. We will prove that the series (19) converges absolutely and uniformly on $\mathscr{K}$. There exist $R > 0$ and $M_0 \in \mathbb{N}_0$ such that $\mathscr{K} \subset \{z: |z| \leq R\}$ and $|\lambda_n| \geq 2R$ for any $n \geq M_0 + 1$. Let $N \geq M_0$

$$\sum_{n=N+1}^{\infty} \left(\left| \frac{P(\lambda)}{\lambda - \lambda_n} \right|^2 \sum_{i=1}^{k_n} |\phi_{n,i}(y_0)|^2 \right) \leq |P(\lambda)|^2 \sum_{n=N+1}^{\infty} \left(\frac{\sum_{i=1}^{k_n} |\phi_{n,i}(y_0)|^2}{(|\lambda_n| - R)^2} \right)$$

The series in the right-hand side converges since it has the same character as

$$\sum_{n=0}^{\infty} \left(\frac{\sum_{i=1}^{k_n} |\phi_{n,i}(y_0)|^2}{|\lambda_n|^2} \right),$$

and this series converges applying Parseval's Theorem to $G_0(x, y_0) \in L^2(\Omega)$. Hence, there exists a positive constant $C_{\mathscr{K}}$, independent of $\lambda \in \mathscr{K}$, such that

$$\sum_{n=N+1}^{\infty} \left(\left| \frac{P(\lambda)}{\lambda - \lambda_n} \right|^2 \sum_{i=1}^{k_n} |\phi_{n,i}(y_0)|^2 \right) \leq C_{\mathscr{K}}.$$

Now, applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left|f(\lambda)-\sum_{n=0}^{N} f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\right|^2\\
&=\left|\sum_{n=N+1}^{\infty} f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\right|^2\\
&\le\left(\sum_{n=N+1}^{\infty}\left|f(\lambda_n)\frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\right|\right)^2\\
&\le\left(\sum_{n=N+1}^{\infty}\sum_{i=1}^{k_n}|\langle F,\phi_{n,i}\rangle_{L^2(\Omega)}|\frac{|P(\lambda)||\phi_{n,i}(y_0)|}{|\lambda-\lambda_n|}\right)^2\\
&\le\left(\sum_{n=N+1}^{\infty}\sum_{i=1}^{k_n}|\langle F,\phi_{n,i}\rangle|^2\right)\left(\sum_{n=N+1}^{\infty}\sum_{i=1}^{k_n}\frac{|P(\lambda)|^2|\phi_{n,i}(y_0)|^2}{|\lambda-\lambda_n|^2}\right)\\
&\le C_{\mathscr{K}}\left(\sum_{n=N+1}^{\infty}\sum_{i=1}^{k_n}|\langle F,\phi_{n,i}\rangle|^2\right).
\end{aligned}$$

Since $F\in L^2(\Omega)$, the last series above goes to zero as $N\to\infty$ regardless of $\lambda\in\mathscr{K}$. The uniform convergence of the series in (19) on compact subsets of $\mathbb{C}$ implies that $f(\lambda)$ is an entire function. □

Formally, if we take a sequence $\{g_n\}_{n=0}^{\infty}$ converging to δ_{y_0}, the Dirac delta in y_0, then $R_\lambda g_n(x)\to G_\lambda(x,y_0)$ in (18) and this suggests that Theorem 4 is a limit case of Theorem 1. More precisely, let A: $\mathscr{D}(A)\subset L^2(\Omega)\to L^2(\Omega)$ be a symmetric operator, densely defined and whose resolvent operator, R_λ, is of Hilbert–Schmidt type. Let $G_\lambda(x,y)\in L^2(\Omega\times\Omega)$ be the resolvent kernel of R_λ in (18).

Fixing $y_0\in\Omega$, we have that $G_\lambda(x,y_0)\in L^2(\Omega)$ and since $\mathscr{D}(\mathscr{A})$ is a dense subset of $L^2(\Omega)$, there exists a sequence $\{g_n\}_{n=0}^{\infty}\subset L^2(\Omega)$ such that

$$R_\lambda g_n\to G_\lambda(\cdot,y_0)\qquad\text{in } L^2(\Omega)\text{ for any }\lambda\in\mathbb{C}-\{\lambda_n\}_{n=0}^{\infty}. \tag{22}$$

Let $\Phi(x,\lambda)=P(\lambda)G_\lambda(x,y_0)$ and $f(\lambda)=\int_\Omega\Phi(x,\lambda)F(x)\,dx$, where $F\in L^2(\Omega)$.

If we consider

$$f_n(\lambda)=\int_\Omega\Phi_n(x,\lambda)F(x)\,dx,$$

where $\Phi_n(x,\lambda)=P(\lambda)R_\lambda g_n(x)$, using Theorem 1 for each $n\in\mathbb{N}_0$ we have

$$f_n(\lambda)=\sum_{m=0}^{\infty}f_n(\lambda_m)\frac{P(\lambda)}{((\lambda-\lambda_m)P'(\lambda_m)}\qquad\text{for any }\lambda\in\mathbb{C}. \tag{23}$$

From (22) we have that $\Phi_n(\cdot,\lambda)\to\Phi(\cdot,\lambda)$ in $L^2(\Omega)$ as $n\to\infty$ for $\lambda\in\mathbb{C}-\{\lambda_n\}_{n=0}^{\infty}$. Applying the Cauchy–Schwarz inequality, we obtain that $\lim_{n\to\infty} f_n(\lambda)=f(\lambda)$ for $\lambda\in\mathbb{C}-\{\lambda_n\}_{n=0}^{\infty}$.

Next, we prove that $\lim_{n\to\infty} f_n(\lambda_m)=f(\lambda_m)$ for any eigenvalue λ_m. Taking into account (7) and (20), we can choose the sequence $\{g_n\}_{n=0}^{\infty}\subset L^2(\Omega)$ verifying (22) and such that

$$\langle g_n,\phi_{m,i}\rangle=\begin{cases}0, & \text{if } m>n,\\ \phi_{m,i}(y_0), & \text{if } m\le n,\end{cases}\qquad \text{for any } 1\le i\le k_m.$$

Now, from (7) and (20) we have that

$$f(\lambda_m)=\lim_{\lambda\to m}\int_0^{\infty}\Phi(x,\lambda)F(x)\,dx=P'(\lambda_m)\sum_{i=1}^{k_m}\langle F,\phi_{m,i}\rangle\phi_{m,i}(y_0)$$

and

$$\begin{aligned} f_n(\lambda_m)&=\lim_{\lambda\to\lambda_m}\int_0^{\infty}\Phi_n(x,\lambda)F(x)\,dx\\ &=P'(\lambda_m)\sum_{i=1}^{k_m}\langle g_n,\phi_{m,i}\rangle\langle F,\phi_{m,i}\rangle. \end{aligned}$$

Since $\lim_{n\to\infty}\langle g_n,\phi_{m,i}\rangle=\phi_{m,i}(y_0)$ for any m and $1\le i\le k_m$, it follows that $\lim_{n\to\infty} f_n(\lambda_m)=f(\lambda_m)$ for any m. Taking limits as $n\to\infty$ in (23) we obtain

$$\begin{aligned} f(\lambda)&=\lim_{n\to\infty}\sum_{m=0}^{\infty} f_n(\lambda_m)\frac{P(\lambda)}{(\lambda-\lambda_m)P'(\lambda_m)}\\ &=\sum_{m=0}^{\infty} f(\lambda_m)\frac{P(\lambda)}{(\lambda-\lambda_m)P'(\lambda_m)}. \end{aligned}\tag{24}$$

Again the interchange of the limit and the series in (24) follows from the Moore–Smith Theorem as in the subsection 3.1.

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