Semi-direct product of groups, filter banks and sampling

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Abstract

An abstract sampling theory associated to a unitary representation of a countable discrete non abelian group G, which is a semi-direct product of groups, on a separable Hilbert space is studied. A suitable expression of the data samples and the use of a filter bank formalism allows to fix the mathematical problem to be solved: the search of appropriate dual frames for $\ell^2(G)$. An example involving crystallographic groups illustrates the obtained results by using average or pointwise samples.

Keywords: Semi-direct product of groups; unitary representation of a group; LCA groups; dual frames; sampling expansions.
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1 Statement of the problem

In this paper an abstract sampling theory associated to non abelian groups is derived for the specific case of a unitary representation of a semi-direct product of groups on a separable Hilbert space. Semi-direct product of groups provide important examples of non abelian groups such as dihedral groups, infinite dihedral group, euclidean motion groups or crystallographic groups. Concretely, let $(n,h) \mapsto U(n,h)$ be a unitary representation on a separable Hilbert space $\mathcal{H}$ of a semi-direct product $G = N \rtimes_\phi H$, where N is a countable discrete LCA (locally compact abelian) group, H is a finite group, and ϕ denotes the action of the group

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H on the group N (see Section 2 infra for the details); for a fixed $a \in \mathcal{H}$ we consider the U-invariant subspace in $\mathcal{H}$

$$\mathcal{A}_a = \Big\{ \sum_{(n,h)\in G} \alpha(n,h)\, U(n,h)a \;:\; \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G) \Big\} ,$$

where we assume that $\{U(n,h)a\}$ is a Riesz sequence for $\mathcal{H}$, i.e., a Riesz basis for $\mathcal{A}_a$ (see Ref. [2] for a necessary and sufficient condition). Given K elements b_k in $\mathcal{H}$, which do not belong necessarily to $\mathcal{A}_a$, the main goal in this paper is the stable recovery of any $x \in \mathcal{A}$ from the given data (generalized samples)

$$\mathcal{L}_k x(n) := \big\langle x, U(n, 1_H) b_k \big\rangle_{\mathcal{H}} , \qquad n \in N \text{ and } k = 1, 2, \dots, K ,$$

where 1_H denotes the identity element in H. These samples are nothing but a generalization of average sampling in shift-invariant subspaces of $L^2(\mathbb{R}^d)$; see, among others, Refs. [1, 6, 8, 9, 13, 14, 15, 16]. The case where G is a discrete LCA group and the samples are taken at a uniform lattice of G has been solved in Ref. [11]; this work relies on the use of the Fourier analysis in the LCA group G. In the case involved here a Fourier analysis is not available and, consequently, we need to overcome this drawback.

Having in mind the filter bank formalism in discrete LCA groups (see, for instance, Refs. [3, 5, 10]), the given data $\{\mathcal{L}_k x(n)\}_{n\in N;\, k=1,2,\dots K}$ can be expressed as the output of a suitable K-channel analysis filter bank corresponding to the input $\boldsymbol{\alpha} = \{\alpha(n,h)\}_{(n,h)\in G}$ in $\ell^2(G)$. As a consequence, the problem consists of finding a synthesis part of the former filter bank allowing perfect reconstruction; besides only Fourier analysis on the LCA group N is needed. Then, roughly speaking, substituting the output of the synthesis part in $x = \sum_{(n,h)\in G} \alpha(n,h)\, U(n,h)a$ we will obtain the corresponding sampling formula in $\mathcal{A}_a$.

This said, as it could be expected the problem can be mathematically formulated as the search of dual frames for $\ell^2(G)$ having the form

$$\big\{T_n \mathsf{h}_k\big\}_{n\in N;\, k=1,2,\dots K} \quad \text{and} \quad \big\{T_n \mathsf{g}_k\big\}_{n\in N;\, k=1,2,\dots K} .$$

Here $\mathsf{h}_k, \mathsf{g}_k \in \ell^2(G)$, $T_n\mathsf{h}_k(m,h) = \mathsf{h}_k(m-n,h)$ and $T_n\mathsf{g}_k(m,h) = \mathsf{g}_k(m-n,h)$, $(m,h)\in G$, where $n \in N$ and $k = 1, 2, \dots, K$. Besides, for any $x \in \mathcal{A}_a$ we have the expression for its samples

$$\mathcal{L}_k x(n) = \big\langle \boldsymbol{\alpha}, T_n \mathsf{h}_k \big\rangle_{\ell^2(G)} , \quad n \in N \text{ and } k = 1, 2, \dots, K .$$

Needless to say that frame theory plays a central role in what follows; the necessary background on Riesz bases or frame theory in a separable Hilbert space can be found, for instance, in Ref. [4]. Finally, sampling formulas in $\mathcal{A}_a$ having the form

$$x = \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, U(n, 1_H) c_k \quad \text{ in } \mathcal{H} ,$$

for some $c_k \in \mathcal{A}_a$, $k = 1, 2, \dots, K$, will come out by using, for $\mathsf{g} \in \ell^2(G)$ and $n \in N$, the shifting property $\mathcal{T}_{U,a}\big(T_n \mathsf{g}\big) = U(n, 1_H)\big(\mathcal{T}_{U,a}\mathsf{g}\big)$ that satisfies the natural isomorphism $\mathcal{T}_{U,a} : \ell^2(G) \to \mathcal{A}_a$ which maps the usual orthonormal basis $\{\boldsymbol{\delta}_{(n,h)}\}_{(n,h)\in G}$ for $\ell^2(G)$ onto the Riesz basis $\big\{U(n,h)a\big\}_{(n,h)\in G}$ for $\mathcal{A}_a$. All these steps will be carried out throughout the remaining sections. For the sake of completeness, Section 2 includes some basic preliminaries on semi-direct product of groups and Fourier analysis on LCA groups. The paper ends with an illustrative example involving the quasi regular representation of a crystallographic group on $L^2(\mathbb{R}^d)$; sampling formulas involving average or pointwise samples are obtained for the corresponding U-invariant subspaces in $L^2(\mathbb{R}^d)$.

2 Some mathematical preliminaries

In this section we introduce the basic tools in semi-direct product of groups and in harmonic analysis in a discrete LCA group that they will be used in the sequel.

2.1 Preliminaries on semi-direct product of groups

Given groups $(N, \cdot)$ and $(H, \cdot)$, and a homomorphism $\phi : H \to Aut(N)$ their semi-direct product $G := N \rtimes_\phi H$ is defined as follows: The underlying set of G is the set of pairs (n, h) with $n \in N$ and $h \in H$, along with the multiplication rule

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1\phi_{h_1}(n_2), h_1h_2)\,, \quad (n_1, h_1),\, (n_2, h_2) \in G\,,$$

where we denote $\phi(h) := \phi_h$; usually the homomorphism ϕ is referred as the action of the group H on the group N. Thus we obtain a new group with identity element $(1_N, 1_H)$, and inverse $(n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$.

Besides, we have the isomorphisms $N \simeq N \times \{1_H\}$ and $H \simeq \{1_N\} \times H$. Unless ϕ_h equals the identity for all $h \in H$, the group $G = N \rtimes_\phi H$ is not abelian, even for abelian N and H groups. In case N is an abelian group, it is a normal subgroup in G. Next we list some examples of semi-direct product of groups:

1. The dihedral group D_{2N} is the group of symmetries of a regular N-sided polygon; it is the semi-direct product $D_{2N} = \mathbb{Z}_N \rtimes_\phi \mathbb{Z}_2$ where $\phi_{\bar{0}} \equiv Id_{\mathbb{Z}_N}$ and $\phi_{\bar{1}}(\bar{n}) = -\bar{n}$ for each $\bar{n} \in \mathbb{Z}_N$. The infinite dihedral group D_∞ defined as $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ for the similar homomorphism ϕ is the group of isometries of $\mathbb{Z}$.

2. The Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d \rtimes_\phi O(d)$, where $O(d)$ is the orthogonal group of order d and $\phi_A(x) = Ax$ for $A \in O(d)$ and $x \in \mathbb{R}^d$. It contains as a subgroup any crystallographic group $M\mathbb{Z}^d \rtimes_\phi \Gamma$, where $M\mathbb{Z}^d$ denotes a full rank lattice of $\mathbb{R}^d$ and Γ is any finite subgroup of $O(d)$ such that $\phi_\gamma(M\mathbb{Z}^d) = M\mathbb{Z}^d$ for each $\gamma \in \Gamma$.

3. The orthogonal group $O(d)$ of all orthogonal real $d \times d$ matrices is isomorphic to the semi-direct product $SO(d) \rtimes_\phi C_2$, where $SO(d)$ consists of all orthogonal matrices with determinant 1 and $C_2 = \{I, R\}$ a cyclic group of order 2; ϕ is the homomorphism given by $\phi_I(A) = A$ and $\phi_R(A) = RAR^{-1}$ for $A \in SO(d)$.

Suppose that N is an LCA group with Haar measure μ_N and H is a locally compact group with Haar measure μ_H. Then, the semi-direct product $G = N \rtimes_\phi H$ endowed with the product topology becomes also a topological group. For the left Haar measure on G see Ref. [2].

2.2 Some preliminaries on harmonic analysis on discrete LCA groups

The results about harmonic analysis on locally compact abelian (LCA) groups are borrowed from Ref. [7]. Notice that, in particular, a countable discrete abelian group is a second countable Hausdorff LCA group.

For a countable discrete group $(N,\cdot)$, non necessarily abelian, the *convolution* of $x,y : N \to \mathbb{C}$ is formally defined as $(x*y)(m) := \sum_{n\in N} x(n)y(n^{-1}m),\ m \in N$. If in addition the group is abelian, therefore denoted by $(N,+)$, the convolution reads as

$$(x*y)(m) := \sum_{n\in N} x(n)y(m-n)\,, \quad m\in N\,.$$

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unidimensional torus. We said that $\xi : N \mapsto \mathbb{T}$ is a character of N if $\xi(n+m) = \xi(n)\xi(m)$ for all $n,m \in N$. We denote $\xi(n) = \langle n,\xi\rangle$. Defining $(\xi+\gamma)(n) = \xi(n)\gamma(n)$, the set of characters $\widehat{N}$ with the operation $+$ is a group, called the dual group of N; since N is discrete $\widehat{N}$ is compact [7, Prop. 4.4]. For $x \in \ell^1(N)$ we define its *Fourier transform* as

$$X(\xi) = \widehat{x}(\xi) := \sum_{n\in N} x(n)\overline{\langle n,\xi\rangle} = \sum_{n\in N} x(n)\langle -n,\xi\rangle\,, \quad \xi\in\widehat{N}\,.$$

It is known [7, Theorem 4.5] that $\widehat{\mathbb{Z}} \cong \mathbb{T}$, with $\langle n,z\rangle = z^n$, and $\widehat{\mathbb{Z}}_s \cong \mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}$, with $\langle n,m\rangle = W_s^{nm}$, where $W_s = e^{2\pi i/s}$.

There exists a unique measure, the Haar measure μ on $\widehat{N}$ satisfying $\mu(\xi+E) = \mu(E)$, for every Borel set $E \subset \widehat{N}$ [7, Section 2.2], and $\mu(\widehat{N}) = 1$. We denote $\int_{\widehat{N}} X(\xi)d\xi = \int_{\widehat{N}} X(\xi)d\mu(\xi)$. If $N = \mathbb{Z}$,

$$\int_{\widehat{N}} X(\xi)d\xi = \int_{\mathbb{T}} X(z)dz = \frac{1}{2\pi}\int_0^{2\pi} X(e^{iw})dw\,,$$

and if $N = \mathbb{Z}_s$,

$$\int_{\widehat{N}} X(\xi)d\xi = \int_{\mathbb{Z}_s} X(n)dn = \frac{1}{s}\sum_{n\in\mathbb{Z}_s} X(n)\,.$$

If $N_1, N_2, \dots N_d$ are abelian discrete groups then the dual group of the product group is $\big(N_1\times N_2\times\ldots\times N_d\big)^\wedge \cong \widehat{N}_1\times\widehat{N}_2\times\ldots\times\widehat{N}_d$ (see [7, Prop. 4.6]) with

$$\big\langle\, (n_1,n_2,\dots,n_d)\,,\,(\xi_1,\xi_2\dots,\xi_d)\,\big\rangle = \langle n_1,\xi_1\rangle\langle n_2,\xi_2\rangle\cdots\langle n_d,\xi_d\rangle\,.$$

The Fourier transform on $\ell^1(N)\cap\ell^2(N)$ is an isometry on a dense subspace of $L^2(\widehat{N})$; Plancherel theorem extends it in a unique manner to a unitary operator of $\ell^2(N)$ onto $L^2(\widehat{N})$ [7, p. 99]. The following lemma, giving a relationship between Fourier transform and convolution, will be used later:

Lemma 1. *Assume that $a,b\in\ell^2(N)$ and $\widehat{a}(\xi)\,\widehat{b}(\xi)\in L^2(\widehat{N})$. Then the convolution $a*b$ belongs to $\ell^2(N)$ and $\widehat{a*b}(\xi) = \widehat{a}(\xi)\,\widehat{b}(\xi)$, a.e. $\xi\in\widehat{N}$.*

Proof. By using Plancherel theorem [7, Theorem 4.25] we obtain

$$(a*b)(n) = \sum_{m\in N} a(m)b(n-m) = \langle a,\widetilde{b}(\cdot-n)\rangle_{\ell^2(N)} = \big\langle \widehat{a},\widehat{\widetilde{b}(\cdot-n)}\big\rangle_{L^2(\widehat{N})}$$

$$= \int_{\widehat{N}} \widehat{a}(\xi)\,\overline{\widehat{\widetilde{b}}(\xi)}\,\overline{\langle -n,\xi\rangle}d\xi = \int_{\widehat{N}} \widehat{a}(\xi)\,\widehat{b}(\xi)\,\overline{\langle -n,\xi\rangle}d\xi\,.$$

Since $\{\langle -n,\xi\rangle\}_{n\in N}$ is an orthonormal basis for $L^2(\widehat{N})$ [7, Theorems 4.26 and 4.31] (we are assuming that $\mu_{\widehat{N}}(\widehat{N})=1$) we finally obtain

$$\widehat{a}(\xi)\,\widehat{b}(\xi)=\sum_{n\in N}(a*b)(n)\langle -n,\xi\rangle=\widehat{a*b}(\xi)\,,\quad \text{a.e. } \xi\in\widehat{N}\,.$$

□

3 Filter banks formalism on semi-direct product of groups

In what follows we will assume that $G=N\rtimes_\phi H$ where $(N,+)$ is a countable discrete abelian group and $(H,\cdot)$ is a finite group. Having in mind the operational calculus $(n,h)\cdot(m,l)=(n+\phi_h(m),hl)$, $(n,h)^{-1}=(\phi_{h^{-1}}(-n),h^{-1})$ and $(n,h)^{-1}\cdot(m,l)=(\phi_{h^{-1}}(m-n),h^{-1}l)$, the convolution $\boldsymbol{\alpha}*\mathsf{h}$ of $\boldsymbol{\alpha},\mathsf{h}\in\ell^2(G)$ can be expressed as

$$\begin{aligned}(\boldsymbol{\alpha}*\mathsf{h})(m,l)&=\sum_{(n,h)\in G}\alpha(n,h)\,\mathsf{h}\big[(n,h)^{-1}\cdot(m,l)\big]\\&=\sum_{(n,h)\in G}\alpha(n,h)\,\mathsf{h}\big(\phi_{h^{-1}}(m-n),h^{-1}l\big)\,,\quad (m,l)\in G\,.\end{aligned}\tag{1}$$

For a function $\boldsymbol{\alpha}:G\to\mathbb{C}$, its *$H$-decimation* $\downarrow_H\boldsymbol{\alpha}:N\to\mathbb{C}$ is defined as $(\downarrow_H\boldsymbol{\alpha})(n):=\alpha(n,1_H)$ for any $n\in N$. Thus we have

$$\begin{aligned}\downarrow_H(\boldsymbol{\alpha}*\mathsf{h})(m)=(\boldsymbol{\alpha}*\mathsf{h})(m,1_H)&=\sum_{(n,h)\in G}\alpha(n,h)\,\mathsf{h}\big(\phi_{h^{-1}}(m-n),h^{-1}\big)\\&=\sum_{(n,h)\in G}\alpha(n,h)\,\mathsf{h}[(n-m,h)^{-1}]\,,\quad m\in N\,.\end{aligned}\tag{2}$$

Defining the polyphase components of $\boldsymbol{\alpha}$ and h as $\boldsymbol{\alpha}_h(n):=\alpha(n,h)$ and $\mathsf{h}_h(n):=\mathsf{h}[(-n,h)^{-1}]$ respectively, we write

$$\downarrow_H(\boldsymbol{\alpha}*\mathsf{h})(m)=\sum_{h\in H}\sum_{n\in N}\boldsymbol{\alpha}_h(n)\,\mathsf{h}_h(m-n)=\sum_{h\in H}\big(\boldsymbol{\alpha}_h*_N\mathsf{h}_h\big)(m)\,,\quad m\in N\,.$$

For a function $c:N\to\mathbb{C}$, its *H-expander* $\uparrow_H c:G\to\mathbb{C}$ is defined as

$$(\uparrow_H c)(n,h)=\begin{cases}c(n) & \text{if } h=1_H\\ 0 & \text{if } h\neq 1_H\,.\end{cases}$$

In case $\uparrow_H c$ and g belong to $\ell^2(G)$ we have

$$\begin{aligned}(\uparrow_H c*\mathsf{g})(m,l)&=\sum_{(n,h)\in G}(\uparrow_H c)(n,h)\,\mathsf{g}\big[(n,h)^{-1}\cdot(m,l)\big]\\&=\sum_{(n,h)\in G}(\uparrow_H c)(n,h)\,\mathsf{g}\big(\phi_{h^{-1}}(m-n),h^{-1}l\big)\\&=\sum_{n\in N}c(n)\,\mathsf{g}(m-n,l)=\big(c*_N\mathsf{g}_l\big)(m)\,,\quad m\in N\,,\,l\in H\,,\end{aligned}$$

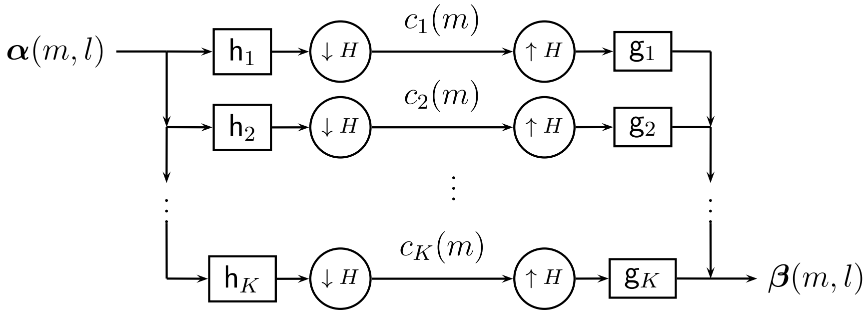

Figure 1: The K-channel filter bank scheme

where $\mathsf{g}_l(n) := \mathsf{g}(n,l)$ is the polyphase component of g.

From now on we will refer to a *K-channel filter bank* with *analysis filters* h_k and *synthesis filters* g_k, $k = 1, 2, \ldots, K$ as the one given by (see Fig. 1)

$$\mathbf{c}_k := \downarrow_H (\boldsymbol{\alpha} * \mathsf{h}_k)\,,\ k = 1, 2, \ldots, K\,, \text{ and } \quad \boldsymbol{\beta} = \sum_{k=1}^{K} (\uparrow_H c_k) * \mathsf{g}_k\,, \tag{3}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote, respectively, the input and the output of the filter bank. In polyphase notation,

$$\begin{aligned} \mathbf{c}_k(m) &= \sum_{h\in H} \big(\boldsymbol{\alpha}_h *_N \mathsf{h}_{k,h}\big)(m)\,, \quad m \in N\,, \quad k = 1, 2, \ldots, K\,, \\ \boldsymbol{\beta}_l(m) &= \sum_{k=1}^{K} \big(\mathbf{c}_k *_N \mathsf{g}_{l,k}\big)(m)\,, \quad m \in N\,,\ l \in H\,, \end{aligned} \tag{4}$$

where $\boldsymbol{\alpha}_h(n) := \alpha(n,h)$, $\boldsymbol{\beta}_l(n) := \beta(n,l)$, $\mathsf{h}_{k,h}(n) := \mathsf{h}_k[(-n,h)^{-1}]$ and $\mathsf{g}_{l,k}(n) := \mathsf{g}_k(n,l)$ are the *polyphase components* of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, h_k and g_k, $k = 1, 2, \ldots, K$, respectively. We also assume that $\mathsf{h}_k, \mathsf{g}_k \in \ell^2(G)$ with $\widehat{\mathsf{h}}_{k,h}, \widehat{\mathsf{g}}_{h,k} \in L^\infty(\widehat{N})$ for $k = 1, 2, \ldots, K$ and $h \in H$; from Lemma 1 the filter bank (3) is well defined in $\ell^2(G)$.

The above K-channel filter bank (3) is said to be a *perfect reconstruction* filter bank if and only if it satisfies $\boldsymbol{\alpha} = \sum_{k=1}^{K} (\uparrow_H c_k) * \mathsf{g}_k$ for each $\boldsymbol{\alpha} \in \ell^2(G)$, or equivalently, $\boldsymbol{\alpha}_h = \sum_{k=1}^{K} \big(\mathbf{c}_k *_N \mathsf{g}_{h,k}\big)$ for each $h \in H$.

Since N is an LCA group where a Fourier transform is available, the polyphase expression (4) of the filter bank (3) allows us to carry out its polyphase analysis.

3.1 Polyphase analysis: Perfect reconstruction condition

For notational ease, we denote $L := |H|$, the order of the group H, and its elements as $H = \{h_1, h_2, \ldots, h_L\}$. Having in mind Lemma 1, the N-Fourier transform in $\mathbf{c}_k(m) = \sum_{h\in H} \big(\boldsymbol{\alpha}_h *_N \mathsf{h}_{k,h}\big)(m)$ gives $\widehat{\mathbf{c}}_k(\gamma) = \sum_{h\in H} \widehat{\mathsf{h}}_{k,h}(\gamma)\, \widehat{\boldsymbol{\alpha}}_h(\gamma)$ a.e. $\gamma \in \widehat{N}$ for each $k = 1, 2, \ldots, K$. In matrix notation,

$$\mathbf{C}(\gamma) = \mathbf{H}(\gamma)\, \mathbf{A}(\gamma) \quad \text{a.e. } \gamma \in \widehat{N}\,,$$

where $\mathbf{C}(\gamma) = \big(\widehat{\mathbf{c}}_1(\gamma), \widehat{\mathbf{c}}_2(\gamma), \dots, \widehat{\mathbf{c}}_K(\gamma)\big)^\top$, $\mathbf{A}(\gamma) = \big(\widehat{\boldsymbol{\alpha}}_{h_1}(\gamma), \widehat{\boldsymbol{\alpha}}_{h_2}(\gamma), \dots, \widehat{\boldsymbol{\alpha}}_{h_L}(\gamma)\big)^\top$, and $\mathbf{H}(\gamma)$ is the $K \times L$ matrix

$$\mathbf{H}(\gamma) = \begin{pmatrix} \widehat{\mathsf{h}}_{1,h_1}(\gamma) & \widehat{\mathsf{h}}_{1,h_2}(\gamma) & \cdots & \widehat{\mathsf{h}}_{1,h_L}(\gamma) \\ \widehat{\mathsf{h}}_{2,h_1}(\gamma) & \widehat{\mathsf{h}}_{2,h_2}(\gamma) & \cdots & \widehat{\mathsf{h}}_{2,h_L}(\gamma) \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{\mathsf{h}}_{K,h_1}(\gamma) & \widehat{\mathsf{h}}_{K,h_2}(\gamma) & \cdots & \widehat{\mathsf{h}}_{K,h_L}(\gamma) \end{pmatrix}, \tag{5}$$

where $\widehat{\mathsf{h}}_{k,h_i} \in L^2(\widehat{N})$ is the Fourier transform of $\mathsf{h}_{k,h_i}(n) := \mathsf{h}_k[(-n, h_i)^{-1}] \in \ell^2(N)$.

The same procedure for $\boldsymbol{\beta}_l(m) = \sum_{k=1}^K \big(\mathbf{c}_k *_N \mathsf{g}_{l,k}\big)(m)$ gives $\widehat{\boldsymbol{\beta}}_l(\gamma) = \sum_{k=1}^K \widehat{\mathsf{g}}_{l,k}(\gamma)\, \widehat{\mathbf{c}}_k(\gamma)$ a.e. $\gamma \in \widehat{N}$. In matrix notation,

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma)\, \mathbf{C}(\gamma) \quad \text{a.e.} \quad \gamma \in \widehat{N}\,,$$

where $\mathbf{B}(\gamma) = \big(\widehat{\boldsymbol{\beta}}_{h_1}(\gamma), \widehat{\boldsymbol{\beta}}_{h_2}(\gamma), \dots, \widehat{\boldsymbol{\beta}}_{h_L}(\gamma)\big)^\top$, $\mathbf{C}(\gamma) = \big(\widehat{\mathbf{c}}_1(\gamma), \widehat{\mathbf{c}}_2(\gamma), \dots, \widehat{\mathbf{c}}_K(\gamma)\big)^\top$ and $\mathbf{G}(\gamma)$ is the $L \times K$ matrix

$$\mathbf{G}(\gamma) = \begin{pmatrix} \widehat{\mathsf{g}}_{h_1,1}(\gamma) & \widehat{\mathsf{g}}_{h_1,2}(\gamma) & \cdots & \widehat{\mathsf{g}}_{h_1,K}(\gamma) \\ \widehat{\mathsf{g}}_{h_2,1}(\gamma) & \widehat{\mathsf{g}}_{h_2,2}(\gamma) & \cdots & \widehat{\mathsf{g}}_{h_2,K}(\gamma) \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{\mathsf{g}}_{h_L,1}(\gamma) & \widehat{\mathsf{g}}_{h_L,2}(\gamma) & \cdots & \widehat{\mathsf{g}}_{h_L,K}(\gamma) \end{pmatrix}, \tag{6}$$

where $\widehat{\mathsf{g}}_{h_i,k} \in L^2(\widehat{N})$ is the Fourier transform of $\mathsf{g}_{h_i,k}(n) := \mathsf{g}_k(n, h_i) \in \ell^2(N)$.

Thus, in terms of the *polyphase matrices* $\mathbf{G}(\gamma)$ and $\mathbf{H}(\gamma)$ the filter bank (3) can be expressed as

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma)\, \mathbf{H}(\gamma)\, \mathbf{A}(\gamma) \quad \text{a.e.} \quad \gamma \in \widehat{N}\,. \tag{7}$$

As a consequence of (7) we have:

Teorema 2. *The K-channel filter bank given in* (3)*, where $\mathsf{h}_k, \mathsf{g}_k$ belong to $\ell^2(G)$ and $\widehat{\mathsf{h}}_{k,h_i}, \widehat{\mathsf{g}}_{h_i,k}$ belong to $L^\infty(\widehat{N})$ for $k = 1, 2, \dots, K$ and $i = 1, 2, \dots, L$, satisfies the perfect reconstruction property if and only if $\mathbf{G}(\gamma)\, \mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$, where $\mathbf{I}_L$ denotes the identity matrix of order L.*

Proof. First of all, note that the mapping $\boldsymbol{\alpha} \in \ell^2(G) \mapsto \mathbf{A} \in L^2_L(\widehat{N})$ is a unitary operator. Indeed, for each $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \ell^2(G)$ we have the isometry property

$$\begin{aligned} \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\ell^2(G)} &= \sum_{(m,h)\in G} \alpha(m,h)\, \overline{\beta(m,h)} = \sum_{h\in H} \langle \boldsymbol{\alpha}_h, \boldsymbol{\beta}_h \rangle_{\ell^2(N)} \\ &= \sum_{h\in H} \langle \widehat{\boldsymbol{\alpha}}_h, \widehat{\boldsymbol{\beta}}_h \rangle_{L^2(\widehat{N})} = \langle \mathbf{A}, \mathbf{B} \rangle_{L^2_L(\widehat{N})}\,. \end{aligned}$$

It is also surjective since the N-Fourier transform is a surjective isometry between $\ell^2(N)$ and $L^2(\widehat{N})$. Having in mind this property, Eq. (7) tell us that the filter bank satisfies the perfect reconstruction property if and only if $\mathbf{G}(\gamma)\, \mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$. □

Notice that, in the perfect reconstruction setting, the number of channels K must be necessarily bigger or equal that the order L of the group H, i.e., $K \geq L$.

4 Frame analysis

For $m \in N$ the *translation operator* $T_m : \ell^2(G) \to \ell^2(G)$ is defined as

$$T_m \boldsymbol{\alpha}(n,h) := \alpha\big((m,1_H)^{-1} \cdot (n,h)\big) = \alpha(n-m,h)\,,\ (n,h) \in G\,. \tag{8}$$

The *involution operator* $\boldsymbol{\alpha} \in \ell^2(G) \mapsto \widetilde{\boldsymbol{\alpha}} \in \ell^2(G)$ is defined as $\widetilde{\alpha}(n,h) := \overline{\alpha\big((n,h)^{-1}\big)}$, $(n,h) \in G$. As expected, the classical relationship between convolution and translation operators holds. Thus, for the K-channel filter bank (3) we have (see (2)):

$$\mathbf{c}_k(m) = \downarrow_H (\boldsymbol{\alpha} * \mathsf{h}_k)(m) = \big\langle \boldsymbol{\alpha}, T_m \widetilde{\mathsf{h}}_k \big\rangle_{\ell^2(G)}\,, \quad m \in N\,,\ k = 1,2,\dots,K\,.$$

Besides,

$$(\uparrow_H \mathbf{c}_k * \mathsf{g}_k)(m,h) = \sum_{n\in N} \mathbf{c}_k(n)\, \mathsf{g}_k(m-n,h) = \sum_{n\in N} \langle \boldsymbol{\alpha}, T_n \widetilde{\mathsf{h}}_k \rangle_{\ell^2(G)}\, T_n \mathsf{g}_k(m,h)\,.$$

In the perfect reconstruction setting, for any $\boldsymbol{\alpha} \in \ell^2(G)$ we have

$$\boldsymbol{\alpha} = \sum_{k=1}^{K} \sum_{n\in N} \langle \boldsymbol{\alpha}, T_n \widetilde{\mathsf{h}}_k \rangle_{\ell^2(G)}\, T_n \mathsf{g}_k \quad \text{in } \ell^2(G)\,. \tag{9}$$

Given K sequences $\mathsf{f}_k \in \ell^2(G)$, $k = 1,2,\dots,K$, our main tasks now are: (i) to characterize the sequence $\big\{T_n \mathsf{f}_k\big\}_{n\in N;\, k=1,2,\dots K}$ as a frame for $\ell^2(G)$, and (ii) to find its dual frames having the form $\big\{T_n \mathsf{g}_k\big\}_{n\in N;\, k=1,2,\dots K}$.

To the first end we consider a K-channel analysis filter bank with analysis filters $\mathsf{h}_k := \widetilde{\mathsf{f}}_k$, $k = 1,2,\dots,K$; let $\mathbf{H}(\gamma)$ be its associated $K \times L$ polyphase matrix (5). First, we check that (5) is:

$$\mathbf{H}(\gamma) = \Big(\overline{\widehat{\mathsf{f}}_{k,h_i}(\gamma)}\Big)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}}\,. \tag{10}$$

Indeed, for $k = 1,2,\dots,K$ and $i = 1,2,\dots,L$ we have

$$\begin{aligned}
\widehat{\mathsf{h}}_{k,h_i}(\gamma) &= \sum_{n\in N} \mathsf{h}_{k,h_i}(n) \langle -n, \gamma\rangle = \sum_{n\in N} \mathsf{h}_k[(-n,h_i)^{-1}] \langle -n, \gamma\rangle = \sum_{n\in N} \tilde{\mathsf{f}}_k[(-n,h_i)^{-1}] \langle -n, \gamma\rangle \\
&= \sum_{n\in N} \overline{\mathsf{f}_k(-n,h_i)} \langle -n, \gamma\rangle = \overline{\sum_{n\in N} \mathsf{f}_k(n,h_i) \langle -n, \gamma\rangle} = \overline{\widehat{\mathsf{f}}_{k,h_i}(\gamma)}\,, \quad \gamma \in \widehat{N}\,.
\end{aligned}$$

Next, we consider its associated constants

$$A_{\mathbf{H}} := \operatorname*{ess\,inf}_{\gamma\in\widehat{N}} \lambda_{\min}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big] \quad \text{and} \quad B_{\mathbf{H}} := \operatorname*{ess\,sup}_{\gamma\in\widehat{N}} \lambda_{\max}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big]\,.$$

Teorema 3. *For f_k in $\ell^2(G)$, $k = 1,2,\dots,K$, consider the associated matrix $\mathbf{H}(\gamma)$ given in* (10)*. Then,*

1. *The sequence $\big\{T_n \mathsf{f}_k\big\}_{n\in N;\, k=1,2,\dots K}$ is a Bessel sequence for $\ell^2(G)$ if and only if $B_{\mathbf{H}} < \infty$.*

2. The sequence $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\dots K}$ *is a frame for* $\ell^2(G)$ *if and only if the inequalities* $0 < A_{\mathbf{H}} \le B_{\mathbf{H}} < \infty$ *hold.*

Proof. Using Plancherel theorem [7, Theorem 4.25], for each $\boldsymbol{\alpha} \in \ell^2(G)$ we get

$$\begin{aligned}\langle \boldsymbol{\alpha}, T_n \mathsf{f}_k\rangle_{\ell^2(G)} &= \sum_{h\in H} \langle \boldsymbol{\alpha}_h, \mathsf{f}_{k,h}(\cdot - n)\rangle_{\ell^2(N)} = \sum_{h\in H} \int_{\widehat{N}} \widehat{\boldsymbol{\alpha}}_h(\gamma)\overline{\widehat{\mathsf{f}}_{k,h}(\gamma)\langle -n,\gamma\rangle}d\gamma \\ &= \int_{\widehat{N}} \sum_{h\in H} \widehat{\boldsymbol{\alpha}}_h(\gamma)\overline{\widehat{\mathsf{f}}_{k,h}(\gamma)}\,\overline{\langle -n,\gamma\rangle}d\gamma = \int_{\widehat{N}} \mathbf{H}_k(\gamma)\mathbf{A}(\gamma)\overline{\langle -n,\gamma\rangle}d\gamma\,,\end{aligned}$$

where $\mathbf{A}(\gamma) = \big(\widehat{\boldsymbol{\alpha}}_{h_1}(\gamma), \widehat{\boldsymbol{\alpha}}_{h_2}(\gamma), \dots, \widehat{\boldsymbol{\alpha}}_{h_L}(\gamma)\big)^\top$ and $\mathbf{H}_k(\gamma)$ denotes the k-th row of $\mathbf{H}(\gamma)$.

Since $\{\langle -n,\gamma\rangle\}_{n\in N}$ is an orthonormal basis for $L^2(\widehat{N})$, in case that $\mathbf{H}(\gamma)\mathbf{A}(\gamma) \in L^2_K(\widehat{N})$ we have

$$\sum_{k=1}^K \sum_{n\in N} |\langle \boldsymbol{\alpha}, T_n \mathsf{f}_k\rangle|^2 = \sum_{k=1}^K \int_{\widehat{N}} \big|\mathbf{H}_k(\gamma)\mathbf{A}(\gamma)\ \big|^2 d\gamma = \int_{\widehat{N}} \big\|\mathbf{H}(\gamma)\mathbf{A}(\gamma)\big\|^2 d\gamma\,.$$

If $B_{\mathbf{H}} < \infty$, having in mind that $\|\boldsymbol{\alpha}\|^2_{\ell^2(G)} = \|\mathbf{A}\|^2_{L^2_L(\widehat{N})} = \int_{\widehat{N}} \big\|\mathbf{A}(\gamma)\big\|^2 d\gamma$, the above equality and the Rayleigh-Ritz theorem [12, Theorem 4.2.2] prove that $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\dots,K}$ is a Bessel sequence for $\ell^2(G)$ with Bessel bound less or equal than $B_{\mathbf{H}}$.

On the other hand, if $K < B_{\mathbf{H}}$ then there exists a set $\Omega \subset \widehat{N}$ having null measure such that $\lambda_{\max}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big] > K$ for $\gamma \in \Omega$. Consider $\boldsymbol{\alpha}$ such that its associated $\mathbf{A}(\gamma)$ is 0 if $\gamma \notin \Omega$, and $\mathbf{A}(\gamma)$ is a unitary eigenvector corresponding to the largest eigenvalue of $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)$ if $\gamma \in \Omega$. Thus we have that

$$\sum_{k=1}^K \sum_{n\in N} |\langle \boldsymbol{\alpha}, T_n \mathsf{f}_k\rangle|^2 = \int_{\widehat{N}} \big\|\mathbf{H}(\gamma)\mathbf{A}(\gamma)\big\|^2 d\gamma > K\int_{\widehat{N}} \big\|\mathbf{A}(\gamma)\big\|^2 d\gamma = K\,\|\boldsymbol{\alpha}\|^2_{\ell^2(G)}$$

As a consequence, if $B_{\mathbf{H}} = \infty$ the sequence is not Bessel, and if $B_{\mathbf{H}} < \infty$ the optimal bound is precisely $B_{\mathbf{H}}$.

Similarly, by using inequality $\big\|\mathbf{H}(\gamma)\mathbf{A}(\gamma)\big\|^2 \ge \lambda_{\min}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big]\big\|\mathbf{A}(\gamma)\big\|^2$, and that equality holds whenever $\mathbf{A}(\gamma)$ is a unitary eigenvector corresponding to the smallest eigenvalue of $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)$ one proves the other inequality in part 2. □

Corollary 4. *The sequence* $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\dots K}$ *is a Bessel sequence for* $\ell^2(G)$ *if and only if for each* $k = 1, 2, \dots, K$ *and* $i = 1, 2, \dots, L$ *the function* $\widehat{\mathsf{f}}_{k,h_i}$ *belongs to* $L^\infty(\widehat{N})$*.*

Proof. It is a direct consequence of the equivalence between the spectral and Frobenius norms for matrices [12]. □

To the second end, a K-channel filter bank formalism allows, in a similar manner, to obtain properties in $\ell^2(G)$ of the sequences $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\dots K}$ and $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\dots K}$. In case they are Bessel sequences for $\ell^2(G)$, the idea is to consider a K-channel filter bank (3) where the analysis filters are $\mathsf{h}_k := \widetilde{\mathsf{f}}_k$ and the synthesis filters are g_k, $k = 1, 2, \dots, K$. As a consequence, the corresponding polyphase matrices $\mathbf{H}(\gamma)$ and $\mathbf{G}(\gamma)$, given in (5) and (6) are,

$$\mathbf{H}(\gamma) = \Big(\overline{\widehat{\mathsf{f}}_{k,h_i}(\gamma)}\Big)_{\substack{k=1,2,\dots,K\\ i=1,2,\dots,L}} \quad \text{and} \quad \mathbf{G}(\gamma) = \Big(\widehat{\mathsf{g}}_{h_i,k}(\gamma)\Big)_{\substack{i=1,2,\dots,L\\ k=1,2,\dots,K}}\,, \quad \gamma \in \widehat{N}\,. \tag{11}$$

Teorema 5. *Let* $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ *and* $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\ldots K}$ *be two Bessel sequences for* $\ell^2(G)$*, and* $\mathbf{H}(\gamma)$ *and* $\mathbf{G}(\gamma)$ *their associated matrices* (11)*. Under the above circumstances we have:*

(a) The sequences $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ *and* $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\ldots K}$ *are dual frames for* $\ell^2(G)$ *if and only if condition* $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ *a.e.* $\gamma \in \widehat{N}$ *holds.*

(b) The sequences $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ *and* $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\ldots K}$ *are biorthogonal sequences in* $\ell^2(G)$ *if and only if condition* $\mathbf{H}(\gamma)\mathbf{G}(\gamma) = \mathbf{I}_K$ *a.e.* $\gamma \in \widehat{N}$ *holds.*

(c) The sequences $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ *and* $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\ldots K}$ *are dual Riesz bases for* $\ell^2(G)$ *if and only if* $K = L$ *and* $\mathbf{G}(\gamma) = \mathbf{H}(\gamma)^{-1}$ *a.e.* $\gamma \in \widehat{N}$*.*

(d) The sequence $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ *is an* A*-tight frame for* $\ell^2(G)$ *if and only if condition* $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma) = A\mathbf{I}_L$ *a.e.* $\gamma \in \widehat{N}$ *holds.*

(e) The sequence $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ *is an orthonormal basis for* $\ell^2(G)$ *if and only if* $K = L$ *and* $\mathbf{H}^*(\gamma) = \mathbf{H}(\gamma)^{-1}$ *a.e.* $\gamma \in \widehat{N}$*.*

Proof. Having in mind (9) and Corollary 4, part (a) is nothing but Theorem 2.
The output of the analysis filter bank (3) corresponding to the input $\mathsf{g}_{k'}$ is a K-vector which k-entry is

$$c_{k,k'}(m) = \downarrow_H (\mathsf{g}_{k'} *_N \mathsf{h}_k)(m) = \langle \mathsf{g}_{k'}, T_m \widetilde{\mathsf{h}}_k \rangle_{\ell^2(G)} = \langle \mathsf{g}_{k'}, T_m \mathsf{f}_k \rangle_{\ell^2(G)}\,,$$

and whose N-Fourier transform is $\mathbf{C}_{k'}(\gamma) = \mathbf{H}(\gamma)\,\mathbf{G}_{k'}(\gamma)$ a.e. $\gamma \in \widehat{N}$, where $\mathbf{G}_{k'}$ is the k'-column of the matrix $\mathbf{G}(\gamma)$. Note that $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ and $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\ldots K}$ are biorthogonal if and only if $\langle \mathsf{g}_{k'}, T_m \mathsf{f}_k \rangle_{\ell^2(G)} = \delta(k-k')\delta(m)$. Therefore, the sequences $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ and $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\ldots K}$ are biorthogonal if and only if $\mathbf{H}(\gamma)\mathbf{G}(\gamma) = \mathbf{I}_K$. Thus, we have proved (b).
Having in mind [4, Theorem 7.1.1], from (a) and (b) we obtain (c).
We can read the frame operator corresponding to the sequence $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots,K}$, i.e.,

$$\mathcal{S}(\boldsymbol{\alpha}) = \sum_{k=1}^{K} \sum_{n\in N} \langle \boldsymbol{\alpha}, T_n \mathsf{f}_k \rangle_{\ell^2(G)}\, T_n \mathsf{f}_k, \quad \boldsymbol{\alpha} \in \ell^2(G)\,,$$

as the output of the filter bank (3), whenever $\mathsf{h}_k = \widetilde{\mathsf{f}}_k$ and $\mathsf{g}_k = \mathsf{f}_k$, for the input $\boldsymbol{\alpha}$. For this filter bank, the (k, h_l)-entry of the analysis polyphase matrix $\mathbf{H}(\gamma)$ is $\overline{\widehat{\mathsf{f}}_{k,h_l}(\gamma)}$ and the (h_l, k)-entry of the synthesis polyphase matrix $\mathbf{G}(\gamma)$ is $\widehat{\mathsf{f}}_{k,h_l}(\gamma)$; in other words, $\mathbf{G}(\gamma) = \mathbf{H}^*(\gamma)$. Hence, the sequence $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots,K}$ is an A-tight frame for $\ell^2(G)$, i.e.,

$$\boldsymbol{\alpha} = \frac{1}{A} \sum_{k=1}^{K} \sum_{n\in N} \langle \boldsymbol{\alpha}, T_n \mathsf{f}_k \rangle_{\ell^2(G)}\, T_n \mathsf{f}_k, \quad \boldsymbol{\alpha} \in \ell^2(G)\,,$$

if and only if $\mathbf{H}^*(\gamma)\,\mathbf{H}(\gamma) = A\mathbf{I}_L$ for all $\gamma \in \widehat{N}$. Thus, we have proved (d).
Finally, from (c) and (d) the sequence $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\ldots K}$ is an orthonormal system if and only if $\mathbf{H}^*(\gamma) = \mathbf{H}(\gamma)^{-1}$ a.e. $\gamma \in \widehat{N}$. □

5 Getting on with sampling

Suppose that $\{U(n,h)\}_{(n,h)\in G}$ is a unitary representation of the group $G = N \rtimes_\phi H$ on a separable Hilbert space $\mathcal{H}$, and assume that for a fixed $a \in \mathcal{H}$ the sequence $\{U(n,h)a\}_{(n,h)\in G}$ is a Riesz sequence for $\mathcal{H}$ (see Ref. [2, Theorem A]). Thus, we consider the U-invariant subspace in $\mathcal{H}$

$$\mathcal{A}_a = \Big\{ \sum_{(n,h)\in G} \alpha(n,h)\, U(n,h)a \ :\ \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G) \Big\}.$$

For K fixed elements $b_k \in \mathcal{H}$, $k = 1,2,\dots,K$, non necessarily in $\mathcal{A}$, we consider for each $x \in \mathcal{A}$ its generalized samples defined as

$$\mathcal{L}_k x(m) := \big\langle x, U(m,1_H)\, b_k \big\rangle_{\mathcal{H}}, \qquad m \in N \text{ and } k = 1,2,\dots,K. \tag{12}$$

The problem is the stable recovery of any $x \in \mathcal{A}$ from the data $\{\mathcal{L}_k x(m)\}_{m\in N;\, k=1,2,\dots,K}$.

In what follows, we propose a solution involving a perfect reconstruction K-channel filter bank. First, we express the samples in a more suitable manner. Namely, for each $x = \sum_{(n,h)\in G} \alpha(n,h)\, U(n,h)\, a$ in $\mathcal{A}_a$ we have

$$\begin{aligned}\mathcal{L}_k x(m) &= \sum_{(n,h)\in G} \alpha(n,h) \big\langle U(n,h)\, a, U(m,1_H)\, b_k \big\rangle \\ &= \sum_{(n,h)\in G} \alpha(n,h) \big\langle a, U\big[(n,h)^{-1}\cdot(m,1_H)\big]\, b_k \big\rangle = \downarrow_H (\boldsymbol{\alpha} * \mathsf{h}_k)(m), \quad m \in N,\end{aligned}$$

where $\boldsymbol{\alpha} = \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G)$, and $\mathsf{h}_k(n,h) := \big\langle a, U(n,h)\, b_k \big\rangle_{\mathcal{H}}$ also belongs to $\ell^2(G)$ for each $k = 1,2,\dots,K$.

Suppose also that there exists a perfect reconstruction K-channel filter-bank with analysis filters the above h_k and synthesis filters g_k, $k = 1,2,\dots,K$, such that the sequences $\{T_n \widetilde{\mathsf{h}}_k\}_{n\in N;\, k=1,2,\dots K}$ and $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\dots K}$ are Bessel sequences for $\ell^2(G)$. Having in mind (9), for each $\boldsymbol{\alpha} = \{\alpha(n,h)\}_{(n,h)\in G}$ in $\ell^2(G)$ we have

$$\boldsymbol{\alpha} = \sum_{k=1}^{K} \sum_{n\in N} \downarrow_H (\boldsymbol{\alpha} * \mathsf{h}_k)(n)\, T_n \mathsf{g}_k = \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, T_n \mathsf{g}_k \quad \text{in } \ell^2(G). \tag{13}$$

In order to derive a sampling formula in $\mathcal{A}_a$, we consider the natural isomorphism $\mathcal{T}_{U,a} : \ell^2(G) \to \mathcal{A}_a$ which maps the usual orthonormal basis $\{\boldsymbol{\delta}_{(n,h)}\}_{(n,h)\in G}$ for $\ell^2(G)$ onto the Riesz basis $\{U(n,h)\, a\}_{(n,h)\in G}$ for $\mathcal{A}_a$, i.e.,

$$\mathcal{T}_{U,a} \,:\, \boldsymbol{\delta}_{(n,h)} \longmapsto U(n,h)a \ \text{ for each } (n,h) \in G.$$

This isomorphism $\mathcal{T}_{U,a}$ possesses the following shifting property:

Lemma 6. *For each $m \in N$, consider the translation operator T_m operator defined in (8). For each $m \in N$, the following shifting property holds*

$$\mathcal{T}_{U,a}\big(T_m \mathsf{f}\big) = U(m,1_H)\big(\mathcal{T}_{U,a}\mathsf{f}\big), \quad \mathsf{f} \in \ell^2(G). \tag{14}$$

Proof. For each $\boldsymbol{\delta}_{(n,h)}$ it is easy to check that $T_m\boldsymbol{\delta}_{(n,h)} = \boldsymbol{\delta}_{(m+n,h)}$. Hence,

$$\mathcal{T}_{U,a}\big(T_m\boldsymbol{\delta}_{(n,h)}\big) = U(m+n,h)\,a = U(m,1_H)U(n,h)\,a = U(m,1_H)\big(\mathcal{T}_{U,a}\boldsymbol{\delta}_{(n,h)}\big)\,.$$

A continuity argument proves the result for all f in $\ell^2(G)$. □

Now for each $x = \mathcal{T}_{U,a}\boldsymbol{\alpha} \in \mathcal{A}_a$, applying the isomorphism $\mathcal{T}_{U,a}$ and the shifting property (14) in (13), we get for each $x \in \mathcal{A}_a$ the expansion

$$\begin{aligned} x &= \sum_{k=1}^{K}\sum_{n\in N}\mathcal{L}_k x(n)\,\mathcal{T}_{U,a}\big(T_n\mathsf{g}_k\big) = \sum_{k=1}^{K}\sum_{n\in N}\mathcal{L}_k x(n)\,U(n,1_H)\big(\mathcal{T}_{U,a}\mathsf{g}_k\big) \\ &= \sum_{k=1}^{K}\sum_{n\in N}\mathcal{L}_k x(n)\,U(n,1_H)c_{k,\mathsf{g}} \quad \text{ in } \mathcal{H}\,, \end{aligned} \tag{15}$$

where $c_{k,\mathsf{g}} = \mathcal{T}_{U,a}\mathsf{g}_k$, $k = 1,2,\dots,K$. In fact, the following sampling theorem in the subspace $\mathcal{A}_a$ holds:

Teorema 7. *For K fixed $b_k \in \mathcal{H}$, let $\mathcal{L}_k : N \to \mathbb{C}$ be its associated U-system defined in* (12) *with corresponding $\mathsf{h}_k \in \ell^2(G)$, $k = 1,2,\dots,K$. Assume that its polyphase matrix $\mathbf{H}(\gamma)$ given in* (5) *has all its entries in $L^\infty(\widehat{N})$. The following statements are equivalent:*

1. *The constant $A_{\mathbf{H}} = \operatorname*{ess\,inf}_{\gamma\in\widehat{N}} \lambda_{\min}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big] > 0$.*
2. *There exist g_k in $\ell^2(G)$, $k = 1,2,\dots,K$, such that the associated polyphase matrix $\mathbf{G}(\gamma)$ given in* (6) *has all its entries in $L^\infty(\widehat{N})$, and it satisfies $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$.*
3. *There exist K elements $c_k \in \mathcal{A}_a$ such that the sequence $\big\{U(n,1_H)c_k\big\}_{n\in N;\,k=1,2,\dots,K}$ is a frame for $\mathcal{A}_a$ and for each $x \in \mathcal{A}_a$ the sampling formula*
$$x = \sum_{k=1}^{K}\sum_{n\in N}\mathcal{L}_k x(n)\,U(n,1_H)c_k \quad \text{in } \mathcal{H} \tag{16}$$
holds.
4. *There exists a frame $\big\{C_{k,n}\big\}_{n\in N;\,k=1,2,\dots,K}$ for $\mathcal{A}_a$ such that for each $x \in \mathcal{A}_a$ the expansion*
$$x = \sum_{k=1}^{K}\sum_{n\in N}\mathcal{L}_k x(n)\,C_{k,n} \quad \text{in } \mathcal{H}$$
holds.

Proof. (1) implies (2). The $L \times K$ Moore-Penrose pseudo-inverse $\mathbf{H}^\dagger(\gamma)$ of $\mathbf{H}(\gamma)$ is given by $\mathbf{H}^\dagger(\gamma) = \big[\mathbf{H}^*(\gamma)\,\mathbf{H}(\gamma)\big]^{-1}\mathbf{H}^*(\gamma)$. Its entries are essentially bounded in $\widehat{N}$ since the entries of $\mathbf{H}(\gamma)$ belong to $L^\infty(\widehat{N})$ and $\det^{-1}\big[\mathbf{H}^*(\gamma)\,\mathbf{H}(\gamma)\big]$ is essentially bounded $\widehat{N}$ since $0 < A_{\mathbf{H}}$. Besides, $\mathbf{H}^\dagger(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$. The inverse N-Fourier transform in $L^2(\widehat{N})$ of the k-th column of $\mathbf{H}^\dagger(\gamma)$ gives g_k, $k = 1,2,\dots,K$.

(2) implies (3). According to Theorems 3 and 5 the sequences $\big\{T_n\widetilde{\mathsf{h}}_k\big\}_{n\in N;\,k=1,2,\dots K}$ and $\big\{T_n\mathsf{g}_k\big\}_{n\in N;\,k=1,2,\dots K}$ form a pair of dual frames for $\ell^2(G)$. We deduce the expansion sampling expansion as in (15). Besides, the sequence $\big\{U(n,1_H)c_{k,\mathsf{g}}\big\}_{n\in N;\,k=1,2,\dots,K}$ is a frame for $\mathcal{A}_a$.

Obviously, (3) implies (4). Finally, (4) implies (1). Applying $\mathcal{T}_{U,a}^{-1}$ we get that the sequences $\big\{T_n\widetilde{\mathsf{h}}_k\big\}_{n\in N;\,k=1,2,\dots K}$ and $\{\mathcal{T}_{U,a}^{-1}(C_{k,n})\}_{n\in N;\,k=1,2,\dots,K}$ form a pair of dual frames for $\ell^2(G)$; in particular, by using Theorem 3 we obtain that $0<A_{\mathbf{H}}$. □

All the possible solutions of $\mathbf{G}(\gamma)\mathbf{H}(\gamma)=\mathbf{I}_L$ a.e. $\gamma\in\widehat{N}$ with entries in $L^\infty(\widehat{N})$ are given in terms of the Moore-Penrose pseudo inverse by the $L\times K$ matrices $\mathbf{G}(\gamma):=\mathbf{H}^\dagger(\gamma)+\mathbf{U}(\gamma)\big[\mathbf{I}_K-\mathbf{H}(\gamma)\mathbf{H}^\dagger(\gamma)\big]$, where $\mathbf{U}(\gamma)$ denotes any $L\times K$ matrix with entries in $L^\infty(\widehat{N})$.

Notice that $K\geq L$ where L is the order of the group H. In case $K=L$, we obtain:

Corollary 8. *In the case $K=L$, assume that its polyphase matrix $\mathbf{H}(\gamma)$ given in (5) has all entries in $L^\infty(\widehat{N})$. The following statements are equivalents:*

1. *The constant $A_{\mathbf{H}}=\operatorname*{ess\,inf}_{\gamma\in\widehat{N}}\lambda_{\min}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big]>0$.*
2. *There exist L unique elements c_k, $k=1,2,\dots,L$, in $\mathcal{A}_a$ such that the associated sequence $\big\{U(n,1_H)c_k\big\}_{n\in N;\,k=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_a$ and the sampling formula*

$$x=\sum_{k=1}^{L}\sum_{n\in N}\mathcal{L}_kx(n)\,U(n,1_H)c_k\qquad\text{in }\mathcal{H}$$

   holds for each $x\in\mathcal{A}_a$.

Moreover, the interpolation property $\mathcal{L}_kc_{k'}(n)=\delta_{k,k'}\delta_{n,0_N}$, where $n\in N$ and $k,k'=1,2,\dots,L$, holds.

Proof. In this case, the square matrix $\mathbf{H}(\gamma)$ is invertible and the result comes out from Theorem 5. From the uniqueness of the coefficients in a Riesz basis we get the interpolation property. □

Denote $H=\{h_1,h_2,\dots,h_L\}$; for a fixed $b\in\mathcal{H}$, we consider the samples

$$\mathcal{L}_kx(m):=\big\langle x,U(m,h_k)b\big\rangle\,,\qquad m\in N\text{ and }k=1,2,\dots,L\,,$$

of any $x\in\mathcal{A}_a$. Since $U(m,h_k)b=U(m,1_H)U(0_N,h_k)b=U(m,1_H)b_k$, where $b_k:=U(0_N,h_k)b$, $k=1,2,\dots,L$, we are in a particular case of (12) with $K=L$.

5.1 An example involving crystallographic groups

The Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d\rtimes_\phi O(d)$ corresponding to the homomorphism $\phi:O(d)\to Aut(\mathbb{R}^d)$ given by $\phi_A(x)=Ax$, where $A\in O(d)$ and $x\in\mathbb{R}^d$. The composition law on $E(d)=\mathbb{R}^d\rtimes_\phi O(d)$ reads $(x,A)\cdot(x',A')=(x+Ax',AA')$.

Let M be a non-singular $d\times d$ matrix and Γ a finite subgroup of $O(d)$ of order L such that $A(M\mathbb{Z}^d)=M\mathbb{Z}^d$ for each $A\in\Gamma$. We consider the *crystallographic group* $\mathcal{C}_{M,\Gamma}:=M\mathbb{Z}^d\rtimes_\phi\Gamma$ and its quasi regular representation (see Ref. [2]) on $L^2(\mathbb{R}^d)$

$$U(n,A)f(t)=f[A^\top(t-n)]\,,\quad n\in M\mathbb{Z}^d,\ A\in\Gamma\text{ and }f\in L^2(\mathbb{R}^d)\,.$$

For a fixed $\varphi \in L^2(\mathbb{R}^d)$ such that the sequence $\{U(n,A)\varphi\}_{(n,A)\in\mathcal{C}_{M,\Gamma}}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ we consider the U-invariant subspace in $L^2(\mathbb{R}^d)$

$$\begin{aligned}\mathcal{A}_\varphi &= \Big\{ \sum_{(n,A)\in\mathcal{C}_{M,\Gamma}} \alpha(n,A)\,\varphi[A^\top(t-n)] \;:\; \{\alpha(n,A)\}\in\ell^2(\mathcal{C}_{M,\Gamma})\Big\} \\ &= \Big\{ \sum_{(n,A)\in\mathcal{C}_{M,\Gamma}} \alpha(n,A)\,\varphi(At-n) \;:\; \{\alpha(n,A)\}\in\ell^2(\mathcal{C}_{M,\Gamma})\Big\}.\end{aligned}$$

Choosing K functions $b_k \in L^2(\mathbb{R}^d)$, $k=1,2,\dots,K$, we consider the average samples of $f \in \mathcal{A}_\varphi$

$$\mathcal{L}_k f(n) = \langle f, U(n,I)b_k\rangle = \langle f, b_k(\cdot - n)\rangle\,, \quad n \in M\mathbb{Z}^d\,.$$

Under the hypotheses in Theorem 7, there exist $K \ge L$ sampling functions $\psi_k \in \mathcal{A}_\varphi$ for $k=1,2,\dots,K$, such that the sequence $\{\psi_k(\cdot - n)\}_{n\in M\mathbb{Z}^d;\, k=1,2,\dots,K}$ is a frame for $\mathcal{A}_\varphi$, and the sampling expansion

$$f(t) = \sum_{k=1}^{K}\sum_{n\in M\mathbb{Z}^d} \big\langle f, b_k(\cdot-n)\big\rangle_{L^2(\mathbb{R}^d)}\,\psi_k(t-n) \quad \text{in } L^2(\mathbb{R}^d) \tag{17}$$

holds.

If the generator $\varphi \in C(\mathbb{R}^d)$ and the function $t \mapsto \sum_n |\varphi(t-n)|^2$ is bounded on $\mathbb{R}^d$, a standard argument shows that $\mathcal{A}_\varphi$ is a reproducing kernel Hilbert space (RKHS) of continuous functions in $L^2(\mathbb{R}^d)$. As a consequence, convergence in $L^2(\mathbb{R}^d)$-norm implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

Notice that the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ is a particular crystallographic group with lattice $\mathbb{Z}$ and $\Gamma = \mathbb{Z}_2$. Its quasi regular representation on $L^2(\mathbb{R})$ reads

$$U(n,0)f(t) = f(t-n) \quad \text{and} \quad U(n,1)f(t) = f(-t+n)\,, \quad n \in \mathbb{Z} \text{ and } f \in L^2(\mathbb{R})\,.$$

So we could obtain sampling formulas as (17) for $K \ge 2$ average functions b_k.

The quasi regular unitary representation of a crystallographic group $\mathcal{C}_{M,\Gamma}$ on $L^2(\mathbb{R}^d)$ motivates the next section:

5.2 The case of pointwise samples

Let $\{U(n,h)\}_{(n,h)\in G}$ be a unitary representation of the group $G = N \rtimes_\phi H$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. If the generator $\varphi \in L^2(\mathbb{R}^d)$ satisfies that, for each $(n,h) \in G$, the function $U(n,h)\varphi$ is continuous on $\mathbb{R}^d$, and the condition

$$\sup_{t\in\mathbb{R}^d} \sum_{(n,h)\in G} \big|[U(n,h)\varphi](t)\big|^2 < \infty\,,$$

then the subspace $\mathcal{A}_\varphi$ is a RKHS of continuous functions in $L^2(\mathbb{R}^d)$; proceeding as in [17] one can prove that the above conditions are also necessary.

For K fixed points $t_k \in \mathbb{R}^d$, $k=1,2,\dots,K$, we consider for each $f \in \mathcal{A}_\varphi$ the new samples given by

$$\mathcal{L}_k f(n) := \big[U(-n,1_H)f\big](t_k)\,, \quad n \in N \text{ and } k = 1,2,\cdots,K\,. \tag{18}$$

For each $f=\sum_{(m,h)\in G}\alpha(m,h)\,U(m,h)\,\varphi$ in $\mathcal{A}_\varphi$ and $k=1,2,\dots,K$ we have

$$\begin{aligned}\mathcal{L}_k f(n)&=\Big[\sum_{(m,h)\in G}\alpha(m,h)\,U[(-n,1_H)\cdot(m,h)]\,\varphi\Big](t_k)\\&=\sum_{(m,h)\in G}\alpha(m,h)\big[U(m-n,h)\varphi\big](t_k)=\big\langle\boldsymbol{\alpha},T_n\mathsf{h}_k\big\rangle_{\ell^2(G)}\,,\quad n\in N\,,\end{aligned}$$

where $\boldsymbol{\alpha}=\{\alpha(m,h)\}_{(m,h)\in G}$ and $\mathsf{h}_k(m,h):=\overline{\big[U(m,h)\varphi\big](t_k)}$, $(m,h)\in G$. Notice that h_k belongs to $\ell^2(G)$, $k=1,2,\cdots,K$. As a consequence, under the hypotheses in Theorem 7 (on these new $\mathsf{h}_k\in\ell^2(G)$, $k=1,2,\dots,K$) a sampling formula as (16) holds for the data sequence $\big\{\mathcal{L}_k f(n)\big\}_{n\in N;\,k=1,2,\dots,K}$ defined in (18).

In the particular case of the quasi regular representation of a crystallographic group $\mathcal{C}_{M,\Gamma}=M\mathbb{Z}^d\rtimes_\phi\Gamma$, for each $f\in\mathcal{A}_\varphi$ the samples (18) read

$$\mathcal{L}_k f(n)=\big[U(-n,I)f\big](t_k)=f(t_k+n)\,,\quad n\in M\mathbb{Z}^d\ \text{ and }\ k=1,2,\dots,K\,.$$

Thus (under hypotheses in Theorem 7), there exist K functions $\psi_k\in\mathcal{A}_\varphi$, $k=1,2,\dots,K$, such that for each $f\in\mathcal{A}_\varphi$ the sampling formula

$$f(t)=\sum_{k=1}^{K}\sum_{n\in M\mathbb{Z}^d}f(t_k+n)\,\psi_k(t-n)\,,\quad t\in\mathbb{R}^d$$

holds. The convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

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