Analytical Sampling, Lagrange-Type Interpolation Series and de Branges Spaces

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Introduction

We consider the classical Whittaker-Shannon-Kotelnikov sampling theorem in the Paley-Wiener space

$$PW_\pi = \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \quad \text{supp}\ \widehat{f} \subseteq [-\pi, \pi] \right\}$$

where $\widehat{f}$ stands for the Fourier transform. Any function f in PW_π can be written as

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(\omega) e^{iz\omega} d\omega = \left\langle \frac{e^{iz\omega}}{\sqrt{2\pi}}, \overline{\widehat{f}} \right\rangle_{L^2[-\pi,\pi]}, \quad z \in \mathbb{C}.$$

with $\widehat{f} \in L^2[-\pi, \pi]$ and the Fourier kernel (denoted K) is given by

$$K : \begin{array}{l} \mathbb{C} \to L^2[-\pi, \pi] \\ z \to K(z) \end{array}, \qquad [K(z)](\omega) = \frac{e^{iz\omega}}{\sqrt{2\pi}}, \quad \omega \in [-\pi, \pi].$$

The following well-known sampling theorem holds:

Whittaker-Shannon-Kotelnikov sampling theorem:

Any function f in the Paley-Wiener space PW_π can be recovered from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ as the cardinal series

$$f(z) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z \in \mathbb{C}.$$

where the convergence in the series is absolute and uniform on horizontal strips of $\mathbb{C}$ since $\|K(z)\|_{L^2[-\pi,\pi]} \leq e^{\pi|y|}$ for all $z = x + iy \in \mathbb{C}$.

The Hilbert space $\mathcal{H}_K$

Given a complex, **separable Hilbert space** $\mathbb{H}$ and an $\mathbb{H}$-valued function

$$K : \begin{array}{rcl} \mathbb{C} & \longrightarrow & \mathbb{H} \\ z & \longmapsto & K(z) \end{array}$$

we define a mapping between $\mathbb{H}$ and the set $\mathcal{F}(\mathbb{C}, \mathbb{C}) := \{f : \mathbb{C} \longrightarrow \mathbb{C}\}$ as follows:

$$\mathcal{T}_K : \begin{array}{rcl} \mathbb{H} & \longrightarrow & \mathcal{F}(\mathbb{C}, \mathbb{C}) \\ x & \longmapsto & \mathcal{T}_K(x) = f_x \end{array}$$

such that, $f_x(z) = \langle K(z), x\rangle_{\mathbb{H}}$, $z \in \mathbb{C}$; and denote by $\mathcal{H}_K$ the linear space of all functions $f_x(z)$ in the range space of $\mathcal{T}_K$; i.e.,

$$\mathcal{H}_K := \mathcal{T}_K(\mathbb{H}) = \{f : \mathbb{C} \longrightarrow \mathbb{C} : f(z) = \langle K(z), x\rangle_{\mathbb{H}},\ x \in \mathbb{H}\}.$$

Some properties of the space $\mathcal{H}_K$

- The space $\mathcal{H}_K$, endowed with the norm

  $$\|f\|_{\mathcal{H}_K} := \inf\{\|x\|_{\mathbb{H}} : f = \mathcal{T}_K(x)\}$$

  becomes a Hilbert Space.
- The mapping $\mathcal{T}_K$ is a bijective isometry from $\mathbb{H}$ to $\mathcal{H}_K$ if and only if $\{K(z) : z \in \mathbb{C}\}$ is complete in $\mathbb{H}$ or equivalently if and only if $\mathcal{T}_K$ is injective. In particular, if there exist $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ such that $\{K(z_n)\}_{n=1}^\infty$ is a basis in $\mathbb{H}$, then $\mathcal{T}_K$ is an antilinear isometry from $\mathbb{H}$ onto $\mathcal{H}_K$.
- $\mathcal{H}_K$ is an **Hilbert space of reproducing kernel** (RKHS in short), i.e., the evaluation functional

  $$E_z : \begin{array}{rcl} \mathcal{H}_K & \longrightarrow & \mathbb{C} \\ f & \longmapsto & f(z) \end{array}$$

  is bounded for each $z \in \mathbb{C}$.
- As a consequence, convergence in the norm $\|\cdot\|_{\mathcal{H}_K}$ implies pointwise convergence which will be uniform on subsets of $\mathbb{C}$ where $\|K(\cdot)\|_{\mathbb{H}}$ is bounded.
- The reproducing kernel of $\mathcal{H}_K$ is given by

  $$\kappa(z, \omega) = \langle K(z), K(\omega)\rangle_{\mathbb{H}}$$

  and it verifies the reproducing property

  $$f(\omega) = \langle f(\cdot), \kappa(\cdot, \omega)\rangle_{\mathcal{H}} \text{ for each } \omega \in \mathbb{C} \text{ and } f \in \mathcal{H}$$

Analyticity of the functions in $\mathcal{H}_K$

Theorem: $\mathcal{H}_K$ is a RKHS of entire functions if and only if the kernel K is analytic in $\mathbb{C}$.

Characterization of the analyticity of the functions in $\mathcal{H}_K$ in terms of Riesz bases.

Suppose that a Riesz basis $\{x_n\}_{n=1}^\infty$ is given and let $\{x_n^*\}_{n=1}^\infty$ be its dual Riesz basis. Expanding $K(z)$, for $z \in \mathbb{C}$ fixed, with respect to this basis we obtain

$$K(z) = \sum_{n=1}^\infty \langle K(z), x_n^*\rangle_{\mathbb{H}}\, x_n$$

where the sequence of coefficients

$$S_n(z) := \langle K(z), x_n^*\rangle_{\mathbb{H}}$$

as functions in z are in $\mathcal{H}_K$. The following result holds:

> **Theorem**
>
> The space $\mathcal{H}_K$ is a RKHS of entire functions if and only if the functions $\{S_n\}_{n=1}^\infty$ are entire and the function $z \mapsto \|K(z)\|_{\mathbb{H}}$ is bounded on compact sets of $\mathbb{C}$.

Sampling in the space $\mathcal{H}_K$

An analytic kernel $K : \mathbb{C} \longrightarrow \mathbb{H}$ is said to be an **analytic Kramer kernel** if there exist sequences $\{z_n\}_{n=1}^\infty$in $\mathbb{C}$, $\{a_n\}_{n=1}^\infty$ in $\mathbb{C}\setminus\{0\}$ and a Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathbb{H}$, such that

$$K(z_n) = a_n x_n \quad \text{for all } n \in \mathbb{N}.$$

> **Analytic Kramer sampling theorem**
>
> Let $K : \mathbb{C} \longrightarrow \mathbb{H}$ be an analytic Kramer kernel as in above definition and $\mathcal{H}_K$ its corresponding RKHS of entire functions.
>
> Then, any $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(z_n)\}_{n=1}^\infty$ by means of the sampling series
>
> $$f(z) = \sum_{n=1}^\infty \frac{f(z_n)}{a_n} S_n(z), \quad z \in \mathbb{C}.$$
>
> This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Lagrange-type interpolation series in $\mathcal{H}_K$

Note that the WSK sampling formula can be written as

$$f(z) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)} = \sum_{n\in\mathbb{Z}} f(n) \frac{G(z)}{(z-n)G'(n)},$$

where $G(z) = (\sin \pi z)/\pi$.

> **Problem:**
> In the analytic Kramer sampling theorem, a more difficult question concerns whether the sampling expansion
>
> $$f(z) = \sum_{n=1}^\infty \frac{f(z_n)}{a_n} S_n(z), \quad z \in \mathbb{C},$$
>
> in $\mathcal{H}_K$ (K an analytic Kramer kernel), can be written as a Lagrange-type interpolation series.

A necessary and sufficient condition involves the following algebraic property:

Zero removing property: A space $\mathcal{H}$ of entire functions has the zero-removing property (ZR in short) if for any $g \in \mathcal{H}$ and any zero ω of g the function $\frac{g(z)}{z-\omega}$ belongs to $\mathcal{H}$.

> **Theorem (Lagrange-type interpolation series)**
>
> Let $\mathcal{H}_K$ be a RKHS of entire functions obtained from an analytic Kramer kernel K with respect to the sequences $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^\infty$ in $\mathbb{C}\setminus\{0\}$, i.e., for some Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathbb{H}$, $K(z_n) = a_n x_n$, $n \in \mathbb{N}$.
>
> Then, the sampling formula $f(z) = \sum_{n=1}^\infty \frac{f(z_n)}{a_n} S_n(z)$, $z \in \mathbb{C}$, for $\mathcal{H}_K$ can be written as a Lagrange-type interpolation series
>
> $$f(z) = \sum_{n=1}^\infty f(z_n) \frac{Q(z)}{Q'(z_n)(z-z_n)}.$$
>
> where Q denotes an entire function having only simple zeros at $\{z_n\}_{n=1}^\infty$, if and only if the space $\mathcal{H}_K$ satisfies the ZR property.

Note: The entire function Q is such that $(z - z_n)S_n(z) = \sigma_n Q(z)$ for some nonzero constants σ_n , $n \in \mathbb{N}$.

The space $\mathcal{H}_K$ as de Branges space

The Paley-Wiener spaces can be seen as special cases of a more general class of Hilbert spaces of entire functions: **de Branges spaces**:

Let E be an entire function verifying $|E(\overline{z})| < |E(z)|$ for $\text{Im}(z) > 0$. A de Branges space $\mathcal{H}(E)$ is the set of all entire functions f such that

$$\|f\|^2_{\mathcal{H}(E)} = \int_{-\infty}^{+\infty} \left|\frac{f(x)}{E(x)}\right|^2 dx < \infty$$

and such that both ratios $\frac{f(z)}{E(z)}$ and $\frac{\overline{f(\overline{z})}}{E(z)}$ are of **bounded type** and **nonpositive mean type** in $\mathbb{C}^+ := \{z : \text{Im}(z) > 0\}$.

Some properties of the de Branges spaces

- Any de Branges function E can be written as $E(z) = A(z) - iB(z)$ where A and B are entire functions which are real when z is real, given by

  $$A(z) = \tfrac{1}{2}(E(z) + \overline{E(\overline{z})}),\ B(z) = \tfrac{i}{2}(E(z) - \overline{E(\overline{z})})$$

  and the functions $A(z)$ and $B(z)$ have only real zeros and these zeros interlace.
- Whenever $\mathcal{H}(E)$ is a RKHS, its reproducing kernel is

  $$\kappa(\omega, z) := \frac{\overline{A(\omega)}B(z) - A(z)\overline{B(\omega)}}{\pi(z - \overline{\omega})}, \quad z, \omega \in \mathbb{C}$$

  This kernel has the property that for each $f(z) \in \mathcal{H}(E)$, there holds

  $$f(\omega) = \langle f(\cdot), \kappa(\omega, \cdot)\rangle_{\mathcal{H}(E)} \qquad \text{for all } \omega \in \mathbb{C}$$
- If E is a strict de Branges function, i.e., E has no zeros in $\mathbb{R}$, then de Branges space $\mathcal{H}(E)$ **satisfies the ZR property**.
- The continuos function $\varphi(x)$ of real x is said be a phase function associated with $E(z)$ if $E(x)e^{i\varphi(x)}$ is real-valued for all $x \in \mathbb{R}$.

Sampling in $\mathcal{H}(E)$

If α is a given real number such that the function $e^{i\alpha}E(z) - e^{-i\alpha}\overline{E(\overline{z})}$ does not belong to $\mathcal{H}(E)$, then the sequence of real numbers $\{t_n\}$ satisfying $\varphi(t_n) = \alpha \mod \pi$ gives an orthogonal basis $\{\kappa(t_n, \cdot)\}$ for $\mathcal{H}(E)$.

Consequently, the following result holds:

> **Theorem**
>
> Let $\mathcal{H}(E)$ be de Branges space, $\{t_n\}$ a sequence of real numbers and $\{\kappa(t_n, \cdot)\}$ an orthogonal basis in $\mathcal{H}(E)$. Then, any function $f \in \mathcal{H}(E)$ can be recovered from its samples $\{f(t_n)\}$ through the sampling formula
>
> $$f(z) = \sum_n f(t_n) \frac{\kappa(t_n, z)}{\kappa(t_n, t_n)} = \sum_n f(t_n) \frac{Q(z)}{(z - t_n)Q'(t_n)}, \quad z \in \mathbb{C}$$
>
> Where Q is an entire function having only simple zeros at $\{t_n\}$. This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Note: The entire function Q is such that $(z - t_n)\kappa(t_n, z) = \sigma_n Q(z)$ for some nonzero constants σ_n , $n \in \mathbb{N}$.

> **Theorem**
>
> A space $\mathcal{H}_K$ is a de Branges space if and only if there exists an orthogonal sampling formula
>
> $$f(z) = \sum_{n=1}^\infty \frac{f(t_n)}{a_n} S_n(z), \quad z \in \mathbb{C},$$
>
> in $\mathcal{H}_K$ such that it can be written as a Lagrange-type interpolation formula
>
> $$f(z) = \sum_{n=1}^\infty f(t_n) \frac{Q(z)}{(z - t_n)Q'(t_n)}, \quad z \in \mathbb{C}$$
>
> where the sampling points $\{t_n\}_{n=1}^\infty$ are real, and Q is an entire function having only simple zeros at $\{t_n\}_{n=1}^\infty$ and satisfying $\overline{Q(\overline{z})} = Q(z)$.

Acknowledgements

This work is supported by Project No. MTM2009-08345 from the Spanish Ministerio de Ciencia y Tecnología.

The proofs of the results above exhibited can be found in the references below. See also references therein for previous and related works.

[1] **"De Branges spaces, Analytic Kramer kernels and Lagrange-type interpolation series"**. *Complex Variables and Elliptic Equations, 2011.* doi:10.1080/17476933.2010.551206

[2] **"The zero-removing property and Lagrange-type interpolation series"** .*Numer. Funct. Anal. Optim., 2011.* doi:10.1080/01630563.2011.587076