On generalized sampling in U-invariant spaces

Héctor R. Fernández Morales

Supervisor: **Prof. Dr. D. Antonio García García**

Departamento de Matemáticas
Universidad Carlos III de Madrid

May 17, 2013

Outline

Outline

1. **Motivation**

2. **Regular case**

Outline

1. Motivation
2. Regular case
3. Irregular case

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi \in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, \varphi(t-n) \ : \ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}$$

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi \in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, \varphi(t-n) \; : \; \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}$$

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a **Riesz sequence** if there exists constants $0 < c \leq C < \infty$ such that

$$c\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \leq \Big\| \sum_{n\in\mathbb{Z}} a_n x_n \Big\|^2 \leq C\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big)$$

for all $\{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi \in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, \varphi(t-n) \; : \; \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}$$

A **Riesz basis** in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator

If we consider

- $\mathcal{L}_j f := f * \mathrm{h}_j, \;\; j = 1, 2, \ldots s$ are convolutions systems (linear time-invariant systems) defined on V_φ^2.

If we consider

- $\mathcal{L}_j f := f * \mathrm{h}_j, \;\; j = 1, 2, \dots s$ are convolutions systems (linear time-invariant systems) defined on V_φ^2.
- The sequence of samples $\{(\mathcal{L}_j f)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ where $r \in \mathbb{N}$, is available for any f in V_φ^2.

If we consider

- $\mathcal{L}_j f := f * \mathrm{h}_j, \;\; j = 1, 2, \dots s$ are convolutions systems (linear time-invariant systems) defined on V_φ^2.
- The sequence of samples $\{(\mathcal{L}_j f)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ where $r \in \mathbb{N}$, is available for any f in V_φ^2.

The generalized sampling problem is to obtain sampling formulas in V_φ^2 having the form

$$f(t) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} (\mathcal{L}_j f)(rm)\, S_j(t - rm)\,, \quad t \in \mathbb{R}\,,$$

where the reconstruction sequence of functions $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for V_φ^2 .

If we consider

- $\mathcal{L}_j f := f * \mathrm{h}_j, \;\; j = 1, 2, \dots s$ are convolutions systems (linear time-invariant systems) defined on V_φ^2.
- The sequence of samples $\{(\mathcal{L}_j f)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ where $r \in \mathbb{N}$, is available for any f in V_φ^2.

A sequence $\{f_k\}_{k=1}^{\infty}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants A, $B > 0$ (frame bounds) such that

$$A\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B\|f\|^2 \quad \text{for all } f \in \mathcal{H}$$

Generalize this problem in the following sense:

Let U be an **unitary operator** in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by

Generalize this problem in the following sense:

Let U be an **unitary operator** in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by

$$\mathcal{A}_a := \overline{\text{span}}\{U^n a,\ n \in \mathbb{Z}\}.$$

Generalize this problem in the following sense:

Let U be an **unitary operator** in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by

$$\mathcal{A}_a := \overline{\operatorname{span}}\{U^n a,\ n \in \mathbb{Z}\}.$$

In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n U^n a : \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}.$$

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta}\, d\mu_a(\theta)\,, \qquad k\in\mathbb{Z}\,,$$

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta)\,, \qquad k \in \mathbb{Z}\,,$$

- The positive Borel spectral measure μ_a can be decomposed as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$.

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta)\,, \qquad k\in\mathbb{Z}\,,$$

- The positive Borel spectral measure μ_a can be decomposed as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$.

- The sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty\,.$$

Given the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ with $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, a challenging problem is:

Given the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ with $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, a challenging problem is:

- Characterize the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ as a **frame (Riesz basis)** in $\mathcal{A}_a$.

Given the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ with $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, a challenging problem is:

- Characterize the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ as a **frame (Riesz basis)** in $\mathcal{A}_a$.
- Look for those **dual frames** having the same form $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ for some $c_j \in \mathcal{A}_a$, so that, for any $x \in \mathcal{A}_a$ the **expansion**

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{rk}b_j\rangle\, U^{rk}c_j \quad \text{in } \mathcal{H}$$

holds.

Remark

In the shift-invariant case, U is defined as the shift operator $U : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$ and we have

$$\langle f, U^{rk} b \rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} f(u) \overline{b(u-rk)} du = (f * h)(rk)\,, \quad u \in \mathbb{R}\,,$$

where $h(u) := \overline{b(-u)}$.

Procedure

For every $j = 1, 2, \dots s$ we have the following representation

$$\langle U^k a, U^{nr} b_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-rn)\theta} \phi_{a,b_j}(e^{i\theta}) d\theta.$$

Procedure

For every $j = 1, 2, \dots s$ we have the following representation

$$\langle U^k a, U^{nr} b_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-rn)\theta} \phi_{a,b_j}(e^{i\theta}) d\theta.$$

Consider the $s \times 1$ matrices of functions defined on the torus $\mathbb{T} := \{e^{i\theta} \, : \, \theta \in [-\pi, \pi)\}$

$$\Phi_{a,b}(e^{i\theta}) := \begin{pmatrix} \phi_{a,b_1}(e^{i\theta}) \\ \phi_{a,b_2}(e^{i\theta}) \\ \vdots \\ \phi_{a,b_s}(e^{i\theta}) \end{pmatrix},$$

Procedure

For every $j = 1, 2, \ldots s$ we have the following representation

$$\langle U^k a, U^{nr} b_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-rn)\theta} \phi_{a,b_j}(e^{i\theta}) d\theta.$$

Consider the $s \times 1$ matrices of functions defined on the torus $\mathbb{T} := \{e^{i\theta} \ : \ \theta \in [-\pi, \pi)\}$

$$\Phi_{a,b}(e^{i\theta}) := \begin{pmatrix} \phi_{a,b_1}(e^{i\theta}) \\ \phi_{a,b_2}(e^{i\theta}) \\ \vdots \\ \phi_{a,b_s}(e^{i\theta}) \end{pmatrix},$$

and

$$\Psi^l_{a,b}(e^{i\theta}) := (D_r S^{-l} \Phi_{a,b})(e^{i\theta}), \quad l = 0, 1, \ldots, r-1.$$

Where $D_r : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the *decimation operator*

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \overset{D_r}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{rk} e^{ik\theta}$$

Where $D_r : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the *decimation operator*

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{D_r}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{rk} e^{ik\theta}$$

and $S : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the *(left) shift operator*

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{S}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{k+1} e^{ik\theta} .$$

Where $D_r : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the *decimation operator*

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{D_r}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{rk} e^{ik\theta}$$

and $S : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the *(left) shift operator*

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{S}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{k+1} e^{ik\theta}\,.$$

Finally, defining the $s \times r$ matrix of functions on the torus $\mathbb{T}$

$$\boldsymbol{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(e^{i\theta}) := \left(\Psi^0_{a,b}(e^{i\theta})\ \Psi^1_{a,b}(e^{i\theta})\ \ldots \Psi^{r-1}_{a,b}(e^{i\theta})\right),$$

Where $D_r : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the *decimation operator*

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{D_r}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{rk} e^{ik\theta}$$

and $S : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the *(left) shift operator*

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{S}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{k+1} e^{ik\theta} .$$

Finally, defining the $s \times r$ matrix of functions on the torus $\mathbb{T}$

$$\boldsymbol{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(e^{i\theta}) := \left(\Psi^0_{a,b}(e^{i\theta})\ \Psi^1_{a,b}(e^{i\theta})\ \dots \Psi^{r-1}_{a,b}(e^{i\theta}) \right),$$

and its related constants,

$$A_{\boldsymbol{\Psi}} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \lambda_{\min}\left[\boldsymbol{\Psi}_{\boldsymbol{a},\boldsymbol{b}}{}^*(\zeta)\boldsymbol{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(\zeta)\right];$$

$$B_{\boldsymbol{\Psi}} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \lambda_{\max}\left[\boldsymbol{\Psi}_{\boldsymbol{a},\boldsymbol{b}}{}^*(\zeta)\boldsymbol{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(\zeta)\right]$$

Theorem.

Let b_j be in $\mathcal{A}_a$ for $j = 1, 2, \ldots, s$ and let $\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}$ be the associated matrix. Then, the following results hold:

- **i)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **complete system** in $\mathcal{A}_a$ if and only if the rank of the matrix $\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(\zeta)$ is r a.e. ζ in $\mathbb{T}$.
- **ii)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **Bessel sequence** for $\mathcal{A}_a$ if and only if the constant $B_{\mathbf{\Psi}} < \infty$.
- **iii)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **frame** for $\mathcal{A}_a$ if and only if constants $A_{\mathbf{\Psi}}$ and $B_{\mathbf{\Psi}}$ satisfy $0 < A_{\mathbf{\Psi}} \le B_{\mathbf{\Psi}} < \infty$. In this case, $A_{\mathbf{\Psi}}$ and $B_{\mathbf{\Psi}}$ are the optimal frame bounds for $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$.
- **iv)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **Riesz basis** for $\mathcal{A}_a$ if and only if it is a frame and $s = r$.

The frame expansion

Taking into account the $r \times s$ matrix $\mathbf{\Gamma}_{\mathbb{U}}$ of functions on $\mathbb{T}$

$$\mathbf{\Gamma}_{\mathbb{U}}(e^{i\theta}) := \mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(e^{i\theta}) + \mathbb{U}(e^{i\theta})\big[\mathbb{I}_s - \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(e^{i\theta})\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(e^{i\theta})\big],$$

The frame expansion

Taking into account the $r \times s$ matrix $\mathbf{\Gamma}_{\mathbb{U}}$ of functions on $\mathbb{T}$

$$\mathbf{\Gamma}_{\mathbb{U}}(\mathrm{e}^{\mathrm{i}\theta}) := \mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) + \mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})\left[\mathbb{I}_s - \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\right],$$

where $\mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})$ is any $r \times s$ matrix with entries in $L^\infty(\mathbb{T})$, and $\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}$ denotes the Moore-Penrose left-inverse of $\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}$,

$$\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := [\mathbf{\Psi}^{*}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})]^{-1}\mathbf{\Psi}^{*}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}).$$

The frame expansion

Taking into account the $r \times s$ matrix $\mathbf{\Gamma}_{\mathbb{U}}$ of functions on $\mathbb{T}$

$$\mathbf{\Gamma}_{\mathbb{U}}(\mathrm{e}^{\mathrm{i}\theta}) := \mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) + \mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})\left[\mathbb{I}_s - \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\right],$$

where $\mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})$ is any $r \times s$ matrix with entries in $L^\infty(\mathbb{T})$, and $\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}$ denotes the Moore-Penrose left-inverse of $\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}$,

$$\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := [\mathbf{\Psi}^{*}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})]^{-1}\mathbf{\Psi}^{*}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}).$$

We can find $c_j \in \mathcal{A}_a$ such that the sequences $\{U^{kr}c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ and $\{U^{kr}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ are a pair of **dual frames** for $\mathcal{A}_a$. Hence, for any $x \in \mathcal{A}_a$ we obtain the following recovery formula

$$x = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}}\langle x, U^{kr}b_j\rangle U^{kr}c_j \quad \text{in } \mathcal{H}.$$

Irregular case

We are going to deal with the following problems:

Irregular case

We are going to deal with the following problems:

- Study the perturbed sequence $\left\{U^{rk+\epsilon_{kj}} b_j\right\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$.

Irregular case

We are going to deal with the following problems:

- Study the perturbed sequence $\left\{U^{rk+\epsilon_{kj}}b_j\right\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.
- Recover $x \in \mathcal{A}_a$ from the perturbed sequence of samples $\left\{\langle x, U^{rk+\epsilon_{kj}}b_j\rangle\right\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

Irregular case

We are going to deal with the following problems:

- Study the perturbed sequence $\left\{U^{rk+\epsilon_{kj}}b_j\right\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.
- Recover $x \in \mathcal{A}_a$ from the perturbed sequence of samples $\left\{\langle x, U^{rk+\epsilon_{kj}}b_j\rangle\right\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

In order to give sense to $U^{rk+\epsilon_{mj}}b_j$ we need to introduce a **continuous group of unitary operators** $\{U^t\}_{t\in\mathbb{R}}$, such that $U = U^1$.

A brief walk on continuous groups of unitary operators

A brief walk on continuous groups of unitary operators

$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\,U^{t'} = U^{t+t'}$,
2. $U^0 = I_{\mathcal{H}}$,
3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

A brief walk on continuous groups of unitary operators

$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\, U^{t'} = U^{t+t'}$,
2. $U^0 = I_{\mathcal{H}}$,
3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

Classical **Stone's theorem** assures us the existence of a self-adjoint operator T (possibly unbounded) such that $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator T, defined on the dense domain of $\mathcal{H}$

$$D_T := \left\{ x \in \mathcal{H} \text{ such that } \int_{-\infty}^{\infty} w^2\, d\|E_w x\|^2 < \infty \right\},$$

admits the *spectral representation* $T = \int_{-\infty}^{\infty} w\, dE_w$ which means:

A brief walk on continuous groups of unitary operators

$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\, U^{t'} = U^{t+t'}$,
2. $U^0 = I_{\mathcal{H}}$,
3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

Classical **Stone's theorem** assures us the existence of a self-adjoint operator T (possibly unbounded) such that $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator T, defined on the dense domain of $\mathcal{H}$

$$D_T := \left\{x \in \mathcal{H} \text{ such that } \int_{-\infty}^{\infty} w^2\, d\|E_w x\|^2 < \infty\right\},$$

admits the *spectral representation* $T = \int_{-\infty}^{\infty} w\, dE_w$ which means:

$$\langle Tx, y\rangle = \int_{-\infty}^{\infty} w\, d\langle E_w x, y\rangle \quad \text{for any } x \in D_T \text{ and } y \in \mathcal{H}\,,$$

where $\{E_w\}_{w\in\mathbb{R}}$ is the corresponding *resolution of the identity*.

Resolution of the identity

is a one-parameter family of projection operators E_w in $\mathcal{H}$ such that

1. $E_{-\infty} := \lim_{w \to -\infty} E_w = O_{\mathcal{H}}, \quad E_{\infty} := \lim_{w \to \infty} E_w = I_{\mathcal{H}},$
2. $E_{w-} = E_w$ for every $-\infty < w < \infty$,
3. $E_u \, E_v = E_w$ where $w = \min\{u, v\}$.

Resolution of the identity

is a one-parameter family of projection operators E_w in $\mathcal{H}$ such that

1. $E_{-\infty} := \lim_{w \to -\infty} E_w = O_{\mathcal{H}}, \;\; E_{\infty} := \lim_{w \to \infty} E_w = I_{\mathcal{H}},$
2. $E_{w-} = E_w$ for every $-\infty < w < \infty$,
3. $E_u \, E_v = E_w$ where $w = \min\{u, v\}$.

Recall that $\|E_w x\|^2$ and $\langle E_w x, y \rangle$, as functions of w, have bounded variation and define, respectively, a positive and a complex Borel measure on $\mathbb{R}$.

Resolution of the identity

is a one-parameter family of projection operators E_w in $\mathcal{H}$ such that

1. $E_{-\infty} := \lim_{w\to-\infty} E_w = O_{\mathcal{H}}$, $E_{\infty} := \lim_{w\to\infty} E_w = I_{\mathcal{H}}$,
2. $E_{w-} = E_w$ for every $-\infty < w < \infty$,
3. $E_u\, E_v = E_w$ where $w = \min\{u, v\}$.

Recall that $\|E_w x\|^2$ and $\langle E_w x, y\rangle$, as functions of w, have bounded variation and define, respectively, a positive and a complex Borel measure on $\mathbb{R}$.

Notice that, whenever the self-adjoint operator T is bounded, $D_T = \mathcal{H}$ and $\mathrm{e}^{\mathrm{i}tT}$ can be defined as the usual exponential series; in any case, $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$ means that

$$\langle U^t x, y\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}wt} d\langle E_w x, y\rangle\,, \quad t \in \mathbb{R}\,,$$

where $x \in D_T$ and $y \in \mathcal{H}$.

The perturbed sequence $\left\{U^{rk+\epsilon_{kj}}b_j\right\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$.

Assuming that

The perturbed sequence $\left\{U^{rk+\epsilon_{kj}} b_j\right\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$.

Assuming that

- $s = r$ and $b_j \in D_T,\ j = 1, 2, \dots, s$.

The perturbed sequence $\{U^{rk+\epsilon_{kj}} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

Assuming that

- $s = r$ and $b_j \in D_T,\ j = 1, 2, \ldots, s$.
- The sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a Riesz basis for $\mathcal{A}_a$.

The perturbed sequence $\left\{U^{rk+\epsilon_{kj}} b_j\right\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

Assuming that

- $s = r$ and $b_j \in D_T,\ j = 1, 2, \ldots, s$.
- The sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a Riesz basis for $\mathcal{A}_a$.

If the following constant

$$R := \|\epsilon\|^2_{\ell^2_r} \max_{j=1,2,\ldots,r} \left\{ \int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2 \right\},$$

The perturbed sequence $\left\{U^{rk+\epsilon_{kj}} b_j\right\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$.

Assuming that

- $s = r$ and $b_j \in D_T,\ j = 1, 2, \dots, s$.
- The sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a Riesz basis for $\mathcal{A}_a$.

If the following constant

$$R := \|\boldsymbol{\epsilon}\|^2_{\ell^2_r} \max_{j=1,2,\dots,r} \left\{ \int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2 \right\},$$

satisfies $R < A_{\boldsymbol{\Psi}}$, then the sequence $\{U^{kr+\epsilon_{kj}} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a **Riesz sequence** in $\mathcal{H}$.

Recover $x \in \mathcal{A}_a$ in a stable way from the perturbed sequence $\{\langle x, U^{rm+\epsilon_{mj}} b_j \rangle_{\mathcal{H}}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

Recover $x \in \mathcal{A}_a$ in a stable way from the perturbed sequence $\big\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

Taking into account the $L^2(0,1)$ functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \langle a, U^k b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k w}\,,\ j = 1, 2, \ldots, s\,,$$

Recover $x \in \mathcal{A}_a$ in a stable way from the perturbed sequence $\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

Taking into account the $L^2(0,1)$ functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \langle a, U^k b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} kw}\,,\ j = 1, 2, \ldots, s\,,$$

we can define the $s \times r$ matrix

$$\mathbb{G}(w) := \begin{bmatrix} \mathbf{g}_1(w) & \mathbf{g}_1(w+\frac{1}{r}) & \cdots & \mathbf{g}_1(w+\frac{r-1}{r}) \\ \mathbf{g}_2(w) & \mathbf{g}_2(w+\frac{1}{r}) & \cdots & \mathbf{g}_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ \mathbf{g}_s(w) & \mathbf{g}_s(w+\frac{1}{r}) & \cdots & \mathbf{g}_s(w+\frac{r-1}{r}) \end{bmatrix}$$

Recover $x \in \mathcal{A}_a$ in a stable way from the perturbed sequence $\big\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\big\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$.

Taking into account the $L^2(0,1)$ functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \langle a, U^k b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k w}\,, \ j = 1, 2, \dots, s\,,$$

we can define the $s \times r$ matrix

$$\mathbb{G}(w) := \begin{bmatrix} \mathbf{g}_1(w) & \mathbf{g}_1(w+\frac{1}{r}) & \cdots & \mathbf{g}_1(w+\frac{r-1}{r}) \\ \mathbf{g}_2(w) & \mathbf{g}_2(w+\frac{1}{r}) & \cdots & \mathbf{g}_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ \mathbf{g}_s(w) & \mathbf{g}_s(w+\frac{1}{r}) & \cdots & \mathbf{g}_s(w+\frac{r-1}{r}) \end{bmatrix}$$

and its related the constants $\alpha_{\mathbb{G}}$ and $\beta_{\mathbb{G}}$ given by

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,.$$

Assuming that we have $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.

Assuming that we have $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.

We can find a positive number γ such that, if the constants

$$\gamma_j := \sup_{m \in \mathbb{Z}} |\epsilon_{mj}|$$

are smaller than γ for all $j = 1, 2, \ldots, s$,

Assuming that we have $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.

We can find a positive number γ such that, if the constants

$$\gamma_j := \sup_{m \in \mathbb{Z}} |\epsilon_{mj}|$$

are smaller than γ for all $j = 1, 2, \ldots, s$, then there exists a **frame** $\{C^{\epsilon}_{m,j}\}_{m \in \mathbb{Z};\, j=1,2,\ldots,s}$ for $\mathcal{A}_a$ allowing the recovery of any $x \in \mathcal{A}_a$ by means of the **sampling expansion**

$$x = \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} \langle x, U^{rm+\epsilon_{mj}} b_j \rangle_{\mathcal{H}} \, C^{\epsilon}_{m,j} \quad \text{in } \mathcal{H}\,.$$

THANKS.

- It has been proved that the sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.

- It has been proved that the sequence $\{\overline{g_j(w)}\, e^{2\pi i r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.
- Consider the sequence $\left\{\overline{g_{m,j}(w)}\, e^{2\pi i r m w}\right\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ as a perturbation of the above frame in $L^2(0,1)$, where

$$g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \langle a, U^{k+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, e^{2\pi i k w}\,,\ \ j = 1, 2, \dots, s\,.$$

- It has been proved that the sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$.
- Consider the sequence $\left\{\overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi \mathrm{i} r m w}\right\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ as a perturbation of the above frame in $L^2(0,1)$, where

$$g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \langle a, U^{k+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k w}\,,\ \ j = 1, 2, \dots, s\,.$$

- Define the functions,

$$M_{a,b_j}(\gamma) := \sum_{k\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} \left|\langle a, U^{k+t} b_j\rangle - \langle a, U^k b_j\rangle\right|,$$

and

- It has been proved that the sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.
- Consider the sequence $\left\{\overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw}\right\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ as a perturbation of the above frame in $L^2(0,1)$, where

$$g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \langle a, U^{k+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} kw}\,,\ j = 1, 2, \dots, s\,.$$

- Define the functions,

$$M_{a,b_j}(\gamma) := \sum_{k\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} \left|\langle a, U^{k+t} b_j\rangle - \langle a, U^k b_j\rangle\right|,$$

and

$$N_{a,b_j}(\gamma) :=$$
$$\max_{k=0,1,\dots,r-1} \sum_{m\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} \left|\langle a, U^{rm+k+t} b_j\rangle - \langle a, U^{rm+k} b_j\rangle\right|.$$

- Notice that $N_{a,b_j}(\gamma) \leq M_{a,b_j}(\gamma)$ and for $r = 1$ the equality holds.

- Notice that $N_{a,b_j}(\gamma) \leq M_{a,b_j}(\gamma)$ and for $r = 1$ the equality holds.
- Assuming that the continuous functions $\varphi_j(t) := \langle a, U^t b_j \rangle$, satisfy a decay condition as $\varphi_j(t) = O\big(|t|^{-(1+\eta_j)}\big)$ when $|t| \to \infty$ for some $\eta_j > 0$, we deduce that the functions $N_{a,b_j}(\gamma)$ and $M_{a,b_j}(\gamma)$ are continuous near to 0.

- Notice that $N_{a,b_j}(\gamma) \leq M_{a,b_j}(\gamma)$ and for $r = 1$ the equality holds.
- Assuming that the continuous functions $\varphi_j(t) := \langle a, U^t b_j \rangle$, satisfy a decay condition as $\varphi_j(t) = O\big(|t|^{-(1+\eta_j)}\big)$ when $|t| \to \infty$ for some $\eta_j > 0$, we deduce that the functions $N_{a,b_j}(\gamma)$ and $M_{a,b_j}(\gamma)$ are continuous near to 0.
- Finally the condition involving the constants γ_j is

$$\sum_{j=1}^{s} M_{a,b_j}(\gamma_j) N_{a,b_j}(\gamma_j) < \alpha_{\mathbb{G}}/r$$

- Notice that $N_{a,b_j}(\gamma) \leq M_{a,b_j}(\gamma)$ and for $r = 1$ the equality holds.
- Assuming that the continuous functions $\varphi_j(t) := \langle a, U^t b_j \rangle$, satisfy a decay condition as $\varphi_j(t) = O\big(|t|^{-(1+\eta_j)}\big)$ when $|t| \to \infty$ for some $\eta_j > 0$, we deduce that the functions $N_{a,b_j}(\gamma)$ and $M_{a,b_j}(\gamma)$ are continuous near to 0.
- Finally the condition involving the constants γ_j is

$$\sum_{j=1}^{s} M_{a,b_j}(\gamma_j) N_{a,b_j}(\gamma_j) < \alpha_{\mathbb{G}}/r$$

Remark

The obtained sampling formula is useless from a practical point of view: it is impossible to determine the involved frame $\{C^{\epsilon}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$. As a consequence, in order to recover $x \in \mathcal{A}_a$ from the sequence of inner products $\big\{\langle x, U^{rm+\epsilon_{mj}} b_j \rangle_{\mathcal{H}}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ we could implement a frame algorithm in the $\ell^2(\mathbb{Z})$ setting.