Sampling Theory in Shift-Invariant Spaces: Generalizations

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Introduction to sampling theory
Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators
Uniform average sampling in frame generated weighted shift-invariant spaces
Sampling theory in U-invariant spaces

Outline

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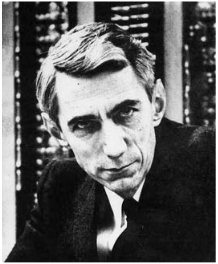
Introduction



Claude E. Shannon, 1916-2001.

Shannon's sampling theorem.

If a function of time is **limited to the band** from 0 to W cycles per second, it is completely determined by giving its ordinates at a series of discrete points spaced $1/2W$ seconds apart in the manner indicated by the following result: If $f(t)$ has no frequencies over W cycles per second, then

$$f(t) = \sum_{n=-\infty}^{\infty} f\Big(\frac{n}{2W}\Big) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

A mathematical theory of comunication, *Bell System Tech. J., 27(1948), 379-423*.

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Uniform average sampling in frame generated weighted shift-invariant spaces

Sampling theory in U-invariant spaces

Edmund T. Whittaker, 1873-1956.

Vladimir A. Kotelnikov, 1908-2005.

Whittaker-Shannon-Kotel'nikov theorem.

If $f(t)$ is a signal (function) **band-limited** to $[-\sigma, \sigma]$, i.e.,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(w) \mathrm{e}^{\mathrm{i}tw} dw, \quad t \in \mathbb{R}$$

for some $F \in L^2(-\sigma, \sigma)$, then it can be reconstructed from its samples values at the points $t_k = k\pi/\sigma, k \in \mathbb{Z}$, via the formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)}, \quad t \in \mathbb{R}$$

with the series being absolutely and uniformly convergent on compact sets.

Drawbacks in WSK theorem

- it relies on the use of **low-pass ideal filters**.
- the **band-limited hypothesis** is in contradiction with the idea of a **finite duration signal**.
- the band-limiting operation generates Gibbs oscillations.
- the sinc function has a **very slow decay at infinity** which makes computation in the signal domain very inefficient.
- the sinc function is well-localized in the frequency domain but it is bad-localized in the time domain.
- in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a *d*-dimensional interval.

Paley-Wiener space

The space of band limited functions to the interval $[-\pi, \pi]$ can be written as

$$PW_\pi = \Big\{ \sum_{n\in\mathbb{Z}} a_n \,\mathrm{sinc}(t-n) \;:\; \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}$$

Furthermore, the coefficients $\{a_n\}_{n\in\mathbb{Z}}$ of $f \in PW_\pi$ are precisely the samples of the function at the integers numbers $\{f(n)\}_{n\in\mathbb{Z}}$.

Chapter 1: Motivation

Paley-Wiener space

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Shift-invariant space

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n \,\varphi(t-n) \;:\; \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}$$

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Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

On the separable Hilbert $L^2(\mathbb{R}^d)$ we can define the shift-invariant subspaces V_Φ^2 in the following way

$$V_\Phi^2 := \overline{\text{span}}_{L^2(\mathbb{R}^d)}\left\{\varphi_k(t-\alpha) \,:\, k = 1, 2, \dots, r \text{ and } \alpha \in \mathbb{Z}^d\right\},$$

where the functions in $\Phi := \{\varphi_1, \dots, \varphi_r\}$ in $L^2(\mathbb{R}^d)$ are called a set of generators for V_Φ^2.

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

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where the functions in $\Phi := \{\varphi_1, \dots, \varphi_r\}$ in $L^2(\mathbb{R}^d)$ are called a set of generators for V_Φ^2. Assuming that the sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, k=1,2\dots,r}$ is a **Riesz sequence**, i.e. a **Riesz basis** for its span, this space can be described as

$$V_\Phi^2 = \Big\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha) \,:\, d_k \in \ell^2(\mathbb{Z}^d), k = 1,2\dots,r \Big\}.$$

Generalized sampling problem

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If we consider

- $\mathcal{L}_j f := f * \mathrm{h}_j,\ \ j = 1, 2, \ldots s$ are **convolutions systems** (linear time-invariant systems) defined on V_φ^2.

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- The samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\dots,s}$ are taken at a lattice

$$\Lambda_M := \{M\alpha : \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d \,.$$

where M is a nonsingular matrix with integer entries.

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where M is a nonsingular matrix with integer entries.

The generalized sampling problem is to obtain sampling formulas in V_φ^2 having the form

$$f = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)\, S_j(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d)\,,$$

where the reconstruction sequence of functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is a **frame** for V_φ^2 .

Definition

A sequence $\{f_k\}_{k\in\mathbb{Z}}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants A, $B > 0$ (frame bounds) such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}$$

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Definition

Two frames sequences $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ which satisfy

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle\, f_k = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle\, g_k, \quad \text{for all } f \in \mathcal{H}.$$

are said to be a pair of **dual frames**.

Our general technique

- Consider an isomorphism $\mathcal{T}_\Phi : L^2 \to V_\Phi^2$ such that

$$Samp(f)_k = \langle F, g_k \rangle_{L^2}\,, \quad f = \mathcal{T}_\Phi F$$

- Characterize $\{g_k\}$ as a frame for L^2
- Find a dual frame $\{h_k\}$

$$F = \sum_k \langle F, g_k \rangle_{L^2} h_k$$

- Apply $\mathcal{T}_\Phi$ to get the sampling formula

$$f = \sum_k Samp(f)_k \underbrace{\mathcal{T}_\Phi h_k}_{frame}$$

Sketch of the procedure

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(i) Consider the **isomorphism**

$$\begin{array}{rccc} \mathcal{T}_\Phi : & L^2_r[0,1)^d & \longrightarrow & V^2_\Phi \\ & \{\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top w}\mathbf{e}_k\} & \longmapsto & \{\varphi_k(t-\alpha)\} \end{array}$$

Verify the **shifting property** for $\mathbf{F} \in L^2_r[0,1)^d$ and $\alpha \in \mathbb{Z}^d$

$$\mathcal{T}_\Phi\big[\mathbf{F}(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top\cdot}\big](t) = \mathcal{T}_\Phi\mathbf{F}(t-\alpha)\,, \quad t \in \mathbb{R}^d\,.$$

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(ii) Deduce the **expression for the samples**

$$\big(\mathcal{L}_j f\big)(M\alpha) = \langle \mathbf{F}, \overline{g_j(\cdot)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\rangle_{L_r^2[0,1)^d}\,,$$

where $\mathbf{F} = \mathcal{T}_\Phi^{-1} f \in L_r^2[0,1)^d$.

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where $\mathbf{F} = \mathcal{T}_\Phi^{-1} f \in L_r^2[0,1)^d$.

(iii) Characterize the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ as a **frame** in $L_r^2[0,1)^d$

(iv) Find a **dual frame** of the form $\left\{(\det M)\mathbf{a}_j(x)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$, wich implies

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\langle \mathbf{F}, \overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\rangle \mathbf{a}_j(x)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}$$

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in $L^2_r[0,1)^d$.

(v) Applying the isomorphism $\mathcal{T}_\Phi$ to the expansion we get the desired

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)\, S_{j,\mathbf{a}}(\cdot - M\alpha) \quad \text{in} \;\; L^2(\mathbb{R}^d)\,,$$

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where $S_{j,\mathbf{a}} := \mathcal{T}_\Phi \mathbf{a}_j$ for $j = 1, 2, \dots, s$.

(vi) As V^2_Φ is a **RKHS**, we have

$$f(t) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)\, S_{j,\mathbf{a}}(t - M\alpha) \qquad t\in\mathbb{R}^d\,.$$

Details of the procedure

Lemma

Let $\mathcal{L}$ be a convolution system. Then, for each $f \in V_\Phi^2$ we have

$$(\mathcal{L}f)(t) = \langle \mathbf{F}, (\overline{\mathbf{Z}\mathcal{L}\Phi})(t,\cdot)\rangle_{L_r^2[0,1)^d}\,, \quad t \in \mathbb{R}^d\,,$$

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where $\mathbf{F} = \mathcal{T}_\Phi^{-1} f$.

Here $\mathbf{Z}\Psi$ denotes the **Zak transform** of Ψ, i.e.,

$$(\mathbf{Z}\Psi)(t,w) := \sum_{\alpha\in\mathbb{Z}^d} \Psi(t+\alpha)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top w}\,.$$

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The expression for the samples

$$\big(\mathcal{L}_j f\big)(M\alpha) = \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(0,\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\rangle_{L_r^2[0,1)^d}\,,$$

Denote

$$\mathbf{g}_j(x) := \mathbf{Z}\mathcal{L}_j\Phi(0,x)\,, \quad j = 1,2,\dots,s\,;$$

The sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ in $L^2_r[0,1)^d$

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The set
$$\#\{\mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\}\} = \det M$$
from now on the elements in this set will be denoted
$$\{i_1 = 0, i_2, \ldots, i_{\det M}\} \subset \mathbb{Z}^d\,.$$

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Consider the $s \times r(\det M)$ matrix of functions

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_1^\top(x + M^{-\top} i_{\det M}) \\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_2^\top(x + M^{-\top} i_{\det M}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_s^\top(x + M^{-\top} i_{\det M}) \end{bmatrix},$$

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and its related constants

$$A_{\mathbb{G}} := \operatorname*{ess\,inf}_{x\in[0,1)^d}\lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)],\quad B_{\mathbb{G}} := \operatorname*{ess\,sup}_{x\in[0,1)^d}\lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)]$$

Lemma

Let $\mathbf{g}_j$ *be in* $L^2_r[0,1)^d$ *for* $j = 1, 2, \ldots, s$ *and let* $\mathbb{G}(x)$ *be its associated matrix. Then,*

(a) *The sequence* $\left\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ *is a* **complete system** *for* $L^2_r[0,1)^d$ *if and only if the rank of the matrix* $\mathbb{G}(x)$ *is* $r(\det M)$ *a.e. in* $[0,1)^d$.

(b) *The sequence* $\left\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ *is a* **Bessel sequence** *for* $L^2_r[0,1)^d$ *if and only if* $\mathbf{g}_j \in L^\infty_r[0,1)^d$ *(or equivalently* $B_{\mathbb{G}} < \infty$*). In this case, the optimal Bessel bound is* $B_{\mathbb{G}}/(\det M)$.

(c) *The sequence* $\left\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ *is a* **frame** *for* $L^2_r[0,1)^d$ *if and only if* $0 < A_{\mathbb{G}} \le B_{\mathbb{G}} < \infty$. *In this case, the optimal frame bounds are* $A_{\mathbb{G}}/(\det M)$ *and* $B_{\mathbb{G}}/(\det M)$.

(d) *The sequence* $\left\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ *is a* **Riesz basis** *for* $L^2_r[0,1)^d$ *if and only if it is a frame and* $s = r(\det M)$.

The sampling result

Theorem

Assume that the functions $\mathbf{g}_j$ *belong to* $L_r^\infty[0,1)^d$ *for each* $j = 1, 2, \ldots, s$. *The following statements are equivalents:*

(a) $A_{\mathbb{G}} > 0$.

(b) *There exists an* $r \times s$ *matrix* $\mathbf{a}(x) := \big[\mathbf{a}_1(x), \ldots, \mathbf{a}_s(x)\big]$ *with columns* $\mathbf{a}_j \in L_r^\infty[0,1)^d$, *and satisfying*

$$\big[\mathbf{a}_1(x), \ldots, \mathbf{a}_s(x)\big]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M - 1)r \times r}] \quad a.e.\ in\ [0,1)^d .$$

(c) *There exists a frame for* V_Φ^2 *having the form* $\{S_{j,\mathbf{a}}(\cdot - M\alpha)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\ldots,s}$ *such that for any* $f \in V_\Phi^2$

$$f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)\, S_{j,\mathbf{a}}(\cdot - M\alpha) \quad in\ L^2(\mathbb{R}^d) .$$

(d) *There exists a frame* $\{S_{j,\alpha}(\cdot)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\ldots,s}$ *for* V_Φ^2 *such that*

$$f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)\, S_{j,\alpha} \quad in\ L^2(\mathbb{R}^d) .$$

Remark

Having in mind the Moore-Penrose pseudo inverse

$$\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x).$$

All matrices $\mathbf{a}(x)$ with entries in $L^\infty[0,1)^d$, and satisfying (b) in the previous theorem correspond to the first r rows of the matrices of the form

$$\mathbb{A} = \mathbb{G}^\dagger(x) + \mathbb{U}(x)\left[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\right],$$

where $\mathbb{U}(x)$ is any $r(\det M) \times s$ matrix with entries in $L^\infty[0,1)^d$.

Reconstruction functions with prescribed properties

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Theorem

- *If the generators φ_k and the functions $\mathcal{L}_j\varphi_k$ have* **compact support**, *then the reconstruction functions $S_{j,a}$ have compact support if and only if*

$$\text{rank } G(\mathbf{z}) = r(\det M) \quad \text{for all } \mathbf{z} \in (\mathbb{C} \setminus \{0\})^d .$$

- *If the generators φ_k and the functions $\mathcal{L}_j\varphi_k$ have* **exponential decay**, *then the reconstruction functions $S_{j,a}$ have exponential decay if and only if*

$$\text{rank } G(\mathbf{z}) = r(\det M) \quad \text{for all } \mathbf{z} \in \mathbb{T}^d .$$

$$g_{j,k}(\mathbf{z}) := \sum_{\mu \in \mathbb{Z}^d} \mathcal{L}_j\varphi_k(\mu)\mathbf{z}^{-\mu}, \quad g_j^\top(\mathbf{z}) := \big(g_{j,1}(\mathbf{z}), g_{j,2}(\mathbf{z}), \dots, g_{j,r}(\mathbf{z})\big) ,$$

$$G(\mathbf{z}) := \Big[g_j^\top(z_1 e^{2\pi i m_1^\top i_l}, \dots, z_d e^{2\pi i m_d^\top i_l})\Big]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r;\ l=1,2,\dots,\det M}}$$

Note also that for the values $x = (x_1, \dots, x_d) \in [0,1)^d$ and $\mathbf{z} = (e^{2\pi i x_1}, \dots, e^{2\pi i x_d}) \in \mathbb{T}^d$ we have $\mathbb{G}(x) = G(\mathbf{z})$.

L^2-approximation properties

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$$\Gamma_{\mathbf{a}} f(t) := \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha), \quad t \in \mathbb{R}^d,$$

Introduction to sampling theory
Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators
Uniform average sampling in frame generated weighted shift-invariant spaces
Sampling theory in U-invariant spaces

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Theorem

Under the following conditions:

- The set of generators $\Phi = \{\varphi_k\}_{k=1}^{r}$ satisfies the **Strang-Fix conditions of order** ℓ
- The **decay condition** $\varphi_k(t) = O\big([1 + |t|]^{-d-\ell-\epsilon}\big)$ for some $\epsilon > 0$,
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we get

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \leq C\,|f|_{\ell,2}\, h^\ell \quad \text{for all } f \in W_2^\ell(\mathbb{R}^d),$$

where the constant C does not depend on h and f.

Irregular sampling in V_Φ^2: time-jitter error

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Introduction to sampling theory
Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators
Uniform average sampling in frame generated weighted shift-invariant spaces
Sampling theory in U-invariant spaces

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Leads us to study the sequence

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as a perturbation of the previous frame

$$\{(\overline{Z\mathcal{L}_j\Phi})(0,\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$$

Theorem

For sufficiently small errors there exists a frame $\{S^{\varepsilon}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ for V^2_Φ such that, for any $f \in V^2_\Phi$

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha}) S^{\varepsilon}_{j,\alpha}(t)\,, \quad t \in \mathbb{R}^d\,,$$

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Remark

Notice that the frame $\{S^{\varepsilon}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$, depends on the error sequence.

We implemented a **frame algorithm** in the $\ell^2(\mathbb{Z}^d)$ setting which approximates the sequence $\{a_{k\alpha}\}_{\alpha\in\mathbb{Z}^d;\, k=1,\ldots,r}\in\ell^2(\mathbb{Z}^d)$ defining $f\in V^2_\Phi$ by a sequence $\{a^{(n)}_{k\alpha}\}_{\alpha\in\mathbb{Z}^d;\, k=1,\ldots,r}\in\ell^2(\mathbb{Z}^d)$ in such a way that

$$f_n(t)=\sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d}a^{(n)}_{k\alpha}\,\varphi_k(t-\alpha)\longrightarrow f(t)=\sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d}a_{k\alpha}\,\varphi_k(t-\alpha)$$

Chapter 2: Motivation

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Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

Sampling theory in U-invariant spaces

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$$V_\Phi^2 := \Big\{ \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} d_j(\alpha)\, \varphi_j(t-\alpha) \; : \; d_j \in \ell^2(\mathbb{Z}^d),\, k = 1, 2 \ldots, r \Big\}$$

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Weighted shif-invariant spaces

$$V_\nu^p(\Phi) := \Big\{ \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(t-\alpha) \;:\; a_j \in \ell_\nu^p(\mathbb{Z}^d),\, j = 1, 2, \ldots, r \Big\}$$

Chapter 2: Motivation

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Weight functions ν control the decay or growth of the signals $f \in V_\nu^p(\Phi)$.

Uniform average sampling in frame generated weighted shift-invariant spaces

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- f belongs to $L^p_\nu(\mathbb{R}^d)$ if νf belongs to $L^p(\mathbb{R}^d)$
- $\|f\|_{L^p_\nu(\mathbb{R}^d)} = \|\nu f\|_{L^p(\mathbb{R}^d)}$
- weight function ν satisfies

$$0 < \nu(x+y) \le \nu(x)\nu(y), \quad \text{for all } x, y \in \mathbb{R}^d \,.$$

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Some typical examples

- Subexponential weight $\nu(x) = e^{\alpha|x|^\beta}$ with $\alpha \geq 0$, $\beta \in [0,1]$
- Sobolev weight $\nu(x) = (1+|x|)^\alpha$, with $\alpha \geq 0$.

Wiener amalgam spaces of measurable functions

For $1 \leq p < \infty$

$$W(L^p_\nu) := \left\{ f \ : \ \|f\|^p_{W(L^p_\nu)} := \sum_{\alpha \in \mathbb{Z}^d} \operatorname*{ess\,sup}_{x \in [0,1]^d} \left\{ |f(x+\alpha)|^p \nu(\alpha)^p \right\} < \infty \right\}$$

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Definition

A collection $\{\phi_j(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\ldots,r}$ is said to be a ***p*-frame** for $V^p_\nu(\Phi)$ if there exists a positive constant C (depending on Φ, p and ν) such that for every $f \in V^p_\nu(\Phi)$

$$C^{-1} \|f\|_{L^p_\nu} \leq \sum_{j=1}^{r} \left\| \left\{ \int_{\mathbb{R}^d} f(x) \overline{\phi_j(x-\alpha)} dx \right\}_{\alpha \in \mathbb{Z}^d} \right\|_{\ell^p_\nu} \leq C \|f\|_{L^p_\nu} .$$

Weighted multiply generated shift-invariant space

Given a set of functions $\Phi := \{\phi_j\}_{j=1}^r$, the weighted multiply generated shift-invariant space $V_\nu^p(\Phi)$ is formally defined as

$$V_\nu^p(\Phi) := \Big\{ \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\,\phi_j(t-\alpha)\ :\ \{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell_\nu^p(\mathbb{Z}^d) \Big\}.$$

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- p-frame condition and $\Phi = \{\phi_j\}_{j=1}^r \subset W(L_\nu^1)$ assure the closedness of $V_\nu^p(\Phi)$ as a subspace of L_ν^p.

Theorem

There exist functions $S_{l,\mathbf{d}}$ such that for any $f \in V_\nu^p(\Phi)$, $1 \le p \le \infty$, the sampling formula

$$f = \sum_{l=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha)\,,$$

holds in the L_ν^p-sense. The convergence is also uniform on $\mathbb{R}^d$.

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Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

Sampling theory in U-invariant spaces

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Wiener's lemma

If $f \in \mathcal{A}_\nu$ and $f(x) \neq 0$ for every $x \in \mathbb{R}^d$, the function $1/f$ is also in $\mathcal{A}_\nu$, where $\mathcal{A}_\nu$ denotes the weighted Wiener algebra of the functions

$$f(x) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\, e^{2\pi i \alpha^\top x}, \qquad \{a(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell_\nu^1(\mathbb{Z}^d)$$

Chapter 3: Motivation

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where $T : f(t) \mapsto f(t-1)$ in $L^2(\mathbb{R})$ is the shift operator.

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U-invariant space

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, U^n a : \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\} .$$

where U is a unitary operator and a is a fixed vector on a separable Hilbert space $\mathcal{H}$.

Generalizing the samples

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The samples in the shift-invariant case were obtained by means of convolution systems $\mathcal{L}_j f := f * \mathrm{h}_j$

Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

Sampling theory in U-invariant spaces

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The *U*-samples

For a fixed $b \in \mathcal{H}$ and a sampling period $r \in \mathbb{N}$ the ***U*-samples** are given by

$$\mathcal{L}_b x(rk) := \langle x, U^{rk} b \rangle_{\mathcal{H}}\,, \quad k \in \mathbb{Z}\,.$$

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In the shift-invariant case, U is defined as the **shift operator** $f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$ and we have

$$\langle f, U^{rk} b \rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} f(u)\overline{b(u-rk)}du = (f * h)(rk)\,, \quad u \in \mathbb{R}\,,$$

where $h(u) := \overline{b(-u)}$.

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Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

Sampling theory in U-invariant spaces

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In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

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Examples: Translation and Modulation operator on $L^2(\mathbb{R})$

$$\big(T_a f\big)(t) = f(t-a) \qquad \big(M_a f\big)(t) = f(t)e^{\mathrm{i}at}$$

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

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- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta}\, d\mu_a(\theta)\,, \qquad k\in\mathbb{Z}\,.$$

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- The positive Borel spectral measure μ_a can be decomposed as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$.

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Theorem

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty\,.$$

If $\{b_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ are the Fourier coefficients of the function $1/\phi_a(\theta) \in L^2(-\pi,\pi)$ then the vector $b = \sum_{k\in\mathbb{Z}} b_k\, U^k a \in \mathcal{A}_a$ generates the dual Riesz basis $\{U^n b\}_{n\in\mathbb{Z}}$ with spectral measure $\phi_b(\theta) = 1/\phi_a(\theta)$.

We define the isomorphism $\mathcal{T}_{U,a}$ which maps the orthonormal basis $\{e^{2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$, that is,

$$\mathcal{T}_{U,a}: \quad \begin{array}{ccc} L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ F = \sum_{n\in\mathbb{Z}} \alpha_n \, e^{2\pi i n w} & \longmapsto & x = \sum_{n\in\mathbb{Z}} \alpha_n \, U^n a \,. \end{array}$$

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$$\mathcal{T}_{U,a}: \quad \begin{array}{ccc} L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ F = \sum_{n\in\mathbb{Z}} \alpha_n \, e^{2\pi i n w} & \longmapsto & x = \sum_{n\in\mathbb{Z}} \alpha_n \, U^n a \,. \end{array}$$

The following U-shift property holds: For any $F \in L^2(0,1)$ and $N \in \mathbb{Z}$, we have

$$\mathcal{T}_{U,a}\Big(F \, e^{2\pi i N w}\Big) = U^N \big(\mathcal{T}_{U,a} F\big) \,.$$

An expression for the generalized samples

For $x \in \mathcal{A}_a$ let $F \in L^2(0,1)$ such that $\mathcal{T}_{U,a}F = x$;

$$\mathcal{L}_j x(rm) = \left\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw} \right\rangle_{L^2(0,1)} \quad \text{for } m \in \mathbb{Z} \text{ and } \; j = 1, 2, \ldots, s\,,$$

where the function

$$g_j(w) := \sum_{k \in \mathbb{Z}} \mathcal{L}_j a(k)\, \mathrm{e}^{2\pi \mathrm{i} kw}$$

belongs to $L^2(0,1)$ for each $j = 1, 2, \ldots, s$.

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belongs to $L^2(0,1)$ for each $j = 1, 2, \dots, s$.

As a consequence, the stable recovery of any $x \in \mathcal{A}_a$ depends on whether the sequence

$$\left\{ \overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw} \right\}_{m \in \mathbb{Z};\, j = 1,2,\dots s}$$

forms a frame for $L^2(0,1)$.

The matrix $\mathbb{G}(w)$

$$\mathbb{G}(w) := \begin{bmatrix} g_1(w) & g_1(w+\frac{1}{r}) & \cdots & g_1(w+\frac{r-1}{r}) \\ g_2(w) & g_2(w+\frac{1}{r}) & \cdots & g_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & \ddots & \vdots \\ g_s(w) & g_s(w+\frac{1}{r}) & \cdots & g_s(w+\frac{r-1}{r}) \end{bmatrix}$$

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$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w \in (0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w \in (0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,,$$

Theorem

Assume that the function g_j, $j = 1, 2, \dots, s$ belongs to $L^\infty(0,1)$. The following statements are equivalent:

(a) $\alpha_{\mathbb{G}} > 0$

(b) There exists a vector $\big[h_1(w), h_2(w), \dots, h_s(w)\big]$ with entries in $L^\infty(0,1)$ satisfying

$$\big[h_1(w), h_2(w), \dots, h_s(w)\big]\mathbb{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0,1)\,.$$

(c) There exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, such that the sequence $\big\{U^{rk} c_j\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$, and for any $x \in \mathcal{A}_a$ we have the expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\, U^{rk} c_j \quad \text{in } \mathcal{H}\,,$$

(d) There exists a frame $\big\{C_{j,k}\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ for $\mathcal{A}_a$ such that, for each $x \in \mathcal{A}_a$ we have the expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\, C_{j,k} \quad \text{in } \mathcal{H}\,,$$

Another approach

$$\langle U^k a, U^{rn} b_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-rn)\theta} \phi_{\mathbf{a},\mathbf{b}_j}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \,.$$

- The *left-shift operator S* defined as

$$S: \begin{array}{ccc} L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}k\theta} & \longmapsto & \sum_{k\in\mathbb{Z}} a_{k+1} \mathrm{e}^{\mathrm{i}k\theta} \end{array},$$

or equivalently, by $(Sf)(\mathrm{e}^{\mathrm{i}\theta}) = f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}\theta}$.

- The *decimation operator D_r*, r a positive integer, defined as

$$D_r: \begin{array}{ccc} L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}k\theta} & \longmapsto & \sum_{k\in\mathbb{Z}} a_{rk} \mathrm{e}^{\mathrm{i}k\theta} \end{array},$$

which can equivalently be written as

$$(D_r f)(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{r} \sum_{k=0}^{r-1} f(\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{r}})\,.$$

Samples taken on real numbers

Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

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The *U*-systems

For any fixed $b \in \mathcal{H}$ we define the U-system $\mathcal{L}_b$ as the linear operator between $\mathcal{H}$ and the set $C(\mathbb{R})$ of the continuous functions on $\mathbb{R}$ given by

$$\begin{array}{rccc} \mathcal{L}_b : & \mathcal{H} & \longrightarrow & C(\mathbb{R}) \\ & x & \longmapsto & \mathcal{L}_b x\,, \end{array}$$

where $\mathcal{L}_b x(t) := \langle x, U^t b\rangle_{\mathcal{H}}\,, \quad t \in \mathbb{R}\,.$

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In this case U should coincide with U^1 on a **continuous group of unitary operators** $\{U^t\}_{t\in\mathbb{R}}$.

A brief walk on continuous groups of unitary operators

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Definition

$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\, U^{t'} = U^{t+t'}$,
2. $U^0 = I_{\mathcal{H}}$,
3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

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Classical **Stone's theorem** assures us the existence of a self-adjoint operator T (possibly unbounded) such that $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator T is defined on the dense domain D_T of $\mathcal{H}$.

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Notice that, whenever the self-adjoint operator T is bounded, $D_T = \mathcal{H}$ and $\mathrm{e}^{\mathrm{i}tT}$ can be defined as the usual exponential series; in any case, $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$ means that

$$\langle U^t x, y\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}wt} d\langle E_w x, y\rangle\,, \quad t \in \mathbb{R}\,,$$

where $x \in D_T$ and $y \in \mathcal{H}$.

Asymmetric sampling

Asymmetric sampling

We have at our disposal the asymmetric samples

$$\big\{\mathcal{L}_j x(\sigma_j + r_j m)\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s}$$

where $\sigma_j \in \mathbb{R}$ and $r_j \in \mathbb{N}$.

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where $\sigma_j \in \mathbb{R}$ and $r_j \in \mathbb{N}$.

The recovery formula

$$x = \sum_{j=1}^{s} \sum_{l_j=1}^{\frac{r}{r_j}} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x\big(\sigma_j + rk + r_j(l_j - 1)\big)\, U^{rk} c_{j,l_j}$$

where $r := \mathrm{lcm}\{r_j\}_{j=1,\dots,s}$

Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

Sampling theory in U-invariant spaces

Time-jitter error: irregular sampling in $\mathcal{A}_a$

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Time-jitter error: irregular sampling in $\mathcal{A}_a$

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$$\{(\mathcal{L}_j x)(rm+\epsilon_{mj})\} = \left\{\left\langle F, \overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw} \right\rangle_{L^2(0,1)}\right\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$$

where

$$g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k+\epsilon_{mj})\, \mathrm{e}^{2\pi \mathrm{i} kw}\,,$$

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where

$$g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k+\epsilon_{mj})\, \mathrm{e}^{2\pi \mathrm{i} kw}\,,$$

We can study

$$\left\{ \overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw} \right\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$$

as a perturbation of the frame

$$\left\{ \overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw} \right\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$$

Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

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Theorem

For **sufficiently small errors** $\epsilon := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$ there exists a frame $\{C^{\epsilon}_{j,m}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x \in \mathcal{A}_a$, the sampling expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm + \epsilon_{mj})\, C^{\epsilon}_{j,m} \quad \text{in } \mathcal{H}\,,$$

holds.

How small should be the errors?

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$$\widetilde{M}_{a,b_j}(\gamma) := \sum_{n\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} \left|\mathcal{L}_j a(n+t) - \mathcal{L}_j a(n)\right|,$$

$$\widetilde{N}_{a,b_j}(\gamma) := \max_{k=0,1,\dots,r-1} \sum_{n\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} \left|\mathcal{L}_j a(rn+k+t) - \mathcal{L}_j a(rn+k)\right|.$$

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$$\widetilde{M}_{a,b_j}(\gamma) := \sum_{n\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} |\mathcal{L}_j a(n+t) - \mathcal{L}_j a(n)| \,,$$

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Theorem

Given an error sequence $\epsilon := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$, define the constant $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}|$ for each $j = 1, 2, \dots, s$. The condition

$$\sum_{j=1}^{s} \widetilde{M}_{a,b_j}(\gamma_j)\, \widetilde{N}_{a,b_j}(\gamma_j) < \frac{\alpha_{\mathbf{G}}}{r}$$

ensures that **reconstruction is possible**.

$b_j \in D_T \;\Rightarrow\; \mathcal{L}_j a(t) \in C^1(\mathbb{R})$ and condition $(\mathcal{L}_j a)'(t) = O(|t|^{-(1+\eta_j)})$ implies that $\widetilde{N}_{a,b_j}(\gamma)$ and $\widetilde{M}_{a,b_j}(\gamma)$ are continuous near to 0.

Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

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The perturbed sequence $\{U^{rk+\epsilon_{kj}} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$

Theorem

Assume that for certain $b_j \in D_T$, $j = 1, 2, \dots, r$, the sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a **Riesz basis** for $\mathcal{A}_a$ with Riesz bounds $0 < A_\Psi \le B_\Psi < \infty$. For a sequence $\epsilon := \{\epsilon_{kj}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ of errors, let R be the constant given by

$$R := \|\epsilon\|^2 \max_{j=1,2,\dots,r} \left\{ \int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2 \right\},$$

where $\|\epsilon\|$ denotes the ℓ_r^2-norm of the sequence ϵ.
If $R < A_\Psi$, then the perturbed sequence

$$\{U^{kr+\epsilon_{kj}} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$$

is a **Riesz sequence** in $\mathcal{H}$ with Riesz bounds $A_\Psi\left(1-\sqrt{R/A_\Psi}\right)^2$ and $B_\Psi\left(1+\sqrt{R/B_\Psi}\right)^2$.

The case of multiple generators

Sampling in multiple generated U- invariant subspaces can be analogously derived, $\mathcal{A}_{\mathbf{a}} := \overline{\text{span}}\{U^n a_l,\ n \in \mathbb{Z};\ l = 1, 2, \ldots, L\}$.

The sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\ldots,L}$ can be thought as an L-dimensional stationary sequence. Its *covariance matrix* $\mathbf{R}_{\mathbf{a}}(k)$ is the $L \times L$ matrix

$$\mathbf{R}_{\mathbf{a}}(k) = \Big[\langle U^k a_m, a_n\rangle_{\mathcal{H}}\Big]_{m,n=1,2,\ldots,L} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta}\, d\boldsymbol{\mu}_{\mathbf{a}}(\theta)\,, \quad k \in \mathbb{Z}\,.$$

Theorem

Let $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\ldots,L}$ be a sequence obtained from an unitary operator with spectral measure $d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \boldsymbol{\Phi}_{\mathbf{a}}(\theta)d\theta + d\boldsymbol{\mu}^s_{\mathbf{a}}(\theta)$, and let $\mathcal{A}_{\mathbf{a}}$ be the closed subspace spanned by $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\ldots,L}$. Then the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\ldots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ if and only if the singular part $\boldsymbol{\mu}^s_{\mathbf{a}} \equiv 0$ and

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] < \infty\,.$$

Summarizing:

	V_Φ^2	$V_\nu^p(\Phi)$	$\mathcal{A}_a$
Reconstruction formula	✓	✓	✓
Time-jitter error	✓		✓
prescribed properties	✓		✓
Other results	L^2-approximation properties	Dirac sampling case	Asymmetric sampling $\{U^{rk+\epsilon_{kj}} b_j\}$ $\{U^n a_l\}$

Future Work

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- ***U*-irregular sampling: the general case.** Consider a non-uniform sampling set of points $\{t_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}$, and try to recover any $x \in \mathcal{A}_a$ from the sequence of non-uniform samples

$$\{\mathcal{L}_j\, x(t_n) := \langle x, U^{t_n} b_j\rangle\}_{n\in\mathbb{Z};\, j=1,2,\dots s}\,,$$

where $\{b_j\}_{j=1,2,\dots,s}$ are s fixed vectors in $\mathcal{H}$.

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- Sampling in **finite** U-invariant subspaces with multiple generators. We have assumed that the stationary sequence $\{U^n a\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ has infinite different elements. It could happen that for some $a \in \mathcal{H}$ there exists $N \in \mathbb{N}$ such that $U^N a = a$.

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Publications

- H. R. Fernández-Morales, A. G. García and G. Pérez-Villalón *Generalized sampling in $L^2(\mathbb{R})$ shift-invariant subspaces with multiple stable generators.* **In Multiscale Signal Analysis and Modelling, Lecture Notes in Electrical Engineering, pp. 51–80, Springer, 2013.** (Ch. 2)
- H. R. Fernández-Morales, A. G. García and M. A. Hernández-Medina *Generalized sampling in U-invariant subspaces.* **Proceedings of the 10th International Conference on Sampling Theory and Applications, Eurasip Open Library, 208-211, 2013.** (Ch. 4)
- H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo. *On some sampling-related frames in U-invariant spaces.* **Abstr. Appl. Anal., Vol. 2013, Article ID 761620, 14 pp., 2013.** (Ch. 4)
- H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo. *Generalized sampling: from shift-invariant to U-invariant spaces*. **Anal. Appl., Vol.15(3): 303–329, 2015.** (Ch. 4)
- H. R. Fernández-Morales and A. G. García. *Uniform average sampling in frame-generated weighted shift-invariant spaces*. Preprint 2015. (Ch. 3)

THANKS

Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

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