Semi-direct product of groups, filter banks and sampling

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Summary

- Statement of the sampling problem
- A brief on semi-direct product of groups
- The filter banks formalism
- Frame analysis
- Getting on with sampling
- Some examples involving crystallographic groups

Statement of the sampling problem

Ingredients

- Let $(n,h) \mapsto U(n,h)$ be a unitary representation on a separable Hilbert space $\mathcal{H}$ of a group $G = N \rtimes_\phi H \quad (\#H = L)$, i.e., homomorphism between G and $\mathcal{U}(\mathcal{H})$
- Fixed $a \in \mathcal{H}$, consider the U-invariant subspace in $\mathcal{H}$

$$\mathcal{A}_a = \Big\{ \sum_{(n,h)\in G} \alpha(n,h) \underbrace{U(n,h)a}_{\textit{Riesz sequence}} \; : \; \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G) \Big\}$$

- For each $x \in \mathcal{A}_a$ consider the data (samples)

$$\mathcal{L}_k x(n) := \big\langle x, U(n, 1_H) b_k \big\rangle_{\mathcal{H}}, \quad n \in N, \; k = 1, 2, \dots, K \, (\geq L)$$

where $b_k \in \mathcal{H}$ are fixed non necessarily in $\mathcal{A}_a$

Goal

The stable recovery of any $x \in \mathcal{A}_a$ from the data sequence $\{\mathcal{L}_k x(n)\}_{n\in N;\, k=1,2,\dots,K}$

- There exist constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_{k=1}^{K} \sum_{n\in N} |\mathcal{L}_k x(n)|^2 \leq B\|x\|^2 \,, \qquad \text{for all } x \in \mathcal{A}_a$$

- Recovery by means of a sampling formula like

$$x = \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, U(n, 1_H) c_k \qquad \text{in } \mathcal{H}$$

Procedure

- Since there exist $\mathsf{h}_k \in \ell^2(G)$, $k = 1, 2, \ldots, K$, such that

$$\mathcal{L}_k x(n) = \left\langle \alpha, T_n \mathsf{h}_k \right\rangle_{\ell^2(G)}, \quad n \in N\,,\ k = 1, 2, \ldots, K$$

where $T_n \mathsf{h}_k(m, h) = \mathsf{h}_k(m - n, h)$, $(m, h) \in G$, $k = 1, 2, \ldots, K$

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- Instead of $x \in \mathcal{A}_a$ proceed to the stable recovery of $\boldsymbol{\alpha} \in \ell^2(G)$ from given data

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- To find its dual frames $\big\{T_n \mathsf{g}_k\big\}_{n \in N;\, k=1,2,\ldots K}$ having the same form

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  where $T_n \mathsf{h}_k(m, h) = \mathsf{h}_k(m - n, h)$, $(m, h) \in G$, $k = 1, 2, \dots, K$
- Instead of $x \in \mathcal{A}_a$ proceed to the stable recovery of $\alpha \in \ell^2(G)$ from given data
- The sequence $\{T_n \mathsf{h}_k\}_{n \in N;\, k=1,2,\dots K}$ should be a frame for $\ell^2(G)$: characterize these frames
- To find its dual frames $\{T_n \mathsf{g}_k\}_{n \in N;\, k=1,2,\dots K}$ having the same form
- Then, in $\ell^2(G)$

$$\alpha = \sum_{k=1}^{K} \sum_{n \in N} \langle \alpha, T_n \mathsf{h}_k \rangle_{\ell^2(G)}\, T_n \mathsf{g}_k = \sum_{k=1}^{K} \sum_{n \in N} \mathcal{L}_k x(n)\, T_n \mathsf{g}_k$$

Procedure

- Introducing in $x = \sum_{(m,h)\in G} \alpha(m,h)\, U(m,h)a$ the expansion for $\boldsymbol{\alpha}$

$$x = \sum_{(m,h)\in G} \Big[\sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, T_n \mathsf{g}_k(m,h) \Big]\, U(m,h)a = \dots$$

$$= \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n) \Big[\sum_{(\tilde{n},h)\in G} \mathsf{g}_k(\tilde{n},h)\, \underbrace{U(n,1_H)U(\tilde{n},h)}_{U(m,h)=U(n+\tilde{n},h)} a \Big]$$

$$= \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, U(n,1_H) c_{k,\mathsf{g}}$$

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$$= \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, U(n,1_H) c_{k,\mathsf{g}}$$

- Next, let's go to do it ...

A brief on semi direct product of groups

The semi-direct product $G := N \rtimes_\phi H$

Given groups $(N, \cdot)$ and $(H, \cdot)$, and a homomorphism $\phi : H \to Aut(N)$ ($\phi(h) := \phi_h$) their semi-direct product $G := N \rtimes_\phi H$ has, in the underlying set $N \times H$, the composition law:

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \phi_{h_1}(n_2), h_1 h_2)\,, \quad (n_1, h_1), (n_2, h_2) \in G\,,$$

The homomorphism ϕ is referred as the action of the group H on the group N. Some basic facts:

- $(1_N, 1_H)$, and $(n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$
- $N \simeq N \times \{1_H\}$ and $H \simeq \{1_N\} \times H$
- $G = N \rtimes_\phi H$ is not abelian, even for abelian N and H groups.
- N is a normal subgroup in G.

Some examples of semi-direct product of groups

1. The dihedral group D_{2N} is the group of symmetries of a regular N-sided polygon; it is the semi-direct product $D_{2N} = \mathbb{Z}_N \rtimes_\phi \mathbb{Z}_2$ where $\phi_{\bar{0}} \equiv Id_{\mathbb{Z}_N}$ and $\phi_{\bar{1}}(\bar{n}) = -\bar{n}$ for each $\bar{n} \in \mathbb{Z}_N$.
   The infinite dihedral group D_∞ defined as $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ for the similar homomorphism ϕ is the group of isometries of $\mathbb{Z}$.
2. The Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d \rtimes_\phi O(d)$, where $O(d)$ is the orthogonal group of order d and $\phi_A(x) = Ax$ for $A \in O(d)$ and $x \in \mathbb{R}^d$. It contains as a subgroup any crystallographic group $\mathcal{C}_{M,\Gamma} := M\mathbb{Z}^d \rtimes_\phi \Gamma$, where $M\mathbb{Z}^d$ denotes a full rank lattice of $\mathbb{R}^d$ and Γ is any finite subgroup of $O(d)$ such that $\phi_\gamma(M\mathbb{Z}^d) = M\mathbb{Z}^d$ for each $\gamma \in \Gamma$.

a. Suppose that N is an locally compact abelian (LCA) group with Haar measure μ_N and H is a locally compact group with Haar measure μ_H. Then, the semi-direct product $G = N \rtimes_\phi H$ endowed with the product topology becomes also a topological group

b. In our case, we will assume that $G = N \rtimes_\phi H$ where $(N, +)$ is a countable discrete abelian group and $(H, \cdot)$ is a finite group. Recall the operational calculus:
   - $(n, h) \cdot (m, l) = (n + \phi_h(m), hl)$
   - $(n, h)^{-1} = (\phi_{h^{-1}}(-n), h^{-1})$
   - $(n, h)^{-1} \cdot (m, l) = (\phi_{h^{-1}}(m - n), h^{-1}l)$
   - $(n, h)^{-1} \cdot (m, 1_H) = (\phi_{h^{-1}}(m - n), h^{-1})$
   - $(n, 1_H)^{-1} \cdot (m, l) = (m - n, l)$

The filter banks formalism

- The convolution $\boldsymbol{\alpha} * \mathsf{h}$ of $\boldsymbol{\alpha}, \mathsf{h} \in \ell^2(G)$ can be expressed as

$$(\boldsymbol{\alpha} * \mathsf{h})(m,l) = \sum_{(n,h)\in G} \alpha(n,h)\, \mathsf{h}\big[(n,h)^{-1} \cdot (m,l)\big]$$
$$= \sum_{(n,h)\in G} \alpha(n,h)\, \mathsf{h}\big(\phi_{h^{-1}}(m-n), h^{-1}l\big)\,, \quad (m,l) \in G$$

- *H*-decimation

$$\downarrow_H (\boldsymbol{\alpha} * \mathsf{h})(m) := (\boldsymbol{\alpha} * \mathsf{h})(m, 1_H) = \sum_{(n,h)\in G} \alpha(n,h)\, \mathsf{h}\big(\phi_{h^{-1}}(m-n), h^{-1}\big)$$
$$= \sum_{(n,h)\in G} \alpha(n,h)\, \mathsf{h}[(n-m,h)^{-1}]\,, \quad m \in N$$

$$\downarrow_H (\boldsymbol{\alpha} * \mathsf{h})(m) = \sum_{h\in H}\sum_{n\in N} \boldsymbol{\alpha}_h(n)\, \mathsf{h}_h(m-n) = \sum_{h\in H} \big(\boldsymbol{\alpha}_h *_N \mathsf{h}_h\big)(m)\,,\ m \in N$$

where $\boldsymbol{\alpha}_h(n) := \alpha(n,h)$ and $\mathsf{h}_h(n) := \mathsf{h}[(-n,h)^{-1}]$

For a function $c : N \to \mathbb{C}$, its H-expander $\uparrow_H c : G \to \mathbb{C}$ is

$$(\uparrow_H c)(n,h) = \begin{cases} c(n) & \text{if } h = 1_H \\ 0 & \text{if } h \neq 1_H \,. \end{cases}$$

In case $\uparrow_H c$ and g belong to $\ell^2(G)$ we have

$$\begin{aligned} (\uparrow_H c * \mathsf{g})(m,l) &= \sum_{(n,h)\in G} (\uparrow_H c)(n,h)\, \mathsf{g}\big[(n,h)^{-1}\cdot(m,l)\big] \\ &= \sum_{(n,h)\in G} (\uparrow_H c)(n,h)\, \mathsf{g}\big(\phi_{h^{-1}}(m-n), h^{-1}l\big) \\ &= \sum_{n\in N} c(n)\, \mathsf{g}(m-n,l) = \big(c *_N \mathsf{g}_l\big)(m)\,, \quad m\in N\,,\, l \in H \end{aligned}$$

where $\mathsf{g}_l(n) := \mathsf{g}(n,l)$

A K-channel filter bank with analysis filters h_k and synthesis filters g_k, $k = 1, 2, \dots, K$

$$\mathbf{c}_k := \downarrow_H (\boldsymbol{\alpha} * \mathsf{h}_k), \; k = 1, 2, \dots, K, \; \text{and} \quad \boldsymbol{\beta} = \sum_{k=1}^{K} (\uparrow_H c_k) * \mathsf{g}_k ,$$

In polyphase notation

$$\mathbf{c}_k(m) = \sum_{h \in H} (\boldsymbol{\alpha}_h *_N \mathsf{h}_{k,h})(m), \quad m \in N, \quad k = 1, 2, \dots, K,$$

$$\boldsymbol{\beta}_l(m) = \sum_{k=1}^{K} (\mathbf{c}_k *_N \mathsf{g}_{l,k})(m), \quad m \in N, \; l \in H,$$

where $\boldsymbol{\alpha}_h(n) := \alpha(n,h)$, $\boldsymbol{\beta}_l(n) := \beta(n,l)$, $\mathsf{h}_{k,h}(n) := \mathsf{h}_k[(-n,h)^{-1}]$ and $\mathsf{g}_{l,k}(n) := \mathsf{g}_k(n,l)$ are the polyphase components of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, h_k and g_k, $k = 1, 2, \dots, K$

Our K-channel filter bank

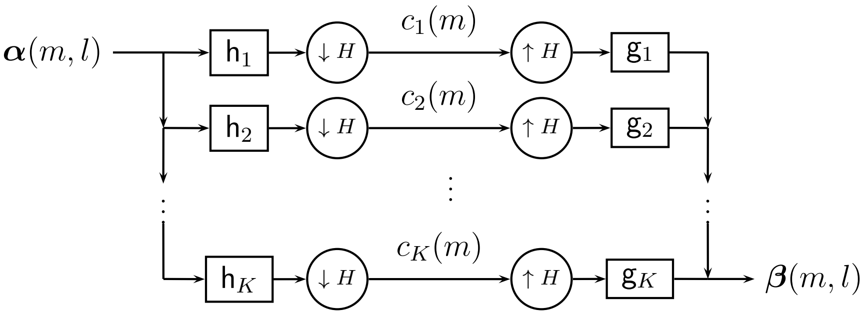

Figure : The K-channel filter bank scheme

Denoting $H = \{h_1, h_2, \dots, h_L\}$, assume that $\mathsf{h}_k, \mathsf{g}_k \in \ell^2(G)$ with $\widehat{\mathsf{h}}_{k,h_i}, \widehat{\mathsf{g}}_{h_i,k} \in L^\infty(\widehat{N})$ for $k = 1, 2, \dots, K$ and $i = 1, 2, \dots, L$.

Recall, $\mathsf{h}_{k,h_i}(n) := \mathsf{h}_k[(-n, h_i)^{-1}]$ and $\mathsf{g}_{h_i,k}(n) := \mathsf{g}_k(n, h_i)$

The N-Fourier transform in

$\mathbf{c}_k(m) = \sum_{h \in H} \big(\boldsymbol{\alpha}_h *_N \mathsf{h}_{k,h}\big)(m) \rightsquigarrow \widehat{\mathbf{c}}_k(\gamma) = \sum_{h \in H} \widehat{\mathsf{h}}_{k,h}(\gamma)\, \widehat{\boldsymbol{\alpha}}_h(\gamma)$ a.e. $\gamma \in \widehat{N}$

In matrix notation,

$$\mathbf{C}(\gamma) = \mathbf{H}(\gamma)\, \mathbf{A}(\gamma) \ a.e.\ \gamma \in \widehat{N}$$

$\mathbf{C}(\gamma) = \big(\widehat{\mathbf{c}}_1(\gamma), \widehat{\mathbf{c}}_2(\gamma), \dots, \widehat{\mathbf{c}}_K(\gamma)\big)^\top$,

$\mathbf{A}(\gamma) = \big(\widehat{\boldsymbol{\alpha}}_{h_1}(\gamma), \widehat{\boldsymbol{\alpha}}_{h_2}(\gamma), \dots, \widehat{\boldsymbol{\alpha}}_{h_L}(\gamma)\big)^\top$, and $\mathbf{H}(\gamma)$ is the $K \times L$ matrix

$$\mathbf{H}(\gamma) = \Big(\, \widehat{\mathsf{h}}_{k,h_i}(\gamma) \,\Big)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}}, \quad \gamma \in \widehat{N}$$

$$\beta_l(m) = \sum_{k=1}^{K} \left(\mathbf{c}_k *_N \mathsf{g}_{l,k}\right)(m) \rightsquigarrow \widehat{\beta}_l(\gamma) = \sum_{k=1}^{K} \widehat{\mathsf{g}}_{l,k}(\gamma)\,\widehat{\mathbf{c}}_k(\gamma) \text{ a.e. } \gamma \in \widehat{N}$$

In matrix notation,

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma)\,\mathbf{C}(\gamma) \; a.e.\ \gamma \in \widehat{N}$$

$\mathbf{B}(\gamma) = \left(\widehat{\beta}_{h_1}(\gamma), \widehat{\beta}_{h_2}(\gamma), \ldots, \widehat{\beta}_{h_L}(\gamma)\right)^\top,$
$\mathbf{C}(\gamma) = \left(\widehat{\mathbf{c}}_1(\gamma), \widehat{\mathbf{c}}_2(\gamma), \ldots, \widehat{\mathbf{c}}_K(\gamma)\right)^\top$
$\mathbf{G}(\gamma)$ is the $L \times K$ matrix

$$\mathbf{G}(\gamma) = \left(\widehat{\mathsf{g}}_{h_i,k}(\gamma)\right)_{\substack{i=1,2,\ldots,L \\ k=1,2,\ldots,K}}, \quad \gamma \in \widehat{N}$$

$\widehat{\mathsf{g}}_{h_i,k} \in L^2(\widehat{N})$ is the Fourier transform of $\mathsf{g}_{h_i,k}(n) := \mathsf{g}_k(n, h_i) \in \ell^2(N)$

Perfect reconstruction

In terms of the polyphase matrices $\mathbf{G}(\gamma)$ and $\mathbf{H}(\gamma)$

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma)\,\mathbf{H}(\gamma)\,\mathbf{A}(\gamma) \quad \text{a.e. } \gamma \in \widehat{N}$$

Theorem

A K-channel filter bank with $\mathsf{h}_k, \mathsf{g}_k$ belong to $\ell^2(G)$ and $\widehat{\mathsf{h}}_{k,h_i}, \widehat{\mathsf{g}}_{h_i,k}$ belong to $L^\infty(\widehat{N})$ for $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, L$, satisfies the perfect reconstruction property if and only if $\mathbf{G}(\gamma)\,\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$, where $\mathbf{I}_L$ denotes the identity matrix of order L.

$$\boldsymbol{\alpha} \in \ell^2(G) \to \mathbf{A} \in L^2_L(\widehat{N}) \quad \text{is a unitary operator}$$

Frame analysis

For $m \in N$ the translation T_m ($m \in N$) and the involution operators are defined in $\ell^2(G)$ as

$$T_m \boldsymbol{\alpha}(n,h) := \alpha\big((m,1_H)^{-1}\cdot(n,h)\big) = \alpha(n-m,h)$$
$$\widetilde{\alpha}(n,h) := \overline{\alpha\big((n,h)^{-1}\big)}\,, \quad (n,h)\in G$$

For a K-channel filter bank we have

$$\mathbf{c}_k(m) = \downarrow_H (\boldsymbol{\alpha} * \mathsf{h}_k)(m) = \big\langle \boldsymbol{\alpha}, T_m \widetilde{\mathsf{h}}_k \big\rangle_{\ell^2(G)}\,,$$
$$(\uparrow_H \mathbf{c}_k * \mathsf{g}_k)(m,h) = \sum_{n\in N} \mathbf{c}_k(n)\,\mathsf{g}_k(m-n,h) = \sum_{n\in N} \langle \boldsymbol{\alpha}, T_n \widetilde{\mathsf{h}}_k \rangle_{\ell^2(G)}\, T_n \mathsf{g}_k(m,h)$$

In the perfect reconstruction setting, for any $\boldsymbol{\alpha} \in \ell^2(G)$ we have

$$\boldsymbol{\alpha} = \sum_{k=1}^{K} \sum_{n\in N} \langle \boldsymbol{\alpha}, T_n \widetilde{\mathsf{h}}_k \rangle_{\ell^2(G)}\, T_n \mathsf{g}_k \quad \text{in } \ell^2(G)$$

Theorem

For f_k in $\ell^2(G)$, $k = 1, 2, \ldots, K$, denote $\mathsf{f}_{k,h_i}(n) := \mathsf{f}_k(n, h_i)$, $h_i \in H$, and consider the associated matrix $\mathbf{H}(\gamma)$. Then,

1. The sequence $\{T_n \mathsf{f}_k\}_{n \in N;\, k=1,2,\ldots,K}$ is a Bessel sequence for $\ell^2(G)$ if and only if $B_{\mathbf{H}} < \infty$ (if and only if the function $\widehat{\mathsf{f}}_{k,h_i} \in L^\infty(\widehat{N})$ for each $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, L$)
2. The sequence $\{T_n \mathsf{f}_k\}_{n \in N;\, k=1,2,\ldots,K}$ is a frame for $\ell^2(G)$ if and only $0 < A_{\mathbf{H}} \le B_{\mathbf{H}} < \infty$

where $\mathbf{H}(\gamma)$ is the $K \times L$ matrix (taking analysis filters $\mathsf{h}_k = \widetilde{\mathsf{f}}_k$)

$$\mathbf{H}(\gamma) = \left(\overline{\widehat{\mathsf{f}}_{k,h_i}(\gamma)} \right)_{\substack{k=1,2,\ldots,K \\ i=1,2,\ldots,L}}, \quad \gamma \in \widehat{N}$$

and its associated constants

$$A_{\mathbf{H}} := \operatorname*{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min} \left[\mathbf{H}^*(\gamma) \mathbf{H}(\gamma) \right]; \quad B_{\mathbf{H}} := \operatorname*{ess\,sup}_{\gamma \in \widehat{N}} \lambda_{\max} \left[\mathbf{H}^*(\gamma) \mathbf{H}(\gamma) \right]$$

Theorem

Let $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\dots,K}$ and $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\dots,K}$ be two Bessel sequences for $\ell^2(G)$ with associated matrices $\mathbf{H}(\gamma)$ and $\mathbf{G}(\gamma)$. Then,

(a) The sequences $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\dots,K}$ and $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\dots,K}$ are dual frames for $\ell^2(G)$ if and only if condition $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$ holds.

(b) The sequences $\{T_n \mathsf{f}_k\}_{n\in N;\, k=1,2,\dots,K}$ and $\{T_n \mathsf{g}_k\}_{n\in N;\, k=1,2,\dots,K}$ are dual Riesz bases for $\ell^2(G)$ if and only if $K = L$ and $\mathbf{G}(\gamma) = \mathbf{H}(\gamma)^{-1}$ a.e. $\gamma \in \widehat{N}$

where the matrices $\mathbf{H}(\gamma)$ and $\mathbf{G}(\gamma)$ are:
(taking analysis filters $\mathsf{h}_k = \widetilde{\mathsf{f}_k}$ and synthesis filters g_k)

$$\mathbf{H}(\gamma) = \left(\overline{\widehat{\mathsf{f}}_{k,h_i}(\gamma)}\right)_{\substack{k=1,2,\dots,K\\ i=1,2,\dots,L}}; \quad \mathbf{G}(\gamma) = \left(\widehat{\mathsf{g}}_{h_i,k}(\gamma)\right)_{\substack{i=1,2,\dots,L\\ k=1,2,\dots,K}}, \quad \gamma \in \widehat{N}$$

Getting on with sampling

For each $x \in \mathcal{A}_a$ in the U-invariant subspace in $\mathcal{H}$

$$\mathcal{A}_a = \Big\{ \sum_{(n,h)\in G} \alpha(n,h)\, U(n,h)a \;:\; \{\alpha(n,h)\}_{(n,h)\in G} \in \ell^2(G) \Big\}$$

we consider its generalized samples

$$\begin{aligned}
\mathcal{L}_k x(m) &:= \big\langle x, U(m, 1_H)\, b_k \big\rangle_{\mathcal{H}} \\
&= \sum_{(n,h)\in G} \alpha(n,h) \big\langle a, U\big[(n,h)^{-1} \cdot (m, 1_H)\big]\, b_k \big\rangle \\
&= \downarrow_H (\boldsymbol{\alpha} * \mathsf{h}_k)(m)\,, \quad m \in N\,,\ k = 1, 2, \dots, K
\end{aligned}$$

where

$$\mathsf{h}_k(n,h) := \big\langle a, U(n,h)\, b_k \big\rangle_{\mathcal{H}}\,,\ (n,h) \in G$$

belongs to $\ell^2(G)$, $k = 1, 2, \dots, K$

Suppose that there exists a perfect reconstruction K-channel filter-bank with analysis filters the above h_k and synthesis filters g_k, $k = 1, 2, \ldots, K$. Then, we have

$$\alpha = \sum_{k=1}^{K} \sum_{n \in N} \downarrow_H (\alpha * \mathsf{h}_k)(n)\, T_n \mathsf{g}_k = \sum_{k=1}^{K} \sum_{n \in N} \mathcal{L}_k x(n)\, T_n \mathsf{g}_k \quad \text{in } \ell^2(G)\,.$$

We consider the natural isomorphism $\mathcal{T}_{U,a} : \ell^2(G) \to \mathcal{A}_a$

$$\mathcal{T}_{U,a} : \delta_{(n,h)} \longmapsto U(n,h)a \ \text{ for each } (n,h) \in G\,.$$

For each $m \in N$, $\mathcal{T}_{U,a}$ possesses the following shifting property

$$\mathcal{T}_{U,a}\big(T_m \mathsf{f}\big) = U(m, 1_H)\big(\mathcal{T}_{U,a}\mathsf{f}\big)\,, \quad \mathsf{f} \in \ell^2(G)\,.$$

Applying the isomorphism $\mathcal{T}_{U,a}$, for each $x \in \mathcal{A}_a$ we get the expansion

$$x = \mathcal{T}_{U,a}\boldsymbol{\alpha} = \sum_{k=1}^{K} \sum_{n \in N} \mathcal{L}_k x(n)\, \mathcal{T}_{U,a}\big(T_n \mathsf{g}_k\big)$$
$$= \sum_{k=1}^{K} \sum_{n \in N} \mathcal{L}_k x(n)\, U(n, 1_H)\big(\mathcal{T}_{U,a}\mathsf{g}_k\big)$$
$$= \sum_{k=1}^{K} \sum_{n \in N} \mathcal{L}_k x(n)\, U(n, 1_H) c_{k,\mathsf{g}} \quad \text{in } \mathcal{H},$$

where $c_{k,\mathsf{g}} = \mathcal{T}_{U,a}\mathsf{g}_k$, $k = 1, 2, \dots, K$.

Notice that the sequence $\big\{U(n, 1_H)c_{k,\mathsf{g}}\big\}_{n \in N;\, k=1,2,\dots,K}$ is a frame for $\mathcal{A}_a$ is a frame for $\mathcal{A}_a$.

In fact, the following result holds:

For h_k such that $\mathsf{h}_k(n,h) := \big\langle a, U(n,h)\, b_k \big\rangle_{\mathcal{H}}\,, \quad (n,h) \in G$,
consider the $K \times L$ matrix

$$\mathbf{H}(\gamma) = \Big(\overline{\widehat{\mathsf{f}}_{k,h_i}(\gamma)} \Big)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}}\,, \quad \gamma \in \widehat{N}$$

where $\mathsf{f}_k := \widetilde{\mathsf{h}}_k$ and $\mathsf{f}_{k,h_i}(n) = \mathsf{f}_k(n,h_i)$, $k = 1,2,\dots,K$, $i = 1,2,\dots,L$
Consider also its associated constants

$$A_{\mathbf{H}} := \operatorname*{ess\,inf}_{\gamma\in\widehat{N}} \lambda_{\min}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big]\,; \quad B_{\mathbf{H}} := \operatorname*{ess\,sup}_{\gamma\in\widehat{N}} \lambda_{\max}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big]$$

Assume that the matrix $\mathbf{H}(\gamma)$ has all its entries in $L^\infty(\widehat{N})$

The sampling result

Theorem

The following statements are equivalent:

1. The constant $A_{\mathbf{H}} = \operatorname*{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big] > 0$.
2. There exist g_k in $\ell^2(G)$, $k = 1, 2, \dots, K$, such that the associated polyphase matrix $\mathbf{G}(\gamma)$ has all its entries in $L^\infty(\widehat{N})$, and it satisfies $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$.
3. There exist K elements $c_k \in \mathcal{A}_a$ such that the sequence $\big\{U(n, 1_H)c_k\big\}_{n\in N;\, k=1,2,\dots,K}$ is a frame for $\mathcal{A}_a$ and for each $x \in \mathcal{A}_a$ we have the sampling formula

$$x = \sum_{k=1}^{K} \sum_{n \in N} \mathcal{L}_k x(n)\, U(n, 1_H)c_k \quad \text{in } \mathcal{H}$$

The sampling result

Theorem

3. There exist K elements $c_k \in \mathcal{A}_a$ such that the sequence $\{U(n, 1_H)c_k\}_{n\in N;\, k=1,2,\ldots,K}$ is a frame for $\mathcal{A}_a$ and for each $x \in \mathcal{A}_a$ we have the sampling formula

$$x = \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, U(n, 1_H)c_k \quad \text{in } \mathcal{H}$$

4. There exists a frame $\{C_{k,n}\}_{n\in N;\, k=1,2,\ldots,K}$ for $\mathcal{A}_a$ such that for each $x \in \mathcal{A}_a$ we have the expansion

$$x = \sum_{k=1}^{K} \sum_{n\in N} \mathcal{L}_k x(n)\, C_{k,n} \quad \text{in } \mathcal{H}$$

Notice that $K \geq L$ where L is the order of the group H. In case $K = L$, we obtain:

Corollary

In the case $K = L$, assume that the matrix $\mathbf{H}(\gamma)$ has all entries in $L^\infty(\widehat{N})$. The following statements are equivalents:

1. The constant $A_{\mathbf{H}} = \operatorname*{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min}\big[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)\big] > 0$.
2. There exist L unique elements c_k, $k = 1, 2, \ldots, L$, in $\mathcal{A}_a$ such that the associated sequence $\big\{U(n, 1_H)c_k\big\}_{n \in N;\, k=1,2,\ldots,L}$ is a Riesz basis for $\mathcal{A}_a$, and for each $x \in \mathcal{A}_a$ we have the sampling formula

$$x = \sum_{k=1}^{L} \sum_{n \in N} \mathcal{L}_k x(n)\, U(n, 1_H)c_k \quad \text{in } \mathcal{H}$$

Moreover, the interpolation property $\mathcal{L}_k c_{k'}(n) = \delta_{k,k'}\delta_{n,0_N}$, where $n \in N$ and $k, k' = 1, 2, \ldots, L$, holds.

Some examples involving crystallographic groups

The Euclidean motion group $E(d) = \mathbb{R}^d \rtimes_\phi O(d)$: for the homomorphism $\phi : O(d) \to Aut(\mathbb{R}^d)$ given by $\phi_A(x) = Ax$, we have the composition law $(x, A) \cdot (x', A') = (x + Ax', AA')$

We consider the crystallographic group $\mathcal{C}_{M,\Gamma} := M\mathbb{Z}^d \rtimes_\phi \Gamma$ where M be a non-singular $d \times d$ matrix and Γ a finite subgroup of $O(d)$ of order L such that $A(M\mathbb{Z}^d) = M\mathbb{Z}^d$ for each $A \in \Gamma$ and its quasi regular representation on $L^2(\mathbb{R}^d)$: for $n \in M\mathbb{Z}^d$, $A \in \Gamma$ and $f \in L^2(\mathbb{R}^d)$

$$U(n, A)f(t) = f[A^\top(t - n)], \quad t \in \mathbb{R}^d$$

For a fixed $\varphi \in L^2(\mathbb{R}^d)$ such that the sequence $\{U(n, A)\varphi\}_{(n,A)\in\mathcal{C}_{M,\Gamma}}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ we consider the U-invariant subspace in $L^2(\mathbb{R}^d)$

$$\mathcal{A}_\varphi = \Big\{ \sum_{(n,A)\in\mathcal{C}_{M,\Gamma}} \alpha(n, A)\, \varphi[A^\top(t - n)] \ : \ \{\alpha(n, A)\} \in \ell^2(\mathcal{C}_{M,\Gamma}) \Big\}$$

Average samples

Choosing K functions $b_k \in L^2(\mathbb{R}^d)$ we consider the average samples of $f \in \mathcal{A}_\varphi$

$$\mathcal{L}_k f(n) = \langle f, U(n, I) b_k \rangle = \langle f, b_k(\cdot - n) \rangle, \quad n \in M\mathbb{Z}^d.$$

Under the hypotheses in our sampling theorem, there exist $K \geq L$ sampling functions $\psi_k \in \mathcal{A}_\varphi$ for $k = 1, 2, \ldots, K$, such that the sequence $\{\psi_k(\cdot - n)\}_{n \in M\mathbb{Z}^d;\, k=1,2,\ldots,K}$ is a frame for $\mathcal{A}_\varphi$, and we get the sampling expansion

$$f(t) = \sum_{k=1}^{K} \sum_{n \in M\mathbb{Z}^d} \big\langle f, b_k(\cdot - n) \big\rangle_{L^2(\mathbb{R}^d)} \psi_k(t - n) \quad \text{in } L^2(\mathbb{R}^d)$$

If the generator $\varphi \in C(\mathbb{R}^d)$ and the function $t \mapsto \sum_n |\varphi(t - n)|^2$ is bounded on $\mathbb{R}^d$, then $\mathcal{A}_\varphi$ is a reproducing kernel Hilbert space (RKHS) of continuous functions in $L^2(\mathbb{R}^d)$ $\rightsquigarrow$ pointwise convergence

Pointwise samples

$\{U(n,h)\}_{(n,h)\in G}$ a unitary representation of $G = N \rtimes_{\phi} H$ on $L^2(\mathbb{R}^d)$

If the generator $\varphi \in L^2(\mathbb{R}^d)$ of $\mathcal{A}_\varphi$ satisfies

- For each $(n,h) \in G$, the function $U(n,h)\varphi$ is continuous on $\mathbb{R}^d$
- $\sup_{t\in\mathbb{R}^d} \sum_{(n,h)\in G} \big|[U(n,h)\varphi](t)\big|^2 < +\infty$

Then the subspace $\mathcal{A}_\varphi = \Big\{ \sum_{(n,h)\in G} \alpha(n,h)\, U(n,h)\, \varphi \Big\}$ is a RKHS of bounded continuous functions in $L^2(\mathbb{R}^d)$.

For K fixed points $t_k \in \mathbb{R}^d$, $k = 1, 2, \ldots, K$, we consider for each $f \in \mathcal{A}_\varphi$ the new samples given by

$$\mathcal{L}_k f(n) := \big[U(-n, 1_H) f\big](t_k)\,, \quad n \in N \text{ and } k = 1, 2, \ldots, K\,.$$

For any $f = \sum_{(m,h)\in G} \alpha(m,h)\, U(m,h)\, \varphi$ in $\mathcal{A}_\varphi$ we have

$$\mathcal{L}_k f(n) = \Big[\sum_{(m,h)\in G} \alpha(m,h)\, U[(-n, 1_H)\cdot(m,h)]\, \varphi \Big](t_k)$$

$$= \sum_{(m,h)\in G} \alpha(m,h) \big[U(m-n,h)\varphi \big](t_k) = \big\langle \boldsymbol{\alpha}, T_n \mathsf{f}_k \big\rangle_{\ell^2(G)}, \quad n \in N$$

where $\mathsf{f}_k(m,h) := \overline{\big[U(m,h)\varphi\big](t_k)}$, $(m,h) \in G$, belongs to $\ell^2(G)$, $k = 1, 2, \cdots, K$. Under the hypotheses in sampling theorem (on the new $\mathsf{h}_k := \widetilde{\mathsf{f}_k} \in \ell^2(G)$, $k = 1, 2, \ldots, K$) we will get a sampling formula for the new data sequence $\big\{\mathcal{L}_k f(n)\big\}_{n\in N;\, k=1,2,\ldots,K}$

In the particular case of the quasi regular representation of a crystallographic group $\mathcal{C}_{M,\Gamma} = M\mathbb{Z}^d \rtimes_\phi \Gamma$, for each $f \in \mathcal{A}_\varphi$ these new samples read

$$\mathcal{L}_k f(n) = \big[U(-n, I)f\big](t_k) = f(t_k + n)\,, \quad n \in M\mathbb{Z}^d\,;\; k = 1, 2, \ldots, K$$

Thus, under the hypotheses in our sampling theorem, there exist K functions $\psi_k \in \mathcal{A}_\varphi$, $k = 1, 2, \ldots, K$, such that for each $f \in \mathcal{A}_\varphi$ the sampling formula

$$f(t) = \sum_{k=1}^{K} \sum_{n \in M\mathbb{Z}^d} f(t_k + n)\, \psi_k(t - n)\,, \quad t \in \mathbb{R}^d$$

holds. The convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$

That's all!