First order spectral perturbation theory of square singular matrix polynomials*

Fernando De Terán[a†] and Froilán M. Dopico[b]

December 2, 2008

[a] Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain. (fteran@math.uc3m.es)

[b] Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain. (dopico@math.uc3m.es).

Abstract

We develop first order eigenvalue expansions of uniparametric perturbations of square singular matrix polynomials. Although the eigenvalues of a singular matrix polynomial $P(\lambda)$ are not continuous functions of the entries of the coefficients of the polynomial, we show that for most perturbations they are indeed continuous. Given an eigenvalue λ_0 of $P(\lambda)$ we prove that, for generic perturbations $M(\lambda)$ with degree less than or equal to the degree of $P(\lambda)$, the eigenvalues of $P(\lambda) + \epsilon M(\lambda)$ admit covergent series expansions near λ_0 and we describe the first order term of these expansions in terms of $M(\lambda_0)$ and certain particular bases of the left and right null spaces of $P(\lambda_0)$. In the important case of λ_0 being a semisimple eigenvalue of $P(\lambda)$ any bases of the left and right null spaces of $P(\lambda_0)$ can be used, and the first order term of the eigenvalue expansions takes a simple form. In this situation we also obtain the limit vector of the associated eigenvector expansions.

Key words. matrix polynomial eigenvalue problem, perturbation, Puiseux expansions, singular matrix polynomials

AMS subject classification. 15A18, 47A56, 65F15, 65F22

1 Introduction

We consider square matrix polynomials with degree ℓ

$$P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^\ell A_\ell ,$$

with $A_i \in \mathbb{C}^{n\times n}$ and $A_\ell \neq 0$. The matrix polynomial $P(\lambda)$ is *singular* if $\det P(\lambda)$ is identically zero as a polynomial in λ. Otherwise $P(\lambda)$ is *regular*. The *normal rank* of $P(\lambda)$—from now

*This work was partially supported by the Ministerio de Educación y Ciencia of Spain through grant MTM-2006-06671 and the PRICIT program of Comunidad de Madrid through grant SIMUMAT (S-0505/ESP/0158) (Froilán M. Dopico), and by the Ministerio de Educación y Ciencia of Spain through grant MTM-2006-05361 (Fernando De Terán).

†Corresponding author

on, denoted by $\operatorname{nrank} P(\lambda)$—is the dimension of the largest non identically zero minor of $P(\lambda)$, and a *finite eigenvalue* of $P(\lambda)$ is a number $\lambda_0 \in \mathbb{C}$ such that

$$\operatorname{rank} P(\lambda_0) < \operatorname{nrank} P(\lambda)\,.$$

If $P(\lambda)$ is regular then the finite eigenvalues of $P(\lambda)$ are the roots of the polynomial $\det P(\lambda)$, but this is no longer true for singular matrix polynomials. As a consequence, the eigenvalues of regular matrix polynomials are continuous functions of the entries of the matrix coefficients of the polynomial, because the roots of a scalar polynomial are continuous functions of the coefficients of the polynomial. This continuity is lost for the eigenvalues of singular matrix polynomials. See [3] for examples in the case of polynomials of degree one.

We think that this lack of continuity in the eigenvalues is one of the main reasons why eigenvalue perturbation theory for singular matrix polynomials—and, in particular, for singular matrix pencils—has not been addressed jointly with eigenvalue perturbation theory for regular polynomials. In fact, the literature about perturbations of the regular polynomial eigenvalue problem has increased appreciably in the past few years (see, for example [15, 8, 1, 7] and the references therein). This has not had a counterpart for singular matrix polynomials, and we do not know any reference dealing with perturbations of eigenvalues of singular matrix polynomials of degree greater than one. On the other hand, the eigenvalues of singular matrix polynomials appear in several applications, as for instance in Linear Systems and Control Theory (see, for example [4, 18, 17]), and, therefore, the study of their perturbations is of interest.

We will see in this paper that we can reduce the perturbation analysis of the eigenvalues of a singular matrix polynomial to the study of the perturbations of the roots of a scalar polynomial as in the regular case. In order to explain this fact we will make use of the notions of algebraic geometry underlying in the eigenvalue perturbation theory. We will consider small perturbations of the singular $n \times n$ matrix polynomial $P(\lambda)$ of the form $\epsilon M(\lambda)$, where ϵ is a small parameter and $M(\lambda)$ is also an $n \times n$ matrix polynomial. For most perturbations $M(\lambda)$ the perturbed polynomial

$$P(\lambda, \epsilon) = P(\lambda) + \epsilon M(\lambda) \tag{1}$$

is regular, so its eigenvalues are the roots (in λ) of $\det P(\lambda, \epsilon)$. We may see the polynomial equation $f(\lambda, \epsilon) := \det P(\lambda, \epsilon) = 0$ as an algebraic curve in $\mathbb{C}^2$. If $f(\lambda, 0)$ were not identically zero, then around each root, λ_0, of $f(\lambda, 0)$ there would exist a certain number of *branches*, i.e., power series expansions in ϵ, denoted by $\lambda(\epsilon)$, satisfying $\lambda(0) = \lambda_0$ and $f(\lambda(\epsilon), \epsilon) \equiv 0$ [9, § 12.1]. These branches are usually called *Puiseux branches.* However, in our case, $f(\lambda, 0) \equiv 0$ because $P(\lambda)$ is singular, and we need to *regularize* the problem before considering *Puiseux branches.* To this purpose, we will prove that there exists a natural number k such that the polynomial $f(\lambda, \epsilon)$ can be written as

$$f(\lambda, \epsilon) = \epsilon^k \widetilde{f}(\lambda, \epsilon)\,,$$

where $\widetilde{f}(\lambda, \epsilon)$ is a polynomial such that $\widetilde{f}(\lambda, 0)$ is not identically zero for generic perturbations $M(\lambda)$. Then, the problem is regularized by considering, for $\epsilon \neq 0$, the polynomial equation $\widetilde{f}(\lambda, \epsilon) = 0$ instead of $f(\lambda, \epsilon) = 0$. More precisely, we will show that the set of roots of $\widetilde{f}(\lambda, \epsilon)$ includes branches $\lambda(\epsilon)$ such that $\lambda(0)$ are the eigenvalues of $P(\lambda)$. The set of perturbations $M(\lambda)$ such that the eigenvalues of $P(\lambda)$ change continuously with ϵ consists of those perturbations for which $\widetilde{f}(\lambda, 0)$ is not identically zero for a certain value of the exponent k. We will see that this set is *generic* in the set of all perturbations $M(\lambda)$ with degree less than or equal to the degree of $P(\lambda)$. The precise meaning of this sentence is that this set is the complementary of a certain proper algebraic manifold in the vector space of matrix polynomials with degree less than or equal to the degree of $P(\lambda)$.

Once the existence of expansions around the eigenvalues of $P(\lambda)$ is established through the condition $\widetilde{f}(\lambda, 0) \not\equiv 0$, we will study the first order terms in these expansions. We will consider first the most important case of a semisimple eigenvalue λ_0 of $P(\lambda)$ with geometric multiplicity g. This case is covered in Theorem 3, which is the main result in this paper. There, we will see that, generically, there are g eigenvalues of $P(\lambda) + \epsilon M(\lambda)$ with expansions

$$\lambda(\epsilon) = \lambda_0 + c\,\epsilon + o(\epsilon)\,, \tag{2}$$

where the leading coefficients $c \in \mathbb{C}$ in (2) are the eigenvalues of a certain regular matrix pencil that can be easily constructed through $M(\lambda_0)$ and arbitrary bases of the left and right nullspaces of the matrix $P(\lambda_0)$. If $P(\lambda)$ is a regular matrix polynomial and λ_0 is simple, then Theorem 3 reduces to the well-known formula $c = -(vM(\lambda_0)u)/(vP'(\lambda_0)u)$, where v and u are, respectively, left and right eigenvectors of $P(\lambda)$ associated with λ_0 [15, p. 345]. We defer to the last section, i.e., Section 6, the study of the first order terms of the perturbation expansions of arbitrary defective eigenvalues. The reason is that these terms are obtained by means of the eigenvalues of certain matrix pencils that are very difficult to construct in practice. Therefore the applicability of these results is limited, although they are interesting from a theoretical point view.

Eigenvectors are not defined in singular polynomials, even for simple eigenvalues. In the case of polynomials with degree one, it is known that the concept of *reducing subspace* is the correct one to be used [16]. A counterpart idea for singular polynomials of higher degree has not been established. As a consequence, a *generic* perturbation theory for eigenvectors of singular matrix polynomials cannot be developed. However, by taking into account that the perturbed polynomial (1) is generically regular, its eigenvectors, $v(\epsilon)$, are perfectly defined, and it is natural to ask how are these eigenvectors related to properties of the unperturbed polynomial $P(\lambda)$ when ϵ is close to zero. We answer this question in Section 5 by determining $v(0)$.

Our work is the natural generalization of the results in reference [3] to singular matrix polynomials of degree greater than one. Although the techniques used in [3] are closely related to the ones that we will use in this work, there is also a fundamental difference: the Kronecker Canonical Form of matrix pencils [5] plays a relevant role in [3], while this is not the case in this work because an analogous canonical form is not defined in matrix polynomials. In addition, the present work is based on results contained in the reference by Langer and Najman [11], where the authors determined the first order term of the eigenvalue expansions for uniparametric perturbations of *regular* analytic matrix functions. It can be said briefly that, after regularizing the problem as described above, our work consists in applying the techniques introduced in [11], and in developing some algebraic concepts that allow us to express the first order terms in a compact way in the case of semisimple eigenvalues. Our expression of the first order terms is influenced by the work of Lancaster, Markus, and Zhou [10] on semisimple eigenvalues of regular analytic matrix functions.

Finally, we stress again that although eigenvalue perturbation theory of singular pencils has been studied before in [3], and in a few previous works by Sun [13, 14], Demmel and Kågström [2], and Stewart [12], we do not know any reference about perturbation theory of eigenvalues of singular matrix polynomials of arbitrary degree.

The paper is organized as follows. In Section 2 we introduce the basic definitions and notation. The existence of eigenvalue expansions is studied in Section 3. The generic first order terms of the expansions of semisimple eigenvalues are derived in Section 4. The limits when ϵ tends to zero of the eigenvectors of the perturbed regular polynomial $P(\lambda) + \epsilon M(\lambda)$ are considered in Section 5. Results for arbitrary eigenvalues are discussed in Section 6.

2 Definitions and notation

In this section we introduce the basic tools and definitions used in the paper. Given an $n \times n$ matrix polynomial $P(\lambda)$ there exist two $n \times n$ matrix polynomials $U(\lambda)$ and $V(\lambda)$ with nonzero constant determinant such that

$$U(\lambda)P(\lambda)V(\lambda) = \begin{bmatrix} U_1(\lambda) \\ U_2(\lambda) \end{bmatrix} P(\lambda)\,[V_1(\lambda)\, V_2(\lambda)] \equiv \begin{bmatrix} D_S(\lambda) & 0 \\ 0 & 0_{d\times d} \end{bmatrix}, \tag{3}$$

where $D_S(\lambda) = \operatorname{diag}(h_1(\lambda), \ldots, h_r(\lambda))$, and the dimensions of $U_1(\lambda)$ and $V_1(\lambda)$ are chosen accordingly. The diagonal entries of $D_S(\lambda)$, $h_i(\lambda)$, $i = 1, \ldots, r$, are nonzero monic polynomials satisfying $h_i(\lambda)|h_{i+1}(\lambda)$, i.e., $h_i(\lambda)$ divides $h_{i+1}(\lambda)$, for $i = 1, \ldots, r-1$ [5, Ch. VI] [6, Ch. S1]. These polynomials are called the *invariant polynomials* of $P(\lambda)$, and the diagonal matrix in the right hand side of (3) is called the *Smith canonical form* of $P(\lambda)$. This form is unique. Notice that the normal rank of $P(\lambda)$ is the number, r, of invariant polynomials. If each

$$h_i(\lambda) = (\lambda - \lambda_1)^{\nu_{i1}} \cdot \cdots \cdot (\lambda - \lambda_q)^{\nu_{iq}}, \quad \text{for } i = 1, \ldots, r, \tag{4}$$

is decomposed in powers of different irreducible factors, then those factors among $(\lambda - \lambda_1)^{\nu_{11}}, \ldots, (\lambda - \lambda_q)^{\nu_{1q}}, \ldots, (\lambda - \lambda_1)^{\nu_{r1}}, \ldots, (\lambda - \lambda_q)^{\nu_{rq}}$ with $\nu_{ij} > 0$ are called the *elementary divisors of* $P(\lambda)$. The roots of the invariant polynomials of $P(\lambda)$ are the finite eigenvalues of $P(\lambda)$, so each elementary divisor is associated with a finite eigenvalue (its root). Based on the elementary divisors we introduce the following definitions.

Definition 1 *Let λ_0 be a finite eigenvalue of the matrix polynomial $P(\lambda)$. Then the number of elementary divisors of $P(\lambda)$ associated with λ_0 is the* geometric multiplicity of λ_0*, and the sum of the degrees of these elementary divisors is the* algebraic multiplicity of λ_0*. The eigenvalue λ_0 is* semisimple *if all the elementary divisors associated with λ_0 are linear.*

It is easy to see that the geometric multiplicity of λ_0 is equal to the difference $\operatorname{nrank} P(\lambda) - \operatorname{rank} P(\lambda_0)$.

We will say that $P(\lambda)$ has an *infinite eigenvalue* if the *dual polynomial* $P^\sharp(\lambda) = \lambda^\ell P(1/\lambda)$ has a zero eigenvalue, and the elementary divisors of $P(\lambda)$ associated with the infinite eigenvalue are the elementary divisors of $P^\sharp(\lambda)$ associated with zero. In this paper, we will focus on finite eigenvalues, although similar results can be obtained for the infinite eigenvalue by considering the dual polynomials. The reader is invited to read [3, Section 5.3] to see how this can be accomplished in matrix pencils.

Let g be the geometric multiplicity of the finite eigenvalue λ_0 of $P(\lambda)$, and $0 < m_1 \leq m_2 \leq \cdots \leq m_g$ be the degrees of the elementary divisors associated with λ_0. Then, after an elementary permutation, the Smith canonical form in the right hand side of (3) can be written as

$$\operatorname{diag}((\lambda - \lambda_0)^{m_1} q_1(\lambda), \ldots, (\lambda - \lambda_0)^{m_g} q_g(\lambda), q_{g+1}(\lambda), \ldots, q_r(\lambda), 0 \ldots, 0),$$

where $q_i(\lambda_0) \neq 0$, for $i = 1, \ldots, r$. Then we can premultiply in (3) by the diagonal matrix $\operatorname{diag}(1/q_1(\lambda), \ldots, 1/q_r(\lambda), 1, \ldots, 1)$, which is invertible at λ_0, to achieve

$$W(\lambda)P(\lambda)V(\lambda) = \Delta(\lambda), \tag{5}$$

where $W(\lambda) = \operatorname{diag}(1/q_1(\lambda), \ldots, 1/q_r(\lambda), 1, \ldots, 1)U(\lambda)$ and with

$$\Delta(\lambda) = \begin{bmatrix} (\lambda-\lambda_0)^{m_1} & & & & \\ & \ddots & & & \\ & & (\lambda-\lambda_0)^{m_g} & & \\ & & & I & \\ & & & & 0_{d\times d} \end{bmatrix}, \qquad d = n - \operatorname{nrank}P(\lambda). \tag{6}$$

The matrix $\Delta(\lambda)$ is the *local Smith form of $P(\lambda)$ at λ_0* and it is unique up to permutation of the diagonal entries. Notice that if $P(\lambda)$ is regular then no zeros appear on the main diagonal of $\Delta(\lambda)$. In the important case of λ_0 being semisimple, the local Smith form simplifies to

$$\Delta(\lambda) = \begin{bmatrix} (\lambda-\lambda_0)I_g & & \\ & I & \\ & & 0_{d\times d} \end{bmatrix}, \qquad d = n - \operatorname{nrank}P(\lambda). \tag{7}$$

2.1 Vector subspaces associated with singular polynomials

We denote by $\mathbb{C}(\lambda)$ the field of rational functions with complex coefficients and by $\mathbb{C}^n(\lambda)$ the vector space over $\mathbb{C}(\lambda)$ of n-tuples of rational functions. For brevity, the elements of $\mathbb{C}^n(\lambda)$ are sometimes row vectors and sometimes column vectors. The meaning will be always clear from the context. A matrix polynomial $P(\lambda)$ can be considered as a matrix with entries in $\mathbb{C}(\lambda)$, and the following definitions make sense.

Definition 2 *Let $P(\lambda)$ be a square $n \times n$ matrix polynomial. The vector subspaces of $\mathbb{C}^{1\times n}(\lambda)$ and $\mathbb{C}^{n\times 1}(\lambda)$*

$$\begin{aligned} \mathcal{N}_T(P) &= \{y(\lambda) \in \mathbb{C}^{1\times n}(\lambda) \, : \, y(\lambda)P(\lambda) \equiv 0\} \qquad \textit{and} \\ \mathcal{N}(P) &= \{x(\lambda) \in \mathbb{C}^{n\times 1}(\lambda) \, : \, P(\lambda)x(\lambda) \equiv 0\} \end{aligned}$$

are, respectively, called the left null space of $P(\lambda)$ *and the* right null space of $P(\lambda)$.

The subscript T in the left null space stands for the fact that its elements are row vectors. From now on, we will follow, as in [6], the convention of using row vectors for left null spaces and column vectors for right null spaces. These subspaces contain nonzero elements if and only if $P(\lambda)$ is singular. Note that, since $P(\lambda)$ is square, $\mathcal{N}_T(P)$ and $\mathcal{N}(P)$ have the same dimension.

Given a fixed number $\mu \in \mathbb{C}$, the left and right null spaces of the matrix $P(\mu) \in \mathbb{C}^{n\times n}$ will be of interest, specially if μ is an eigenvalue of $P(\lambda)$. These left and right null spaces are denoted, respectively, by $\mathcal{N}_T(P(\mu))$ $(\subset \mathbb{C}^n)$ and $\mathcal{N}(P(\mu))$ $(\subset \mathbb{C}^n)$.

A vector subspace of $\mathbb{C}^n(\lambda)$—in particular $\mathcal{N}_T(P)$ and $\mathcal{N}(P)$— has always a basis consisting of *vector polynomials*, i.e., vectors whose entries are polynomials in λ. We will refer to these bases as *polynomial bases*. Note that if $v(\lambda) \in \mathcal{N}(P)$ (resp. $u(\lambda) \in \mathcal{N}_T(P)$), $\mu \in \mathbb{C}$ is a fixed number, and $v(\mu)$ (resp. $u(\mu)$) is defined, then $v(\mu) \in \mathcal{N}(P(\mu))$ (resp. $u(\mu) \in \mathcal{N}_T(P(\mu))$). Lemma 1 below shows that if we consider a polynomial basis of $\mathcal{N}(P)$ (resp. of $\mathcal{N}_T(P)$) then the vector subspace of $\mathbb{C}^n$ spanned by the vectors of the polynomial basis evaluated at $\mu \in \mathbb{C}$ is the same for any basis, provided the vectors of the basis evaluated at μ are linearly independent.

Lemma 1 a) *Let $\{v_1(\lambda), \ldots, v_d(\lambda)\}$ and $\{\widetilde{v}_1(\lambda), \ldots, \widetilde{v}_d(\lambda)\}$ be two polynomial bases of $\mathcal{N}(P)$ and $\mu \in \mathbb{C}$ be a fixed number. If the sets $\{v_1(\mu), \ldots, v_d(\mu)\}$ and $\{\widetilde{v}_1(\mu), \ldots, \widetilde{v}_d(\mu)\}$ are linearly independent, then Span$\{v_1(\mu), \ldots, v_d(\mu)\}$ = Span$\{\widetilde{v}_1(\mu), \ldots, \widetilde{v}_d(\mu)\}$.*

b) *Let $\{u_1(\lambda), \ldots, u_d(\lambda)\}$ and $\{\widetilde{u}_1(\lambda), \ldots, \widetilde{u}_d(\lambda)\}$ be two polynomial bases of $\mathcal{N}_T(P)$ and $\mu \in \mathbb{C}$ be a fixed number. If the sets $\{u_1(\mu), \ldots, u_d(\mu)\}$ and $\{\widetilde{u}_1(\mu), \ldots, \widetilde{u}_d(\mu)\}$ are linearly independent, then Span$\{u_1(\mu), \ldots, u_d(\mu)\}$ = Span$\{\widetilde{u}_1(\mu), \ldots, \widetilde{u}_d(\mu)\}$.*

Proof. We will only prove a) because b) is similar. Since $\{v_1(\lambda), \ldots, v_d(\lambda)\}$ spans the same vector space over $\mathbb{C}(\lambda)$ as $\{\widetilde{v}_1(\lambda), \ldots, \widetilde{v}_d(\lambda)\}$ we have that

$$\begin{bmatrix} \widetilde{v}_1(\lambda) & \ldots & \widetilde{v}_d(\lambda) \end{bmatrix} = \begin{bmatrix} v_1(\lambda) & \ldots & v_d(\lambda) \end{bmatrix} A(\lambda) \tag{8}$$

for some $d \times d$ matrix $A(\lambda)$ with entries in $\mathbb{C}(\lambda)$.

First we will prove that $A(\mu)$ is well defined. Assume, on the contrary, that some entries of $A(\lambda)$ have denominators vanishing at μ. Without loss of generality, we may assume that some of these entries are in the first column, $a_1(\lambda)$, of $A(\lambda)$. Let $\alpha(\lambda)$ be the monic polynomial with least degree such that $\alpha(\lambda)a_1(\lambda)$ is a vector polynomial. Note that $\alpha(\mu) = 0$ and $\alpha(\mu)a_1(\mu) \neq 0$ because otherwise $\alpha(\lambda)/(\lambda - \mu)$ would be a polynomial with lower degree such that $(\alpha(\lambda)/(\lambda - \mu))a_1(\lambda)$ is a vector polynomial. Then, since

$$\widetilde{v}_1(\lambda) = \begin{bmatrix} v_1(\lambda) & \ldots & v_d(\lambda) \end{bmatrix} a_1(\lambda),$$

we have

$$\alpha(\lambda)\widetilde{v}_1(\lambda) = \begin{bmatrix} v_1(\lambda) & \ldots & v_d(\lambda) \end{bmatrix} (\alpha(\lambda)a_1(\lambda))$$

and, evaluating at μ we get

$$\alpha(\mu)\widetilde{v}_1(\mu) = \begin{bmatrix} v_1(\mu) & \dots & v_d(\mu) \end{bmatrix} (\alpha(\mu)a_1(\mu)) . \tag{9}$$

Since $\widetilde{v}_1(\lambda)$ is a vector polynomial, we have that $\alpha(\mu)\widetilde{v}_1(\mu) = 0$, and identity (9) is in contradiction with the fact that $\{v_1(\mu), \dots, v_d(\mu)\}$ is a linearly independent set. Then $A(\mu)$ is well defined.

Now, evaluating (8) at μ we have

$$\begin{bmatrix} \widetilde{v}_1(\mu) & \dots & \widetilde{v}_d(\mu) \end{bmatrix} = \begin{bmatrix} v_1(\mu) & \dots & v_d(\mu) \end{bmatrix} A(\mu) ,$$

and, since $\{v_1(\mu), \dots, v_d(\mu)\}$ and $\{\widetilde{v}_1(\mu), \dots, \widetilde{v}_d(\mu)\}$ are linearly independent sets, the matrix $A(\mu)$ is invertible, and both sets of vectors span the same subspace of $\mathbb{C}^n$. □

Lemma 1 is a particular case of implication $2a) \Rightarrow 4a)$ of the Main Theorem in [4], where the modulo $\alpha(\lambda) = \lambda - \mu$ is considered. We have included the proof for completeness.

If μ is not an eigenvalue of the singular matrix polynomial $P(\lambda)$ then the subspaces considered in Lemma 1 are the corresponding null spaces, i.e., $\text{Span}\{v_1(\mu), \dots, v_d(\mu)\} = \mathcal{N}(P(\mu))$ and $\text{Span}\{u_1(\mu), \dots, u_d(\mu)\} = \mathcal{N}_T(P(\mu))$. This is not true if μ is an eigenvalue of $P(\lambda)$, and in this case $\text{Span}\{v_1(\mu), \dots, v_d(\mu)\}$ and $\text{Span}\{u_1(\mu), \dots, u_d(\mu)\}$ are proper subspaces of $\mathcal{N}(P(\mu))$ and $\mathcal{N}_T(P(\mu))$, respectively. Anyway, Lemma 1 allows us to establish the following definition.

Definition 3 *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial and λ_0 be an eigenvalue of $P(\lambda)$. Let $\{v_1(\lambda), \dots, v_d(\lambda)\}$ (resp. $\{u_1(\lambda), \dots, u_d(\lambda)\}$) be a polynomial basis of $\mathcal{N}(P)$ (resp. $\mathcal{N}_T(P)$) such that $\{v_1(\lambda_0), \dots, v_d(\lambda_0)\}$ (resp. $\{u_1(\lambda_0), \dots, u_d(\lambda_0)\}$) is linearly independent. The vector subspace Span$\{v_1(\lambda_0), \dots, v_d(\lambda_0)\} \subset \mathbb{C}^n$ (resp. Span$\{u_1(\lambda_0), \dots, u_d(\lambda_0)\}$) will be called the* right singular space of $P(\lambda)$ at λ_0 *(resp.* left singular space of $P(\lambda)$ at λ_0*).*

Let us relate the singular spaces of $P(\lambda)$ with the matrices $U(\lambda)$ and $V(\lambda)$ transforming $P(\lambda)$ into its Smith canonical form as in (3). Since both $U(\lambda)$ and $V(\lambda)$ are nonsingular, it is immediate to see that the last d rows of $U(\lambda)$ and the last d columns of $V(\lambda)$ are bases of, respectively, $\mathcal{N}_T(P)$ and $\mathcal{N}(P)$. Lemma 2 below uses this fact to obtain bases of the left and right singular spaces of $P(\lambda)$ at λ_0. From these bases we will get bases of the complete spaces $\mathcal{N}_T(P(\lambda_0))$ and $\mathcal{N}(P(\lambda_0))$ by adding some vectors from $U(\lambda_0)$ and $V(\lambda_0)$.

Lemma 2 *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with local Smith form $\Delta(\lambda)$ at the eigenvalue λ_0 given by (6). Let $W(\lambda) = \begin{bmatrix} w_1(\lambda) \\ \vdots \\ w_n(\lambda) \end{bmatrix}$ and $V(\lambda) = \begin{bmatrix} v_1(\lambda) & \dots & v_n(\lambda) \end{bmatrix}$ be the matrices transforming $P(\lambda)$ into $\Delta(\lambda)$, where $w_i(\lambda)$ denotes a row vector and $v_i(\lambda)$ a column vector for $i = 1, \dots, n$. Then the following statements hold.*

a) *The sets of vectors $\{w_{n-d+1}(\lambda_0), \dots, w_n(\lambda_0)\}$ and $\{v_{n-d+1}(\lambda_0), \dots, v_n(\lambda_0)\}$ are bases of, respectively, the left and the right singular spaces of $P(\lambda)$ at λ_0.*

b) *The sets of vectors $\{w_1(\lambda_0), \dots, w_g(\lambda_0), w_{n-d+1}(\lambda_0), \dots, w_n(\lambda_0)\}$ and $\{v_1(\lambda_0), \dots, v_g(\lambda_0), v_{n-d+1}(\lambda_0), \dots, v_n(\lambda_0)\}$ are bases of, respectively, $\mathcal{N}_T(P(\lambda_0))$ and $\mathcal{N}(P(\lambda_0))$.*

Proof. We will only prove the result for the left basis, because the arguments for the right one are similar. The last d rows of $U(\lambda)$ in (3) constitute a polynomial basis of $\mathcal{N}_T(P)$ and, since $U(\lambda_0)$ is nonsingular, Lemma 1 implies that the last d rows of $U(\lambda_0)$ are a basis of the left singular space of $P(\lambda)$ at λ_0. Recall that

$$W(\lambda_0) = \text{diag}(1/q_1(\lambda_0), \dots, 1/q_r(\lambda_0), 1, \dots, 1)U(\lambda_0),$$

so the last d rows of $W(\lambda_0)$ are equal to the last d rows of $U(\lambda_0)$, and therefore they also form a basis of the left singular space of $P(\lambda)$ at λ_0.

The claim b) follows from the definition of the local Smith form at λ_0 in (6) and the fact that $W(\lambda_0)$ and $V(\lambda_0)$ are nonsingular. □

2.2 A particular relationship for semisimple eigenvalues

In Section 4 we will make use of the specific result for semisimple eigenvalues presented in Lemma 3.

Lemma 3 *Let $W(\lambda)$ and $V(\lambda)$ be as in the statement of* Lemma 2 *and assume, in addition, that the eigenvalue λ_0 of $P(\lambda)$ is semisimple. Then*

$$\begin{bmatrix} w_1(\lambda_0) \\ \vdots \\ w_g(\lambda_0) \\ w_{n-d+1}(\lambda_0) \\ \vdots \\ w_n(\lambda_0) \end{bmatrix} P'(\lambda_0) \begin{bmatrix} v_1(\lambda_0) & \dots & v_g(\lambda_0) & v_{n-d+1}(\lambda_0) & \dots & v_n(\lambda_0) \end{bmatrix} = \begin{bmatrix} I_g & \\ & 0_{d\times d} \end{bmatrix}.$$

Proof. Taking derivatives in the identity $W(\lambda)P(\lambda)V(\lambda) = \Delta(\lambda)$, where $\Delta(\lambda)$ is given by (7), we achieve

$$W'(\lambda)P(\lambda)V(\lambda) + W(\lambda)P'(\lambda)V(\lambda) + W(\lambda)P(\lambda)V'(\lambda) = \Delta'(\lambda). \tag{10}$$

By using Lemma 2 b) we obtain $w_i(\lambda_0)P(\lambda_0) = 0$ and $P(\lambda_0)v_i(\lambda_0) = 0$ for $i = 1, \dots, g, n-d+1, \dots, n$, and from (7) that

$$\Delta'(\lambda_0) = \begin{bmatrix} I_g & \\ & 0 \end{bmatrix}.$$

Now the result follows from evaluating at λ_0 the equation (10). □

Let us illustrate the definitions introduced in this section with an example.

Example 1 *Let $P(\lambda)$ be the following 3×3 singular matrix polynomial of degree two:*

$$P(\lambda) = \begin{bmatrix} \lambda^2 & \lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 1 & \lambda^2 \end{bmatrix}.$$

This polynomial has normal rank equal to 2 and the simple finite eigenvalue $\lambda_0 = 0$ with geometric multiplicity 1.

Polynomial bases of the left and the right null spaces $\mathcal{N}_T(P)$ and $\mathcal{N}(P)$ are given, respectively, by $y(\lambda) = \begin{bmatrix} 1 & -\lambda & 0 \end{bmatrix}$ and $x(\lambda) = \begin{bmatrix} \lambda & -\lambda^2 & 1 \end{bmatrix}^T$.

Since $y(0)$ and $x(0)$ are nonzero vectors, the left singular space of $P(\lambda)$ at 0 is spanned by $y(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and the right singular space of $P(\lambda)$ at 0 is spanned by $x(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. We can complete these bases to bases of the whole null spaces of $P(0)$ as follows:

$$\mathcal{N}_T(P(0)) = \operatorname{span}\left\{\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\right\} \quad \text{and} \quad \mathcal{N}(P(0)) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

The local Smith form of $P(\lambda)$ at $\lambda_0 = 0$ is

$$\Delta(\lambda) = \begin{bmatrix} \lambda & & \\ & 1 & \\ & & 0 \end{bmatrix}$$

and we have $W(\lambda)P(\lambda)V(\lambda) = \Delta(\lambda)$, with

$$W(\lambda) = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -\lambda & 0 \end{bmatrix}, \quad V(\lambda) = \begin{bmatrix} -1 & 0 & \lambda \\ \lambda & 1 & -\lambda^2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that the previous bases of $\mathcal{N}_T(P(0))$ and $\mathcal{N}(P(0))$ are given by, respectively, $\{w_1(0), w_3(0)\}$ and $\{v_1(0), v_3(0)\}$.

3 Existence of eigenvalue expansions

This section is devoted to one of the central problems of the present paper, i.e., to study the existence of perturbation expansions near the eigenvalues of a singular $n \times n$ matrix polynomial $P(\lambda)$. Our approach is similar to the one followed in [3, Section 3] for matrix pencils, and the reader is referred to [3] for some details that are omitted here.

Given a singular $n \times n$ matrix polynomial $P(\lambda)$ we will obtain sufficient conditions on the perturbation polynomial $M(\lambda)$ that guarantee that all the eigenvalues of $P(\lambda) + \epsilon M(\lambda)$ can be expanded as (fractional) power series in ϵ, and that when these series are evaluated at $\epsilon = 0$ all the eigenvalues (finite and infinite) of $P(\lambda)$ are obtained, together with some other numbers or infinities that are not eigenvalues of $P(\lambda)$ and are fully determined by the perturbation $M(\lambda)$. This result will be presented in Theorem 1, which is the generalization of Theorem 1 in [3] to square singular matrix polynomials. Before, we need to establish the technical Lemma 4 that proves that the sufficient condition for the existence of expansions holds simultaneously in a matrix polynomial and its dual. This implies that the existence of expansions of finite and infinite eigenvalues are simultaneously guaranteed.

Lemma 4 *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree ℓ with Smith canonical form given by* (3)*, $P^\sharp(\lambda)$ be its dual polynomial, and $M(\lambda)$ be another $n \times n$ matrix polynomial. Let us consider the Smith canonical form of $P^\sharp(\lambda)$,*

$$\widetilde{U}(\lambda)P^\sharp(\lambda)\widetilde{V}(\lambda) = \begin{bmatrix} \widetilde{U}_1(\lambda) \\ \widetilde{U}_2(\lambda) \end{bmatrix} P^\sharp(\lambda)\,[\widetilde{V}_1(\lambda)\,\widetilde{V}_2(\lambda)] \equiv \begin{bmatrix} \widetilde{D}_S(\lambda) & 0 \\ 0 & 0_{d\times d} \end{bmatrix}, \tag{11}$$

partitioned in blocks with dimensions as those in (3)*. Then* $\det(U_2(\lambda)M(\lambda)V_2(\lambda)) \not\equiv 0$ *if and only if* $\det(\widetilde{U}_2(\lambda)M^\sharp(\lambda)\widetilde{V}_2(\lambda)) \not\equiv 0$*.*

Proof. Note first that the definition of dual polynomial implies that $\mathcal{N}(P(\mu)) = \mathcal{N}(P^\sharp(1/\mu))$ and $\mathcal{N}_T(P(\mu)) = \mathcal{N}_T(P^\sharp(1/\mu))$, for any number $0 \neq \mu \in \mathbb{C}$. Recall also that λ_0 is an eigenvalue of $P(\lambda)$ if and only if $1/\lambda_0$ is an eigenvalue of $P^\sharp(\lambda)$.

Let $\mu \in \mathbb{C}$ be a number such that $\mu \neq 0$ and μ is not an eigenvalue of $P(\lambda)$. In this case the columns of $V_2(\mu)$ (resp. the rows of $U_2(\mu)$) form a basis of $\mathcal{N}(P(\mu))$ (resp. of $\mathcal{N}_T(P(\mu))$) and the columns of $\widetilde{V}_2(1/\mu)$ (resp. the rows of $\widetilde{U}_2(1/\mu)$) form a basis of $\mathcal{N}(P^\sharp(1/\mu))$ (resp. of $\mathcal{N}_T(P^\sharp(1/\mu))$). As a consequence there exist nonsingular $d \times d$ matrices T and S such that $\widetilde{V}_2(1/\mu) = V_2(\mu)T$ and $\widetilde{U}_2(1/\mu) = SU_2(\mu)$. So

$$\begin{aligned} \det(\, U_2(\mu)\, M(\mu)\, V_2(\mu)\,) \neq 0 \quad &\Longleftrightarrow \quad \det(\, \widetilde{U}_2(1/\mu)\, M(\mu)\, \widetilde{V}_2(1/\mu)\,) \neq 0 \\ &\Longleftrightarrow \quad \det(\, \widetilde{U}_2(1/\mu)\, M^\sharp(1/\mu)\, \widetilde{V}_2(1/\mu)\,) \neq 0, \end{aligned}$$

where we have used that $M(\mu) = \mu^k M^\sharp(1/\mu)$ with k the degree of $M(\lambda)$.

Finally observe that $p(\lambda) = \det(\, U_2(\lambda)\, M(\lambda)\, V_2(\lambda)\,)$ and $\widetilde{p}(\lambda) = \det(\, \widetilde{U}_2(\lambda)\, M^\sharp(\lambda)\, \widetilde{V}_2(\lambda)\,)$ are polynomials in λ. Therefore $p(\lambda)$ is not the zero polynomial if and only if $p(\mu) \neq 0$ for a number μ such that $\mu \neq 0$ and μ is not an eigenvalue of $P(\lambda)$. Analogously $\widetilde{p}(\lambda)$ is not the zero polynomial if and only if $\widetilde{p}(\gamma) \neq 0$ for a number γ such that $\gamma \neq 0$ and γ is not an eigenvalue of $P^\sharp(\lambda)$ □

Theorem 1 *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with degree ℓ whose Smith canonical form is given by* (3)*, and $M(\lambda)$ be another $n \times n$ matrix polynomial with degree smaller than or equal to ℓ, and such that* $\det(\, U_2(\lambda)\, M(\lambda)\, V_2(\lambda)\,) \not\equiv 0$*. Then*

1. *There exists a constant $b > 0$ such that the matrix polynomial $P(\lambda) + \epsilon M(\lambda)$ is regular whenever $0 < |\epsilon| < b$.*
2. *For $0 < |\epsilon| < b$ the finite eigenvalues of $P(\lambda) + \epsilon M(\lambda)$ are the roots of a polynomial in λ, $p_\epsilon(\lambda)$, whose coefficients are polynomials in ϵ. In addition, when $\epsilon = 0$,*

$$p_0(\lambda) = \det(\, D_S(\lambda)\,)\, \det(\, U_2(\lambda)\, M(\lambda)\, V_2(\lambda)\,). \tag{12}$$

3. *Let ϵ be such that $0 < |\epsilon| < b$. Then the $n\ell$ eigenvalues*[1], $\{\lambda_1(\epsilon), \ldots, \lambda_{n\ell}(\epsilon)\}$, *of* $P(\lambda) + \epsilon M(\lambda)$ *can be expanded as (fractional) power series in* ϵ. *Some of these series may have terms with negative exponents and tend to ∞ as ϵ tends to zero. The rest of the series converge in a neighborhood of* $\epsilon = 0$.
4. *If the finite eigenvalues of $P(\lambda)$ are $\{\mu_1, \ldots, \mu_s\}$, where common elements are repeated according to their algebraic multiplicity, then there exists a subset* $\{\lambda_{i_1}(\epsilon), \ldots, \lambda_{i_s}(\epsilon)\}$ *of* $\{\lambda_1(\epsilon), \ldots, \lambda_{n\ell}(\epsilon)\}$ *such that*

$$\lim_{\epsilon \to 0} \lambda_{i_j}(\epsilon) = \mu_j \quad j = 1, \ldots, s.$$

5. *If the polynomial $P(\lambda)$ has an infinite eigenvalue with algebraic multiplicity p, then there exist* $\{\lambda_{l_1}(\epsilon), \ldots, \lambda_{l_p}(\epsilon)\}$ *such that*

$$\lim_{\epsilon \to 0} \lambda_{l_j}(\epsilon) = \infty \quad j = 1, \ldots, p.$$

Proof. The proof is exactly the same as the one of Theorem 1 in [3], only by changing matrix pencils by matrix polynomials. We include it here for the sake of completitude.

Let us partition $U(\lambda)\, M(\lambda)\, V(\lambda)$ conformally with (3) as

$$U(\lambda)\, M(\lambda)\, V(\lambda) = \begin{bmatrix} B_{11}(\lambda) & B_{12}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) \end{bmatrix}.$$

This means that $B_{22}(\lambda) = U_2(\lambda)\, M(\lambda)\, V_2(\lambda)$. Thus

$$\det(\, P(\lambda) + \epsilon M(\lambda)\,) = C \det \begin{bmatrix} D_S(\lambda) + \epsilon B_{11}(\lambda) & \epsilon B_{12}(\lambda) \\ \epsilon B_{21}(\lambda) & \epsilon B_{22}(\lambda) \end{bmatrix},$$

where C is the nonzero constant $C = 1/\det(U(\lambda)\, V(\lambda))$. Then

$$\det(\, P(\lambda) + \epsilon M(\lambda)\,) = C\, \epsilon^d \det \begin{bmatrix} D_S(\lambda) + \epsilon B_{11}(\lambda) & B_{12}(\lambda) \\ \epsilon B_{21}(\lambda) & B_{22}(\lambda) \end{bmatrix}.$$

Let us define the polynomial in λ

$$p_\epsilon(\lambda) \equiv \det \begin{bmatrix} D_S(\lambda) + \epsilon B_{11}(\lambda) & B_{12}(\lambda) \\ \epsilon B_{21}(\lambda) & B_{22}(\lambda) \end{bmatrix},$$

whose coefficients are polynomials in ϵ, and write

$$\det(\, P(\lambda) + \epsilon M(\lambda)\,) = C\, \epsilon^d\, p_\epsilon(\lambda). \tag{13}$$

It is obvious that when $\epsilon = 0$

$$p_0(\lambda) = \det(\, D_S(\lambda)\,)\, \det(\, B_{22}(\lambda)\,). \tag{14}$$

We know that $\det(\, D_S(\lambda)\,) \not\equiv 0$, and, therefore, $\det(\, B_{22}(\lambda)\,) \not\equiv 0$ implies that $P(\lambda) + \epsilon M(\lambda)$ is regular in a punctured disk $0 < |\epsilon| < b$. This is obvious by continuity: if $\det(\, D_S(\mu)\,)\, \det(\, B_{22}(\mu)\,) \neq 0$ for some fixed number μ, then $p_\epsilon(\mu) \neq 0$ for ϵ small enough, since $p_\epsilon(\mu)$ is continuous as a function of ϵ. In addition, whenever $0 < |\epsilon| < b$, equation (13) implies that z is a finite eigenvalue of $P(\lambda) + \epsilon M(\lambda)$ if and only if $p_\epsilon(z) = 0$. So, the first and second items in Theorem 1 are proved.

Notice that we have reduced the original perturbation eigenvalue problem to the study of the variation of the roots of $p_\epsilon(\lambda)$ as ϵ tends to zero. But since the coefficients are polynomials in ϵ, this is a classical problem solved by Algebraic Function Theory [9]. In particular the

[1] It is well known that any $n \times n$ regular matrix polynomial with degree ℓ has exactly $n\ell$ eigenvalues, if finite and infinite eigenvalues are counted with their multiplicities [6].

third item is a consequence of this theory (for infinite eigenvalues similar arguments can be applied to zero eigenvalues of dual polynomials). We just comment that if the degree of $p_\epsilon(\lambda)$ in λ is δ_1 and the degree of $\det(D_S(\lambda))\det(B_{22}(\lambda))$ is $\delta_2 < \delta_1$, then $\delta_1 - \delta_2$ roots of $p_\epsilon(\lambda)$ tend to infinity when ϵ tends to zero. The fourth item is again a consequence of Algebraic Function Theory and (14), since those roots that remain finite have as limits the roots of $\det(D_S(\lambda))\det(B_{22}(\lambda))$, and the roots of $\det(D_S(\lambda))$ are precisely the finite eigenvalues of $P(\lambda)$.

The last item can be proved by applying the previous results to the zero eigenvalue of the dual polynomial of $P(\lambda) + \epsilon M(\lambda)$, and taking into account that $\lambda_i(\epsilon)$ is an eigenvalue of $P(\lambda) + \epsilon M(\lambda)$ if and only if $1/\lambda_i(\epsilon)$ is an eigenvalue of the dual polynomial. $\square$

It can be seen from the proof of Theorem 1 that if the degree of the perturbation polynomial $M(\lambda)$ in the statement is $\widetilde{\ell} > \ell$ then the result would be true, with the exception that $P(\lambda) + \epsilon M(\lambda)$ would have a number of eigenvalues equal to $n\widetilde{\ell}$. We have enunciated the result with the restriction of $M(\lambda)$ having degree smaller than or equal to the degree of $P(\lambda)$, because we think that it is the natural situation in perturbation theory of matrix polynomials. Moreover, in this case, once $P(\lambda)$ is fixed, $\det(U_2(\lambda)\,M(\lambda)\,V_2(\lambda)) \not\equiv 0$ is a generic condition on the set of perturbation polynomials $M(\lambda) = B_0 + \lambda B_1 + \cdots + \lambda^\ell B_\ell$ of degree at most ℓ, because it does not hold only on the algebraic manifold defined by equating to zero all the coefficients of the polynomial $p(\lambda) = \det(U_2(\lambda)\,M(\lambda)\,V_2(\lambda))$. These coefficient are multivariate polynomials in the entries of $B_0, B_1, \ldots, B_\ell$.

4 Expansions for semisimple eigenvalues

In Section 3 we have obtained a global condition for the existence of expansions near all eigenvalues of $P(\lambda)$. In this section we focus on a given semisimple eigenvalue λ_0 of the singular square matrix polynomial $P(\lambda)$, we obtain a specific generic condition for the existence of eigenvalue expansions near λ_0, and, more important, we derive simple expressions for the first order terms of the expansions of those eigenvalues of the perturbed polynomial $P(\lambda) + \epsilon M(\lambda)$ whose limit is λ_0 when ϵ tends to zero.

Our main results will be based on the following construction. Throughout this section, λ_0 is a semisimple eigenvalue, with geometric multiplicity g, of the square singular matrix polynomial $P(\lambda)$. Let $\{w_{n-d+1}, \ldots, w_n\}$ and $\{v_{n-d+1}, \ldots, v_n\}$ be bases of, respectively, the left and the right singular spaces of $P(\lambda)$ at λ_0. Then we can complete these bases to, respectively, a basis of $\mathcal{N}_T(P(\lambda_0))$ and a basis of $\mathcal{N}(P(\lambda_0))$:

$$\{w_1, \ldots, w_g, w_{n-d+1}, \ldots, w_n\} \quad \textit{and} \quad \{v_1, \ldots, v_g, v_{n-d+1}, \ldots, v_n\} \,.$$

Recall that the vectors w_i are row vectors, whereas v_j are column vectors. Using these vectors we build up the $(g+d) \times (g+d)$ matrices

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \left[\begin{array}{c} w_1 \\ \vdots \\ w_g \\ \hline w_{n-d+1} \\ \vdots \\ w_n \end{array}\right] \quad \textit{and} \quad \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \left[\begin{array}{ccc|ccc} v_1 & \ldots & v_g & v_{n-d+1} & \ldots & v_n \end{array}\right] . \tag{15}$$

Note that the vectors in item b) of Lemma 2 are particular cases of the bases in (15). The following result generalizes Lemma 3.

Lemma 5 *Let* $\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ *and* $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *be the matrices defined in* (15)*. Then* $W_1 P'(\lambda_0) V_2 = 0$*,* $W_2 P'(\lambda_0) V_1 = 0$*,* $W_2 P'(\lambda_0) V_2 = 0$*, and* $W_1 P'(\lambda_0) V_1$ *is nonsingular.*

Proof. Let $W(\lambda)$ and $V(\lambda)$ be as in the statement of Lemma 2. Then, there exist some matrices R_{11}, R_{12}, R_{22} and S_{11}, S_{21}, S_{22}, with $R_{11}, R_{22}, S_{11}, S_{22}$ nonsingular, such that

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \begin{bmatrix} w_1(\lambda_0) \\ \vdots \\ w_g(\lambda_0) \\ w_{n-d+1}(\lambda_0) \\ \vdots \\ w_n(\lambda_0) \end{bmatrix}$$

and

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} v_1(\lambda_0) & \dots & v_g(\lambda_0) & v_{n-d+1}(\lambda_0) & \dots & v_n(\lambda_0) \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix}.$$

Now, using Lemma 3, we have

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} P'(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} R_{11}S_{11} & \\ & 0_{d\times d} \end{bmatrix},$$

and this concludes the proof. □

From the matrices in (15) we define the $(g+d)\times(g+d)$ matrix,

$$\Phi = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} M(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix}. \tag{16}$$

Associated with Φ we introduce the $(g+d)\times(g+d)$ matrix pencil

$$\mathcal{P}(\zeta) = \Phi + \zeta \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} P'(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \tag{17}$$

and note that, by virtue of Lemma 5,

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} P'(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} W_1 P'(\lambda_0) V_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let us illustrate these definitions with an example.

Example 2 *Let $P(\lambda)$ be the same polynomial as in* Example 1 *and set*

$$M(\lambda) = \begin{bmatrix} \lambda & 1 & 1-\lambda^2 \\ 2+\lambda & 5 & \lambda^2 \\ 1 & 2\lambda & 4+\lambda \end{bmatrix}.$$

Let also $W(\lambda)$ and $V(\lambda)$ be as in Example 1*. Then for the unique finite eigenvalue $\lambda_0 = 0$ of $P(\lambda)$,*

$$\Phi = \begin{bmatrix} w_1(0) \\ w_3(0) \end{bmatrix} M(0) \begin{bmatrix} v_1(0) & v_3(0) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathcal{P}(\zeta) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} + \zeta \begin{bmatrix} w_1(0) \\ w_3(0) \end{bmatrix} P'(0) \begin{bmatrix} v_1(0) & v_3(0) \end{bmatrix} = \begin{bmatrix} 1+\zeta & 4 \\ 0 & 1 \end{bmatrix}.$$

Lemma 6 states some relevant properties of the pencil $\mathcal{P}(\zeta)$ that are used in subsequent developments.

Lemma 6 *Let Φ be the matrix defined in* (16) *and $\mathcal{P}(\zeta)$ the pencil in* (17)*. Then the following statements hold.*

1) *$\mathcal{P}(\zeta)$ is regular and has exactly g finite eigenvalues if and only if the $d \times d$ matrix $W_2 M(\lambda_0) V_2$ is nonsingular.*

2) *Assume that $W_2 M(\lambda_0) V_2$ is nonsingular, then the g finite eigenvalues of $\mathcal{P}(\zeta)$ are all different from zero if and only if Φ is nonsingular.*

Proof. Let us express

$$\Phi = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & W_2 M(\lambda_0) V_2 \end{bmatrix}.$$

By Lemma 5 we have

$$\mathcal{P}(\zeta) = \Phi + \zeta \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} P'(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} C_{11} + \zeta W_1 P'(\lambda_0) V_1 & C_{12} \\ C_{21} & W_2 M(\lambda_0) V_2 \end{bmatrix}.$$

Therefore,

$$\det \mathcal{P}(\zeta) = \zeta^g \det(W_1 P'(\lambda_0) V_1) \det(W_2 M(\lambda_0) V_2) + \zeta^{g-1} b_{g-1} + \cdots + \det \Phi, \qquad (18)$$

where the coefficients $b_{g-1}, \ldots, b_1$ in the previous polynomial are of no interest in this argument. Since $W_1 P'(\lambda_0) V_1$ is nonsingular by Lemma 5, both claims follow easily. □

Note that the pencil $\mathcal{P}(\zeta)$ depends on the particular matrices of bases $\{W_1, W_2\}$ and $\{V_1, V_2\}$ of $\mathcal{N}_T(P(\lambda_0))$ and $\mathcal{N}(P(\lambda_0))$ that are used, but the property of $W_2 M(\lambda_0) V_2$ being nonsingular is independent on the particular bases W_2 and V_2 of, respectively, the left and the right singular spaces of $P(\lambda)$ at λ_0. The invertibility of $W_2 M(\lambda_0) V_2$ plays an essential role in Theorem 2 below, and this result implies Theorem 3 which is the main result in this paper. Observe also that $\det(W_2 M(\lambda_0) V_2) \neq 0$ implies that the assumption $\det(\, U_2(\lambda)\, M(\lambda)\, V_2(\lambda)\,) \not\equiv 0$ in Theorem 1 holds.

Theorem 2 *Let $P(\lambda)$ be an arbitrary $n \times n$ matrix polynomial and $M(\lambda)$ be another matrix polynomial with the same dimension. Let λ_0 be a finite semisimple eigenvalue of $P(\lambda)$ with geometric multiplicity g , $W = [W_1^T\, W_2^T]^T$ be a matrix whose rows form a basis of $\mathcal{N}_T(P(\lambda_0))$, and $V = [V_1\, V_2]$ be a matrix whose columns form a basis of $\mathcal{N}(P(\lambda_0))$, where the rows of W_2 (resp. the columns of V_2) form a basis of the left (resp. the right) singular space of $P(\lambda)$ at λ_0. Let also Φ be the matrix defined in* (16) *and $\mathcal{P}(\zeta)$ be the pencil defined in* (17)*. If $W_2 M(\lambda_0) V_2$ is nonsingular, then the perturbed matrix polynomial $P(\lambda) + \epsilon M(\lambda)$ is regular and has exactly g eigenvalues in a neighborhood of $\epsilon = 0$ satisfying*

$$\lambda_j(\epsilon) = \lambda_0 + \zeta_j \epsilon + o(\epsilon) \qquad j = 1, \ldots, g\,, \qquad (19)$$

where $\zeta_1, \ldots, \zeta_g$ are the finite eigenvalues of the pencil $\mathcal{P}(\zeta)$. If, in addition, Φ is nonsingular, then $\zeta_1, \ldots, \zeta_g$ are all nonzero and all the expansions near λ_0 have leading exponent equal to one. If $g = 1$, i.e., λ_0 is simple, then W_1 has only one row vector and V_1 only one column vector, and (19) *simplifies to*

$$\lambda(\epsilon) = \lambda_0 - \frac{\det(W M(\lambda_0) V)}{(W_1 P'(\lambda_0) V_1) \cdot \det(W_2 M(\lambda_0) V_2)}\, \epsilon + O(\epsilon^2). \qquad (20)$$

Proof. The property of $W_2 M(\lambda_0) V_2$ being nonsingular is independent on the choice of bases W_2 and V_2 of the left and the right singular spaces of $P(\lambda)$ at λ_0. In addition, the eigenvalues of the matrix pencil $\mathcal{P}(\zeta)$ are also independent on the bases $\{W_1, W_2\}$ and $\{V_1, V_2\}$. This means that we may consider the particular bases of $\mathcal{N}_T(P(\lambda_0))$ and $\mathcal{N}(P(\lambda_0))$ given by the rows and the columns (respectively) of the matrices $W(\lambda)$ and $V(\lambda)$ in the statement of Lemma 2. Recall also the these bases verify Lemma 3.

With this choice of bases is obvious that the invertibility of $W_2 M(\lambda_0) V_2$ implies that $\det(\, U_2(\lambda)\, M(\lambda)\, V_2(\lambda)\,) \not\equiv 0$ in Theorem 1 holds, and that the polynomial $p_0(\lambda)$ in (12) has exactly g roots equal to λ_0. Therefore, Theorem 1 guarantees that $P(\lambda) + \epsilon M(\lambda)$ is regular in a neighborhood of $\epsilon = 0$ and has exactly g eigenvalues whose expansions tend to λ_0 when ϵ tends to zero. Let us determine the first terms of these expansions.

The proof is similar to the one of Theorem 2 in [3], and it is based on the local Smith form. We restrict ourselves to the case $\lambda_0 = 0$. If $\lambda_0 \neq 0$, we just make a shift $\mu = \lambda - \lambda_0$ in the local Smith form: $W(\lambda - \lambda_0 + \lambda_0)P(\lambda - \lambda_0 + \lambda_0)V(\lambda - \lambda_0 + \lambda_0) = \Delta(\lambda - \lambda_0 + \lambda_0)$, define $\widetilde{W}(\mu) := W(\mu + \lambda_0)$, $\widetilde{V}(\mu) := V(\mu + \lambda_0)$, $\widetilde{P}(\mu) := P(\mu + \lambda_0)$, and $\widetilde{\Delta}(\mu) := \Delta(\mu + \lambda_0)$, and, finally, consider $\widetilde{W}(\mu)\widetilde{P}(\mu)\widetilde{V}(\mu) = \widetilde{\Delta}(\mu)$. Note that $\widetilde{W}(0) = W(\lambda_0)$ and $\widetilde{V}(0) = V(\lambda_0)$.

Assuming that $\lambda_0 = 0$, we consider the transformation to the local Smith form at $\lambda_0 = 0$,

$$W(\lambda)(P(\lambda) + \epsilon M(\lambda))V(\lambda) = \Delta(\lambda) + \epsilon W(\lambda)M(\lambda)V(\lambda) \equiv \widehat{\Delta}(\lambda) + G(\lambda, \epsilon), \tag{21}$$

where

$$\widehat{\Delta}(\lambda) = \begin{bmatrix} \lambda I_g & & \\ & 0 & \\ & & 0_{d\times d} \end{bmatrix} \qquad \text{and} \qquad G(\lambda, \epsilon) = \begin{bmatrix} \epsilon G_{11}(\lambda) & \epsilon G_{12}(\lambda) & \epsilon G_{13}(\lambda) \\ \epsilon G_{21}(\lambda) & I + \epsilon G_{22}(\lambda) & \epsilon G_{23}(\lambda) \\ \epsilon G_{31}(\lambda) & \epsilon G_{32}(\lambda) & \epsilon G_{33}(\lambda) \end{bmatrix}$$

are partitioned conformally, and $[G_{ij}(\lambda)]_{i,j=1}^{3} = W(\lambda)M(\lambda)V(\lambda)$. Therefore, if $P(\lambda) + \epsilon M(\lambda)$ is regular, its finite eigenvalues are the roots of

$$f(\lambda, \epsilon) = \det(P(\lambda) + \epsilon M(\lambda)) = \delta(\lambda)\epsilon^d \widetilde{f}(\lambda, \epsilon),$$

where

$$\widetilde{f}(\lambda, \epsilon) = \det(\widehat{\Delta}(\lambda) + \widetilde{G}(\lambda, \epsilon))$$

and

$$\widetilde{G}(\lambda, \epsilon) = \begin{bmatrix} \epsilon G_{11}(\lambda) & \epsilon G_{12}(\lambda) & G_{13}(\lambda) \\ \epsilon G_{21}(\lambda) & I + \epsilon G_{22}(\lambda) & G_{23}(\lambda) \\ \epsilon G_{31}(\lambda) & \epsilon G_{32}(\lambda) & G_{33}(\lambda) \end{bmatrix}.$$

In addition, the function $\delta(\lambda)$ is given by $\delta(\lambda) = p(\lambda)q(\lambda)$ where, $\det(W(\lambda)) = 1/p(\lambda)$ and $\det(V(\lambda)) = 1/q(\lambda)$. So $\delta(\lambda)$ is a polynomial such that $\delta(0) \neq 0$ and that does not depend on the perturbation $M(\lambda)$. These facts imply that for $\epsilon \neq 0$, the polynomial $P(\lambda) + \epsilon M(\lambda)$ is regular if and only if $\widetilde{f}(\lambda, \epsilon) \not\equiv 0$, and that, in this case, the eigenvalues of $P(\lambda) + \epsilon M(\lambda)$ whose limit is $\lambda_0 = 0$ as ϵ tends to zero are those zeros, $\lambda(\epsilon)$, of $\widetilde{f}(\lambda, \epsilon)$ whose limit is 0. Obviously, $\widetilde{f}(\lambda, \epsilon)$ is a rational function in λ, where the coefficients of the numerator are polynomials in ϵ, and the denominator is precisely $\delta(\lambda)$. So, $\widetilde{f}(\lambda, \epsilon)$ can be also seen as a polynomial in ϵ whose coefficients are rational functions in λ. Let us study more carefully the function $\widetilde{f}(\lambda, \epsilon)$.

In the first place, note that

$$\Phi = \begin{bmatrix} G_{11}(0) & G_{13}(0) \\ G_{31}(0) & G_{33}(0) \end{bmatrix}, \quad \text{and} \quad W_2 M(0) V_2 = G_{33}(0). \tag{22}$$

We now make use of the Lemma in [11, p. 799] on determinants of the type $\det(D + G)$ with D diagonal, to expand $\widetilde{f}(\lambda, \epsilon)$ as

$$\widetilde{f}(\lambda, \epsilon) = \det \widetilde{G}(\lambda, \epsilon) + \sum \lambda^s \det \widetilde{G}(\lambda, \epsilon)(\{\nu_1, \ldots, \nu_s\}^c)\,, \tag{23}$$

where for any matrix C, $C(\{\nu_1, \ldots, \nu_s\}^c)$ denotes the matrix obtained by removing from C the rows and columns with indices $\nu_1, \ldots, \nu_s$. The sum runs over all $s \in \{1, \ldots, g\}$ and all $\nu_1, \ldots, \nu_s$ such that $1 \leq \nu_1 < \ldots < \nu_s \leq g$. Finally, note that

$$\det \widetilde{G}(\lambda, \epsilon) = \epsilon^g(\det \Phi + Q_0(\lambda, \epsilon))\,, \tag{24}$$

for $Q_0(\lambda, \epsilon)$ rational with $Q_0(0, 0) = 0$, and

$$\det \widetilde{G}(\lambda, \epsilon)(\{\nu_1, \ldots, \nu_s\}^c) = \epsilon^{g-s}(\det \Phi(\{\nu_1, \ldots, \nu_s\}^c) + Q_{\nu_1 \ldots \nu_s}(\lambda, \epsilon))\,, \tag{25}$$

with $Q_{\nu_1\ldots\nu_s}(\lambda,\epsilon)$ rational and $Q_{\nu_1\ldots\nu_s}(0,0)=0$. In particular,

$$\det \widetilde{G}(\lambda,\epsilon)(\{1,\ldots,g\}^c) = \det G_{33}(0) + Q_{1,\ldots,g}(\lambda,\epsilon)\,.$$

From now on, it suffices to repeat the arguments in [11, pp. 799-800] by taking into account that in this case, $\det G_{33}(0)\neq 0$ because $W_2M(0)V_2$ is nonsingular and, so, the point $(g,0)$ appears in the Newton Polygon of $\widetilde{f}(\lambda,\epsilon)$. This means that there are g eigenvalue expansions near $\lambda_0=0$. On the other hand, Φ nonsingular implies that also $(0,g)$ is in the Newton Polygon of $\widetilde{f}(\lambda,\epsilon)$, so there is a line segment (whose extremal points are $(g,0)$ and $(0,g)$), with slope equal to -1 and horizontal length equal to g. This implies the existence of exactly g expansions with leading exponent equal to 1 and whose leading coefficients are the ones described in the statement.

The expansion (20) in the case $g=1$ follows from (18). The only point to justify is why $o(\epsilon)$ is replaced by $O(\epsilon^2)$. This follows from the fact that the polynomial (12) has only one root equal to zero, and so the corresponding root of $p_\epsilon(\lambda)$ is analytic in ϵ [9]. □

The leading term of the eigenvalue expansion (20) for simple eigenvalues generalizes the already known expression for simple eigenvalues of regular matrix polynomials [15, p. 345], i.e.,

$$\lambda(\epsilon)=\lambda_0-\epsilon\frac{wM(\lambda_0)v}{wP'(\lambda_0)v}+O(\epsilon^2)\,,$$

where w and v are, respectively, the left and right eigenvectors associated with λ_0.

Let us illustrate the application of Theorem 2 with an example.

Example 3 *We continue with* Example 2. *Here* $W_2M(0)V_2=1$ *is nonsingular, or, equivalently, the pencil* $\mathcal{P}(\zeta)$ *is regular and has only one finite eigenvalue* $\zeta_1=-1$. *This means that there is an unique eigenvalue of* $P(\lambda)+\epsilon M(\lambda)$ *approaching* $\lambda_0=0$ *of the form*

$$\lambda_1(\epsilon)=-\epsilon+O(\epsilon^2)\,.$$

To ascertain the quality of this approximation, we have computed the eigenvalues of the pencil $P(\lambda)+\epsilon M(\lambda)$, *for* $\epsilon=10^{-4}$, 10^{-6}, 10^{-8} *and* 10^{-10}, *solving the polynomial equation* $\det(P(\lambda)+\epsilon M(\lambda))=0$ *in the variable precision arithmetic of MATLAB with* 64 *decimal digits of precision, and rounding the results to* $4,6,8,$ *and* 10 *digits respectively. The root* $\lambda_1(\epsilon)$ *closest to zero in each of these cases is*

$\epsilon=10^{-4}$	$\epsilon=10^{-6}$	$\epsilon=10^{-8}$	$\epsilon=10^{-10}$
$-0.9994\cdot 10^{-4}$	$-0.999994\cdot 10^{-6}$	$-0.99999994\cdot 10^{-8}$	$-0.9999999994\cdot 10^{-10}$

The main assumption and the expansions in Theorem 2 depend on bases of the left and right singular spaces of $P(\lambda)$ at λ_0. This is an important drawback, because these bases may be difficult to calculate since they are defined through bases of certain vector spaces over the field of rational functions $\mathbb{C}(\lambda)$. Fortunately this can be avoided and, based on Lemma 6 and Theorem 2, it is possible to provide sufficient conditions for the existence of expansions near λ_0 and give also an expression for the leading coefficient using any bases of the null spaces of the matrix $P(\lambda_0)$, that can be computed with classical procedures of Numerical Linear Algebra. This is presented in Theorem 3 that is the most useful result in this work.

Theorem 3 *Let* $P(\lambda)$ *be an arbitrary* $n\times n$ *matrix polynomial (singular or not),* $M(\lambda)$ *be another polynomial with the same dimension, and* λ_0 *be a finite semisimple eigenvalue of* $P(\lambda)$ *with geometric multiplicity* g. *Denote by* W *a matrix whose rows form any basis of* $\mathcal{N}_T(P(\lambda_0))$ *and by* V *a matrix whose columns form any basis of* $\mathcal{N}(P(\lambda_0))$. *Then*

1. *The pencil* $WM(\lambda_0)V+\zeta WP'(\lambda_0)V$ *is generically regular and has exactly* g *finite eigenvalues, i.e., this holds for all matrix polynomials* $M(\lambda)$ *whose degree is less than or equal the degree of* $P(\lambda)$ *except those in an algebraic manifold of positive codimension.*

2. *If the pencil* $WM(\lambda_0)V + \zeta\, WP'(\lambda_0)V$ *is regular and has exactly* g *finite eigenvalues equal to* $\zeta_1, \ldots, \zeta_g$*, then there are exactly* g *eigenvalues of* $P(\lambda) + \epsilon M(\lambda)$ *such that*

$$\lambda_j(\epsilon) = \lambda_0 + \zeta_j\, \epsilon + o(\epsilon), \qquad j = 1, \ldots, g\,, \tag{26}$$

as ϵ *tends to zero. If* $g = 1$*, i.e.,* λ_0 *is a simple eigenvalue, then* $o(\epsilon)$ *can be replaced by* $O(\epsilon^2)$ *in the previous expansions.*

Proof. In the first place, notice that the eigenvalues and the regularity of the pencil $WM(\lambda_0)V + \zeta\, WP'(\lambda_0)V$ are independent on the bases W and V of the left and right null spaces of $P(\lambda_0)$, because any change of bases simply transforms the pencil into a strictly equivalent pencil. Therefore, we can choose $W = \left[\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right]$ and $V = \left[\begin{array}{cc} V_1 & V_2 \end{array}\right]$ be bases as the ones in Theorem 2. With this choice $WM(\lambda_0)V + \zeta\, WP'(\lambda_0)V$ is precisely $\mathcal{P}(\zeta)$ in (17). Lemma 6 states that $W_2 M(\lambda_0) V_2$ is nonsingular if and only if $WM(\lambda_0)V + \zeta\, WP'(\lambda_0)V$ is regular with exactly g finite eigenvalues. On the other hand, the condition of $W_2 M(\lambda_0) V_2$ being nonsingular is generic because $\det(W_2 M(\lambda_0) V_2)$ is a multivariate polynomial in the entries of the coefficients of $M(\lambda)$, so the first item is proved. The second item is an immediate consequence of Theorem 2 and Lemma 6. □

It is worth to compare Theorem 3 with the first paragraph of the statement of Theorem 6 in [10]. In both cases, the first order term of the eigenvalue expansions around a semisimple eigenvalue are determined, and in both cases this is done through the eigenvalues of a certain matrix pencil constructed in a similar way. The main difference is that in [10], the perturbed matrix functions $L(\lambda, \epsilon)$ are regular in $\epsilon = 0$ and in a neighborhood of this value (more restrictive than in Theorem 3) and analytic (not necessarily polynomials, and so, more general than our Theorem 3). With the notation in [10], the pencil whose eigenvalues determine the first order coefficients of the eigenvalue expansions near λ_0 is $\mathcal{P}(\zeta) = W(\zeta \frac{\partial L}{\partial \lambda}(\lambda_0, 0) + \frac{\partial L}{\partial \epsilon}(\lambda_0, 0))V$, where W and V are bases of, respectively, $\mathcal{N}_T(L(\lambda_0, 0))$ and $\mathcal{N}(L(\lambda_0, 0))$.

5 Approximate eigenvectors for semisimple eigenvalues

This section is closely related to [3, Section 6], and the reader is referred to this reference for some technical details that are omitted here. We consider in this section only right eigenvectors. Counterpart results for left eigenvectors can be established in a similar way.

Eigenvectors are not well defined in singular matrix polynomials [2, p. 145], [3]. However, for $\epsilon \neq 0$, the perturbed polynomial $P(\lambda) + \epsilon M(\lambda)$ is generically regular, has simple eigenvalues, well defined associated eigenvectors, and, given a semisimple eigenvalue λ_0 of $P(\lambda)$ with geometric multiplicity g, Theorem 3 guarantees the existence of exactly g eigenvalue expansions of $P(\lambda) + \epsilon M(\lambda)$ near λ_0 for most perturbations $M(\lambda)$. The (right) eigenvectors, $v_j(\epsilon)$, $j = 1, \ldots, g$, associated to these g eigenvalues can expanded as power series of ϵ [3, Lemma 7]. We will determine, under generic perturbations, $\lim_{\epsilon \to 0} v_j(\epsilon)$, $j = 1, \ldots, g$, and, as a consequence, we will see that these limits belongs to $\mathcal{N}(P(\lambda_0))$. This is presented in Theorem 4, an analog, for singular matrix polynomials, of the second part of Theorem 6 in [10].

Theorem 4 *Let* $P(\lambda)$ *be an arbitrary* $n \times n$ *matrix polynomial (singular or not),* $M(\lambda)$ *be another polynomial with the same dimension, and* λ_0 *be a finite semisimple eigenvalue of* $P(\lambda)$ *with geometric multiplicity* g*. Denote by* W *a matrix whose rows form any basis of* $\mathcal{N}_T(P(\lambda_0))$ *and by* V *a matrix whose columns form any basis of* $\mathcal{N}(P(\lambda_0))$*. Let us assume that the pencil*

$$WM(\lambda_0)V + \zeta\, WP'(\lambda_0)V$$

is regular, and has exactly g *finite eigenvalues* $\zeta_1, \ldots, \zeta_g$ *different from zero, such that* $\zeta_i \neq \zeta_j$ *if* $i \neq j$*, with associated right eigenvectors* $c_1, \ldots, c_g$*. Then in a punctured neighborhood* $0 < |\epsilon| < b$ *the eigenvectors* $v_1(\epsilon), \ldots, v_g(\epsilon)$ *of* $P(\lambda) + \epsilon M(\lambda)$ *corresponding to the eigenvalues* (26) *satisfy*

$$v_j(\epsilon) = V\, c_j + O(\epsilon), \quad j = 1, \ldots, g.$$

Proof. As in the proof of Theorem 3 the result is independent on the bases W and V. Therefore, we can choose $W = \left[\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right]$ and $V = \left[\begin{array}{cc} V_1 & V_2 \end{array}\right]$ be the bases in Lemma 2, that also satisfy Lemma 3. Observe that the assumptions of Theorem 4 guarantee that the matrices $W_2 M(\lambda_0) V_2$ and Φ defined in (16) are nonsingular by Lemma 6. For each eigenvalue $\lambda_j(\epsilon)$ in (26), we consider, for $\epsilon \neq 0$, the corresponding eigenvector $v_j(\epsilon)$. It can be shown as in [3, Lemma 7] that this eigenvector is analytic at $\epsilon = 0$, so we can write $v_j(\epsilon) = v_j + \sum_{k=1}^{\infty} u_{jk}\epsilon^k$. Our task is to determine v_j.

For simplicity, we take $\lambda_0 = 0$ as in the proof of Theorem 2. Again the proof is based on the local Smith form (7), which is well defined and analytic in a neighborhood of $\lambda_0 = 0$. To take advantage of this local Smith form we replace $v_j(\epsilon)$ with

$$z_j(\epsilon) = V(\lambda_j(\epsilon))^{-1} v_j(\epsilon), \tag{27}$$

which satisfies

$$[\Delta(\lambda_j(\epsilon)) + \epsilon \widetilde{M}(\lambda_j(\epsilon))]\, z_j(\epsilon) = 0, \tag{28}$$

where

$$\widetilde{M}(\lambda_j(\epsilon)) = W(\lambda_j(\epsilon))\, M(\lambda_j(\epsilon))\, V(\lambda_j(\epsilon)).$$

Notice that one can easily recover $v_j = v_j(0)$ from $z_j(0)$, since $v_j(0) = V(0) z_j(0)$. We partition $\widetilde{M}(\lambda_j(\epsilon))$ as a 3×3 block matrix according to the three diagonal blocks of $\Delta(\lambda)$ specified in partition (7), and denote, as in the proof of Theorem 2, $[G_{ik}(\lambda_j(\epsilon))]_{i,k=1}^{3} \equiv \widetilde{M}(\lambda_j(\epsilon))$. The vector $z_j(\epsilon)$ is partitioned accordingly, and (28) can be written as

$$\left(\begin{bmatrix} \lambda_j(\epsilon)\, I_g & & \\ & I & \\ & & 0_{d\times d} \end{bmatrix} + \epsilon \begin{bmatrix} G_{11}(\lambda_j(\epsilon)) & G_{12}(\lambda_j(\epsilon)) & G_{13}(\lambda_j(\epsilon)) \\ G_{21}(\lambda_j(\epsilon)) & G_{22}(\lambda_j(\epsilon)) & G_{23}(\lambda_j(\epsilon)) \\ G_{31}(\lambda_j(\epsilon)) & G_{32}(\lambda_j(\epsilon)) & G_{33}(\lambda_j(\epsilon)) \end{bmatrix} \right) \begin{bmatrix} z_j^{(1)}(\epsilon) \\ z_j^{(2)}(\epsilon) \\ z_j^{(3)}(\epsilon) \end{bmatrix} = 0. \tag{29}$$

For $\epsilon = 0$ this equation reduces to $z_j^{(2)}(0) = 0$. The rows corresponding to the first and third rows of blocks are

$$\begin{aligned} \lambda_j(\epsilon)\, z_j^{(1)}(\epsilon) + \epsilon(G_{11}(\lambda_j(\epsilon))\, z_j^{(1)}(\epsilon) + G_{12}(\lambda_j(\epsilon))\, z_j^{(2)}(\epsilon) + G_{13}(\lambda_j(\epsilon))\, z_j^{(3)}(\epsilon)) &= 0 \quad (30) \\ G_{31}(\lambda_j(\epsilon))\, z_j^{(1)}(\epsilon) + G_{32}(\lambda_j(\epsilon))\, z_j^{(2)}(\epsilon) + G_{33}(\lambda_j(\epsilon))\, z_j^{(3)}(\epsilon) &= 0. \quad (31) \end{aligned}$$

Notice that the terms of lower order in ϵ of $\lambda_j(\epsilon)$ are of the form $\zeta_j\,\epsilon$, for $j = 1, \ldots, g$, with $\zeta_j \neq 0$. So we can divide (30) and (31) by ϵ and take the limit $\epsilon \to 0$ to obtain (see (22))

$$\left(\zeta_j \begin{bmatrix} I_g & 0 \\ 0 & 0 \end{bmatrix} + \Phi \right) \begin{bmatrix} z_j^{(1)}(0) \\ z_j^{(3)}(0) \end{bmatrix} = 0.$$

The result now follows from (27). □

6 Expansions for arbitrary eigenvalues

In this last section, we consider the first order terms of the expansions around arbitrary eigenvalues, i.e., the eigenvalue λ_0 of $P(\lambda)$ is not necessarily semisimple. In this general case it is not possible to prove a counterpart of Theorem 3 valid for any bases of $\mathcal{N}_T(P(\lambda_0))$ and $\mathcal{N}(P(\lambda_0))$, and we cannot avoid the use of specific bases of these subspaces that are defined through certain vectors with entries in the field of rational functions $\mathbb{C}(\lambda)$. As a consequence, the first order terms that we will obtain are very difficult to compute in practice. Additional notation has to be introduced before stating Theorem 5, the main result in this section.

To start with, recall that Theorem 1 is valid for arbitrary eigenvalues, therefore, for generic perturbations $M(\lambda)$, there exist (fractional) power expansions of the eigenvalues of $P(\lambda) + \epsilon M(\lambda)$ near any eigenvalue λ_0 of $P(\lambda)$. We will use again the local Smith form of

$P(\lambda)$ at λ_0 given by (6), and we rename the degrees of the elementary divisors associated with λ_0 as

$$\{\underbrace{n_1, \ldots, n_1}_{r_1}, \ldots, \underbrace{n_q, \ldots, n_q}_{r_q}\} \equiv \{m_1, \ldots, m_g\}, \tag{32}$$

where we assume that,

$$0 < n_1 < n_2 < \cdots < n_q. \tag{33}$$

The natural numbers $n_1, n_2, \ldots, n_q$ are sometimes called the *partial multiplicities* of λ_0. Note that the algebraic and geometric multiplicities of λ_0 are given, respectively, by

$$a = \sum_{i=1}^{q} r_i\, n_i \quad \text{and} \quad g = \sum_{i=1}^{q} r_i.$$

Let us define the sequence

$$f_j = \sum_{i=j}^{q} r_i, \quad j = 1, \ldots, q, \quad \text{and} \quad f_{q+1} = 0,$$

so $f_1 = g$. We consider also the following submatrices of the matrices $W(\lambda_0)$ and $V(\lambda_0)$ in (5):

$$W_{1j} = (W(\lambda_0))(g - f_j + 1 : g, :) \quad \text{and} \quad V_{1j} = (V(\lambda_0))(:, g - f_j + 1 : g), \text{ for } j = 1, \ldots, q,$$
$$W_2 = (W(\lambda_0))(n - d + 1 : n, :) \quad \text{and} \quad V_2 = (V(\lambda_0))(:, n - d + 1 : n),$$

where we use MATLAB's notation for submatrices. According to the notation in (15) and Lemma 2, observe that $W_{11} = W_1$ and $V_{11} = V_1$. Note also that the rows of $[W_1^T\ W_2^T]^T$ (resp. the columns of $[V_1\ V_2]$) form a very specific basis of $\mathcal{N}_T(P(\lambda_0))$ (resp. of $\mathcal{N}(P(\lambda_0))$). Now, we can build up the matrices

$$\Phi_j = \begin{bmatrix} W_{1j} \\ W_2 \end{bmatrix} M(\lambda_0) \begin{bmatrix} V_{1j} & V_2 \end{bmatrix}, \quad j = 1, \ldots, q, \quad \text{and} \quad \Phi_{q+1} = W_2 M(\lambda_0) V_2. \tag{34}$$

The matrix Φ_1 coincides with Φ defined in (16), therefore Φ_j is the $(f_j + d) \times (f_j + d)$ lower right principal submatrix of Φ. Finally, we define

$$E_j = \operatorname{diag}(I_{r_j}, 0_{(f_{j+1}+d)\times(f_{j+1}+d)}), \quad j = 1, \ldots, q. \tag{35}$$

Now, we are in the position to state the main result of this section, which is the generalization to matrix polynomials of [3, Theorem 2], that is only valid for pencils.

Theorem 5 *Let $P(\lambda)$ be an arbitrary $n \times n$ matrix polynomial (singular or not), and $M(\lambda)$ another polynomial with the same dimension. Let λ_0 be a finite eigenvalue of $P(\lambda)$ such that the degrees of the elementary divisors associated with λ_0 satisfy* (32) *and* (33). *Let Φ_j, $j = 1, \ldots, q+1$, and E_j, $j = 1, \ldots, q$, be the matrices defined in* (34) *and* (35). *If* $\det \Phi_{j+1} \neq 0$ *for some $j \in \{1, 2, ..., q\}$, let $\xi_1, \ldots, \xi_{r_j}$ be the r_j finite eigenvalues of the pencil $\Phi_j + \zeta E_j$, and $(\xi_t)_s^{1/n_j}$, $s = 1, \ldots, n_j$, be the n_j determinations of the n_jth root. Then, in a neighborhood of $\epsilon = 0$, the polynomial $P(\lambda) + \epsilon M(\lambda)$ has $r_j n_j$ eigenvalues satisfying*

$$\lambda_j^{rs}(\epsilon) = \lambda_0 + (\xi_t)_s^{1/n_j} \epsilon^{1/n_j} + o(\epsilon^{1/n_j}), \quad t = 1, 2, ..., r_j, \quad s = 1, 2, ..., n_j, \tag{36}$$

where ϵ^{1/n_j} is the principal determination of the n_jth root of ϵ. Moreover, the polynomial $P(\lambda) + \epsilon M(\lambda)$ is regular in the same neighborhood for $\epsilon \neq 0$. If, in addition, $\det \Phi_j \neq 0$, *then all ξ_t in* (36) *are nonzero, and* (36) *are all the expansions near λ_0 with leading exponent $1/n_j$.*

Proof. The proof follows closely the one of [3, Theorem 2]. We omit the details for the sake of brevity. □

Theorem 5, as well as Theorems 2 and 3, can be adapted to cover the perturbation expansions of infinite eigenvalues by considering the dual polynomials, see [3, Corollary 1] for more details.

References

[1] J-P. Dedieu, F. Tisseur, Perturbation theory for homogeneus polynomial eigenvalue problems, Linear Algebra Appl., 358 (2003) 71–94.

[2] J. W. Demmel, B. Kågström, Computing stable eigendecompositions of Matrix Pencils, Linear Algebra Appl., 88/89 (1987) 139–186.

[3] F. De Terán, F. M. Dopico, J. Moro, First order spectral perturbation theory of square singular matrix pencils, Linear Algebra Appl., 429 (2008) 548–576.

[4] G. D. Forney, Minimal bases of rational vector spaces, with applications to multivariable linear systems, SIAM J. Control, 13 (1975) 493–520.

[5] F. R. Gantmacher, The Theory of Matrices, AMS Chelsea, Providence, RI, 1998.

[6] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials , Academic Press, New York, 1982.

[7] N. J. Higham, R-C. Li, F. Tisseur, Backward error of polynomial eigenproblems solved by linearization, SIAM J. Matrix Anal. Appl., 29, no. 4 (2007) 1218–1241.

[8] N. J. Higham, F. Tisseur, Bounds for eigenvalues of matrix polynomials, Linear Algebra Appl., 358 (2003) 5–22.

[9] E. Hille, Analytic Function Theory, Vol. II, 2nd ed., Chelsea, New York, 1987.

[10] P. Lancaster, A. S. Markus, F. Zhou, Perturbation theory for analytic matrix functions: the semisimple case , SIAM J. Matrix Anal. Appl., 25 (2003) 606-626.

[11] H. Langer, B. Najman, Remarks on the perturbation of analytic matrix functions III, Integr. Equat. Oper. Th., 15 (1992) 796–806.

[12] G. W. Stewart, Perturbation theory for rectangular matrix pencils, Linear Algebra Appl., 208/209 (1994) 297–301.

[13] J. G. Sun, Orthogonal projections and the perturbation of the eigenvalues of singular pencils, J. Comp. Math., 1 (1983) 63–74.

[14] J. G. Sun, Gerschgorin type theorem and the perturbation of the eigenvalues of a singular pencil, Chinese J. Num. Math. Appl., 10:1 (1988) 1–13. (Math. Num. Sin., 7:3 (1985) 253–264)

[15] F. Tisseur, Backward error and condition of polynomial eigenvalue problems, Linear Algebra Appl., 309 (2000) 339–361.

[16] P. Van Dooren, Reducing subspaces: Definitions, properties, and algorithms, in: B. Kågström and A. Ruhe (Eds.), Matrix Pencils, Lecture Notes in Mathematics, Vol. 973, Springer-Verlag, Berlin, 1983, pp. 58–73.

[17] P. Van Dooren, P. Dewilde, The eigenstructure of an arbitrary polynomial matrix: Computational aspects, Linear Algebra Appl., 50 (1983) 545–579.

[18] G. Verghese, P. Van Dooren, T. Kailath, Properties of the system matrix of a generalized state-space system, Int. J. Control, 30, no. 2 (1979) 235–243.