Constructing strong ℓ-ifications from dual minimal bases[☆]

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Abstract

We provide an algorithm for constructing strong ℓ-ifications of a given matrix polynomial $P(\lambda)$ of degree d and size $m \times n$ using only the coefficients of the polynomial and the solution of linear systems of equations. A strong ℓ-ification of $P(\lambda)$ is a matrix polynomial of degree ℓ having the same finite and infinite elementary divisors, and the same numbers of left and right minimal indices as the original matrix polynomial $P(\lambda)$. All explicit constructions of strong ℓ-ifications introduced so far in the literature have been limited to the case where ℓ divides d, though recent results on the inverse eigenstructure problem for matrix polynomials show that more general constructions are possible. Based on recent developments on dual polynomial minimal bases, we present a general construction of strong ℓ-ifications for wider choices of the degree ℓ, namely, when ℓ divides one of nd or md (and $d \geq \ell$). In the case where ℓ divides nd (respectively, md), the strong ℓ-ifications we construct allow us to easily recover the minimal indices of $P(\lambda)$. In particular, we show that they preserve the left (resp., right) minimal indices of $P(\lambda)$, and the right (resp., left) minimal indices of the ℓ-ification are the ones of $P(\lambda)$ increased by $d-\ell$ (each). Moreover, in the particular case ℓ divides d, the new method provides a companion ℓ-ification that resembles very much the companion ℓ-ifications already known in the literature.

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1. Introduction

In the literature of linear time-invariant dynamical systems, matrix polynomials were already used in the 1960's but the monograph of Rosenbrock [24] gave in 1970 for the first time a systematic treatment of the fundamental role played by matrix polynomial models in the study and solution of dynamical systems. Rosenbrock showed that any $p \times m$ rational transfer matrix $R(\lambda)$ (where λ is the Laplace variable) of a dynamical system relating an input vector $u(\lambda)$ of dimension m to an output vector $y(\lambda)$ of dimension p can be represented by a set of matrix polynomial equations when introducing a so-called internal state vector $\xi(\lambda)$ of dimension n :

$$y(\lambda) = R(\lambda)u(\lambda) \quad \Longleftrightarrow \begin{bmatrix} 0 \\ y(\lambda) \end{bmatrix} = \begin{bmatrix} T(\lambda) & U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix} \begin{bmatrix} \xi(\lambda) \\ u(\lambda) \end{bmatrix},$$

which is solvable when the $n \times n$ matrix polynomial $T(\lambda)$ is invertible. The elimination of the internal state $\xi(\lambda)$ then also shows the relation between the transfer function $R(\lambda)$ and the quadruple $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$:

$$R(\lambda) = W(\lambda) - V(\lambda)T(\lambda)^{-1}U(\lambda),$$

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which is why this quadruple was also called a *polynomial realization* of the rational matrix $R(\lambda)$. The compound matrix polynomial

$$S(\lambda) := \begin{bmatrix} T(\lambda) & U(\lambda) \\ V(\lambda) & W(\lambda) \end{bmatrix}$$

was called the *system matrix*, and it generalized the notion of state-space realizations studied by Kalman [20], where the system matrix $S(\lambda)$ was linear (i.e. a first degree matrix polynomial), to arbitrary degree polynomial models. It was also shown in [24] that the Smith-McMillan zeros of the transfer function $R(\lambda)$ were the same as the Smith zeros of the system matrix $S(\lambda)$, provided that the matrix polynomial pairs

$$\begin{bmatrix} T(\lambda) & U(\lambda) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T(\lambda) \\ V(\lambda) \end{bmatrix} \tag{1}$$

have, respectively, full row rank and full column rank n, for all λ. It was shown later on that not only the finite zero structure of $R(\lambda)$ is preserved in the polynomial system matrix $S(\lambda)$, but also its infinite zero structure, as well as its left and right null space structure, provided that the rank conditions on the matrices (1) are made more stringent [12], [25]. It was shown, e.g. in [26], that under particular rank conditions there always exist generalized state space realizations of any rational matrix $R(\lambda)$ such that the right and left null space structure of $R(\lambda)$ and $S(\lambda)$ are the same and that their finite and infinite zero structures are also the same, as long as one uses the MacMillan definition for the zero structure at infinity.

A second influential book on matrix polynomials is the one by Gohberg, Lancaster and Rodman [15]. In this book, the structure of a matrix polynomial $P(\lambda)$ of an arbitrary degree d is revisited and much attention is paid to the problem of *linearization*, which is a first degree matrix polynomial (in other words, a *pencil*) that has the same zero structure as $P(\lambda)$ in its finite zeros (i.e., the same finite eigenstructure). But the authors also introduce a new notion of infinite eigenstructure, which differs from the definition of MacMillan. This new definition is by now well accepted and has lead to the notion of *strong linearization*, which was introduced in [14] and named later in [21]. Strong linearizations are those linearizations that preserve also the infinite eigenstructure of the polynomial. Later on that notion was also extended to matrix polynomials of arbitrary degree ℓ, named *strong ℓ-ifications* in [8]. Such constructions have so far been limited to degrees ℓ that divide d, even though recent results [10] on the inverse eigenstructure problem for matrix polynomials show that more general constructions are possible. In particular, it has been proven in [10] that any singular matrix polynomial has a strong ℓ-ification for any positive integer ℓ, while, if $1 \leq \ell \leq d$, regular $n \times n$ matrix polynomials have strong ℓ-ifications if and only if ℓ divides nd.

Constructing strong ℓ-ifications is not merely a theoretical issue. The main interest of these constructions is in potential applications, mainly related to the *Polynomial Eigenvalue Problem* (PEP) or, more in general, in the problem of computing the complete eigenstructure of a general matrix polynomial of arbitrary degree d. The PEP consists of computing the eigenvalues and eigenvectors of matrix polynomials. The usual approach to this problem is through the use of (strong) linearizations and, in particular, through the use of the classical Frobenius companion forms. For an arbitrary matrix polynomial, there are infinitely many strong linearizations, which can be easily constructed from the eigenstructure of the polynomial [3]. However, in the context of the PEP, it is important to derive general constructions, valid for all matrix polynomials, that do not depend on the previous knowledge of the eigenstructure, which is precisely the information to be computed.

In the past few years, considerable effort has been devoted to construct collections of *companion linearizations*, see for instance [1, 2, 5, 6, 7, 8, 27] among many other references. More in general, *companion forms* are uniform templates for building strong ℓ-ifications of matrix polynomials $P(\lambda)$ directly from the coefficients of $P(\lambda)$ without involving any other operations than multiplications by fixed constants, and which are valid for every $P(\lambda)$ (see [8, Def. 5.1] for the precise definition). A particular example of companion linearizations are the classical *Frobenius (companion) linearizations* mentioned above [15]. Though the use of companion linearizations in the PEP is widely extended and produces, in general, good results, linearizations can increase significantly the size of the problem, modify the conditioning of the problem, and loose the original structure. For instance, the linearizations of square matrix polynomials that are commonly used in practice (in particular, the classical Frobenius companion linearizations), as well as all *companion* linearizations (see [8]) have size $nd \times nd$, instead of the size $n \times n$ of the original polynomial of degree d. By contrast, the size of companion ℓ-ifications of a matrix polynomial with degree d and size $n \times n$ introduced in [8, Cor. 7.10] for the case where ℓ divides d is $n(d/\ell) \times n(d/\ell)$. Something similar occurs with the known companion linearizations and ℓ-ifications for rectangular matrix polynomials [7, 8].

For the strong ℓ-ifications to be useful in a practical setting, like the PEP, one would need to have suitable algorithms for computing the eigenstructure of low-degree matrix polynomials that work directly on the polynomial, without the use of linearizations. Though there are not too many references in this direction, algorithms of this kind have been proposed in the literature, in particular for quadratic palindromic matrix polynomials, as well as for even-degree palindromic polynomials using an appropriate quadratification [17, 18]. More in general, to recover the whole eigenstructure of the polynomial and, in particular, the singular structure, one should be able to relate the minimal indices of the ℓ-ifications with the corresponding minimal indices of the polynomial. In the few ℓ-ifications known so far in the literature [8], this relationship is obtained though very simple recovery formulas.

In this paper, we present a general construction for strong ℓ-ifications of arbitrary matrix polynomials of degree d and size $m \times n$ (that is, including rectangular polynomials) for the case when ℓ divides one of nd or md. This construction is general in two senses. First, it is the same for all matrix polynomials of a given size $m \times n$ and degree d. Second, the condition ℓ divides nd or md is the most general condition for which a given construction can provide strong ℓ-ifications for all matrix polynomials with such size and degree, since this is the only case where strong ℓ-ifications can exist for regular matrix polynomials (see, for instance, [8, Th. 7.5] and [10, Th. 4.7, Cor. 4.9]). In the particular case of square $n \times n$ matrix polynomials, the construction has size $(nd)/\ell \times (nd)/\ell$, which is the size of any companion ℓ-ification for matrix polynomials of degree d and size $n \times n$ (see [8, Cor. 7.10] and [10, Cor. 4.9]). We want to emphasize that our construction is not exactly a companion form, but a more general construction that contains, as particular cases, some companion ℓ-ifications which are essentially the ones that have been recently introduced in [8] (see Section 4.4). Moreover, this construction also allows to easily recover the minimal indices of the matrix polynomial. In particular, if ℓ divides nd then our strong ℓ-ification has the same left minimal indices as the matrix polynomial, and the right minimal indices are obtained from the ones of the polynomial by adding a constant shift equal to $d - \ell$. In the case where ℓ divides md, the situation is exactly the same after exchanging the roles of the left and right minimal indices. It is worth to emphasize that in [10], it has been proved that there always exist a strong ℓ-ification of a given singular matrix polynomial for any degree ℓ [10, Remark 4.11], and all possible sizes of strong ℓ-ifications, together with the possible values of their left and right minimal indices have been described [10, Th. 4.10]. Also, necessary and sufficient conditions for the existence of strong ℓ-ifications of regular matrix polynomials have been provided, and it has been shown that, when they exist, they all have the same size, namely $(dn)/\ell \times (dn)/\ell$ [10, Th. 4.7]. Hence, the problem of the complete characterization of the existence, size, and minimal indices of strong ℓ-ifications has been already solved in [10], but without constructing such ℓ-ifications. By contrast, the main contribution of the present paper is to provide an explicit construction of strong ℓ-ifications of arbitrary matrix polynomials.

Our construction strongly relies on the notion of *dual minimal bases* [11]. It is worth to emphasize that some matrix polynomials which are particular pairs of dual minimal bases have been already used in the construction of the Frobenius-like companion strong ℓ-ifications in [8, Section 5.2], valid only when ℓ divides d, and even before in the classical Frobenius linearizations [8, Section 5.1], although the fact that they are minimal bases has not received attention so far. The key contribution of this work is to use much more general pairs of dual minimal bases to construct, from the coefficients of the matrix polynomial $P(\lambda)$, many new strong ℓ-ifications in a process that generalizes significantly the developments in [8, Sections 5.1 and 5.2]. More precisely, a pair of dual minimal bases $\widehat{L}(\lambda), \widehat{N}(\lambda)$, with all degrees of the rows of $\widehat{L}(\lambda)$ (resp. $\widehat{N}(\lambda)$) being equal to ℓ (resp. to $d - \ell$), will be the skeleton of our construction. The matrix polynomial $\widehat{L}(\lambda)$ will be a submatrix of the desired ℓ-ification $L(\lambda)$ of $P(\lambda)$, while $\widehat{N}(\lambda)$ is used to build up the unimodular transformations that take $L(\lambda)$ into $\operatorname{diag}(I, P(\lambda))$, and to construct the remaining part of $L(\lambda)$ as the solution of a linear system of equations. Observe that our results can be seen as a new application of minimal bases to be added to the classical ones mentioned in [11].

The paper is organized as follows. In Section 2 we set the basic notation and recall the basic definitions and previous results used in the paper (in particular, the notion of strong ℓ-ification). In Section 3 we recall the notion of minimal basis, as well as the notion of dual minimal bases, and we state some basic results about them that will be needed later. In particular, we recall the zigzag construction introduced recently in [9]. Section 4 is the central section of the paper, and is devoted to explain the general construction we propose of strong ℓ-ifications. Finally, in Section 5 we summarize the main contributions of the paper.

2. Basic definitions and notation

Along the paper we use the following notation. Given an arbitrary field $\mathbb{F}$, we denote by $\overline{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$. By $\mathbb{F}[\lambda]^{m\times n}$ we denote the set of $m \times n$ matrix polynomials in the variable λ over the field $\mathbb{F}$, and $\mathbb{F}(\lambda)^n$ denotes the vector space with n coordinates in the field of rational functions in λ (this field is denoted by $\mathbb{F}(\lambda)$). The matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ has *degree* d if

$$P(\lambda) = \lambda^d P_d + \lambda^{d-1} P_{d-1} + \cdots + \lambda P_1 + P_0, \tag{2}$$

with $P_i \in \mathbb{F}^{m\times n}$, for $i = 0, 1, \ldots, d$, and $P_d \neq 0$. The *reversal* of the matrix polynomial $P(\lambda)$ in (2) is

$$\operatorname{rev}P(\lambda) := \lambda^d P(1/\lambda) = \lambda^d P_0 + \lambda^{d-1} P_1 + \cdots + \lambda P_{d-1} + P_d.$$

Note that, if $P_0 \neq 0$, then $\operatorname{rev}P(\lambda)$ is again a matrix polynomial of degree d. Otherwise, $\operatorname{rev}P(\lambda)$ has degree smaller than d. By $P(\lambda)^T$ we denote the transpose of $P(\lambda)$.

The *normal rank* of the matrix polynomial $P(\lambda)$ is the size of the largest non-identically zero minor of $P(\lambda)$, in other words, the rank of $P(\lambda)$ when considered as a matrix with entries in the field $\mathbb{F}(\lambda)$.

For the *eigenstructure* of a matrix polynomial $P(\lambda)$ we follow the same definition as in [10, Def. 2.17]. We recall that it consists of the *regular eigenstructure*, that comprises both the *finite structure* (invariant polynomials of $P(\lambda)$), and the *infinite structure* (partial multiplicity sequence at ∞ of $P(\lambda)$), together with the *singular structure* (left and right minimal indices of $P(\lambda)$). The *partial multiplicity sequence of* $P(\lambda)$ *at* $\lambda_0 \in \overline{\mathbb{F}}$ is the sequence containing the exponents of the factors $\lambda - \lambda_0$ in the invariant polynomials of $P(\lambda)$ (see [10, Def. 2.3]). The *partial multiplicity sequence at* ∞ is the partial multiplicity sequence at zero of $\operatorname{rev}P(\lambda)$, as in [10, Def. 2.5]. For those readers familiarized with the definitions from linear systems theory, we want to note that the infinite structure should not be confused with the *structural indices at* ∞, as defined, for instance, in [19], though both notions are related (see [10, Remark 2.8]). To illustrate these notions, and for the sake of a better understanding of the proof of Theorem 4.5, we include the following example.

Example 2.1. Let $P(\lambda)$ be the following 7×7 singular quadratic matrix polynomial

$$P(\lambda) = \begin{bmatrix} \lambda^2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Smith normal form of $P(\lambda)$ is $\Delta(\lambda) = \operatorname{diag}(1, 1, 1, 1, \lambda, \lambda^4, 0)$, so, in particular. the normal rank of $P(\lambda)$ is 6. Moreover, the nontrivial invariant polynomials of $P(\lambda)$ are $p_1(\lambda) = \lambda, p_2(\lambda) = \lambda^4$. Notice that $\lambda = 0$ is the only finite eigenvalue of $P(\lambda)$, and the partial multiplicity sequence at $\lambda = 0$ is $(0, 0, 0, 0, 1, 4)$, whose length is equal to the normal rank of the polynomial. By looking at the polynomial $\operatorname{rev}P(\lambda)$, we see that the partial multiplicity sequence at ∞ of $P(\lambda)$ is $(0, 0, 0, 1, 2, 2)$. Finally, a simple direct computation shows that $P(\lambda)$ has one right minimal index $\varepsilon_1 = 2$, and another left minimal index $\eta_1 = 0$.

A square matrix polynomial $U(\lambda)$ is said to be *unimodular* if $U(\lambda)$ has constant nonzero determinant. Therefore, the inverse of a unimodular matrix polynomial is again a unimodular matrix polynomial. Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are *equivalent* if there are two unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that $Q(\lambda) = U(\lambda)P(\lambda)V(\lambda)$.

The main notion of this paper is the one of strong ℓ-ification. We reproduce here, for completeness, the definition, introduced in [8, Def. 3.3], but in the form presented in [10, Def. 4.1].

Definition 2.1. (Strong ℓ-ification). A matrix polynomial $L(\lambda)$ of degree $\ell > 0$ is said to be an *ℓ-ification* of a given matrix polynomial $P(\lambda)$ if for some $s, t \geq 0$ there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda) \begin{bmatrix} I_s & \\ & L(\lambda) \end{bmatrix} V(\lambda) = \begin{bmatrix} I_t & \\ & P(\lambda) \end{bmatrix}. \tag{3}$$

If, in addition, $\operatorname{rev} L(\lambda)$ is an ℓ-ification of $\operatorname{rev} P(\lambda)$, then $L(\lambda)$ is said to be a *strong* ℓ-ification of $P(\lambda)$.

The main property of (strong) ℓ-ifications from the applied point of view is that strong ℓ-ifications preserve the regular eigenstructure and part of the singular structure of the matrix polynomial. More precisely, if $L(\lambda)$ is a strong ℓ-ification of $P(\lambda)$, then $L(\lambda)$ and $P(\lambda)$ have the same regular eigenstructure, and the same number of left and right minimal indices [8, Th. 4.1]. Motivated by potential applications of ℓ-ifications to the problem of computing the eigenstructure of matrix polynomials (including the PEP), in this work we are only interested in the case where $\ell < d$, though there may exist strong ℓ-ifications with $\ell \geq d$ [10, Th. 4.7, Th. 4.10].

It is shown in [8, Cor. 4.3] that at least one of s, t in Definition 2.1 may be taken to be zero. As the following example shows, it may happen that $t = 0$.

Example 2.2. Set

$$P(\lambda) = \begin{bmatrix} \lambda^2 & 1 & 0 \\ 0 & 0 & \lambda^2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad L(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & 0 \end{bmatrix}.$$

We have that $\left[\begin{smallmatrix} 1 & \\ & L(\lambda) \end{smallmatrix}\right]$ is equivalent to $P(\lambda)$ and $\left[\begin{smallmatrix} 1 & \\ & \operatorname{rev} L(\lambda) \end{smallmatrix}\right]$ is equivalent to $\operatorname{rev} P(\lambda)$, so $L(\lambda)$ is a strong linearization of $P(\lambda)$.

However, when $P(\lambda)$ is regular, has size $n \times n$ and degree d, and $L(\lambda)$ is a strong ℓ-ification of $P(\lambda)$, then neither $P(\lambda)$ nor $L(\lambda)$ have minimal indices (so that $L(\lambda)$ is regular as well). Then, by the Index Sum Theorem [8, Th. 6.5], we have $nd = \widetilde{n}\ell$, where $L(\lambda)$ has size $\widetilde{n} \times \widetilde{n}$. Since we are assuming $\ell < d$, then we should have $\widetilde{n} > n$, so that $t = 0$ in (3) is not possible.

In order to get constructions valid for both square and rectangular matrix polynomials, including regular and singular ones, we look for ℓ-ifications with size $(\widehat{n} + m) \times (\widehat{n} + n)$ (with $\widehat{n} \geq 0$), so that:

$$M(\lambda) L(\lambda) N(\lambda)^T = \begin{bmatrix} I_{\widehat{n}} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{4}$$

with $M(\lambda), N(\lambda)$ unimodular.

The reason for using $N(\lambda)^T$ in (4), instead of $N(\lambda)$, will become clear later (the matrix $N(\lambda)$ will be constructed from a (row) minimal basis, as introduced in Definition 3.3).

The *row degrees* of a matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ are $d_1, \ldots, d_m$ if the maximum degree of the entries of the ith row of $P(\lambda)$ is d_i, for $i = 1, \ldots, m$. We define similarly the *column degrees* of $P(\lambda)$.

3. Dual minimal bases

In this section we revisit the notions of minimal polynomial bases of rational vector subspaces and their minimal indices. The main purpose of this section is to recall the notion of dual minimal bases, which play a crucial role in our constructive algorithms. For this, we first need to introduce Definitions 3.1, 3.2, and 3.3.

Definition 3.1. Let $N(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ be a matrix polynomial whose row degrees are $d_1, \ldots, d_m$, respectively. Then the *highest row degree coefficient matrix* of $N(\lambda)$, denoted by $N_h \in \mathbb{F}^{m \times n}$, is the matrix whose j-th row is the coefficient of λ^{d_j} in the j-th row of $N(\lambda)$, for $j = 1, \ldots, m$.

Note that if $N(\lambda)$ is a matrix polynomial with all row degrees equal to k, then $N(\lambda) = \sum_{i=0}^{k} \lambda^i N_i$ and $N_h = N_k$. This elementary fact will be often used in Section 4 without mentioning it explicitly.

Definition 3.2. The matrix polynomial $N(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ is *row reduced* if the highest row degree coefficient matrix of $N(\lambda)$ has full row rank.

We can define, analogously, the notions of *highest column degree coefficient matrix* of a matrix polynomial and *column reduced* matrix polynomial, by focusing on columns instead of rows.

Definition 3.3. The $m \times n$ matrix polynomial $N(\lambda)$, with $m \leq n$ (respectively, $m \geq n$) is a *minimal basis*, if it has full row rank m (resp., full column rank n) for all $\lambda \in \overline{\mathbb{F}}$ and it is row (resp., column) reduced.

We illustrate the previous definitions with the following example.

Example 3.1. Let $N(\lambda) \in \mathbb{F}[\lambda]^{2\times 3}$ be the cubic matrix polynomial:

$$N(\lambda) = \begin{bmatrix} \lambda^3 & 1 & \lambda \\ \lambda & 3\lambda^2 + 2 & \lambda + 1 \end{bmatrix}.$$

Then the highest row degree coefficient matrix is

$$N_h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix},$$

which has full row rank, so $N(\lambda)$ is row reduced. Moreover, after computing the 2×2 minors is straightforward to check that $N(\lambda)$ has full row rank for all $\lambda \in \overline{\mathbb{F}}$. Hence, $N(\lambda)$ is a minimal basis.

Though the notion of minimal basis goes back to, at least, the 1970's (see [11]), the way we have introduced them in Definition 3.3 follows the characterization presented in Theorem 2.14 in [10] (see the original Main theorem in [11, p. 495]). In [11], minimal bases are introduced as polynomial bases of a vector subspace, V, of $\mathbb{F}(\lambda)^n$ whose sum of the degrees of their vectors is minimal among all polynomial bases of V. In Section 4.2, we will use this approach when considering minimal bases of the nullspace of a matrix polynomial. We are also implicitly using this approach when considering the singular spectral structure of $P(\lambda)$ defined, in the classical way, as the left and right minimal indices of $P(\lambda)$. More precisely, they are the minimal indices of the left and right nullspaces of $P(\lambda)$. We will refer to the minimal bases of the left and right null spaces of $P(\lambda)$ as *left* and *right minimal bases of* $P(\lambda)$, respectively.

We are now in the position to introduce the notion of dual minimal bases.

Definition 3.4. Two matrix polynomials $N_1(\lambda) \in \mathbb{F}[\lambda]^{m_1\times n}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{m_2\times n}$ are *dual minimal bases* if $N_1(\lambda)$ and $N_2(\lambda)$ are both minimal bases and they satisfy:

$$m_1 + m_2 = n, \quad \text{and} \qquad N_1(\lambda)N_2(\lambda)^T = 0. \tag{5}$$

The first condition $m_1 + m_2 = n$ in (5) comes from the fact that we look for bases of subspaces of $\mathbb{F}(\lambda)^n$ that are "orthogonal complements" to each other or, rigorously, dual in the sense of Forney [11, Sec. 6].

It is known since, at least, the 1970's (see, for instance, [11, p. 503]), that if $N_1(\lambda)$ and $N_2(\lambda)$ are dual minimal bases with respective row degrees $\eta_1, \ldots, \eta_{m_1}$, and $\varepsilon_1, \ldots, \varepsilon_{m_2}$, then

$$\sum_{j=1}^{m_1} \eta_j = \sum_{i=1}^{m_2} \varepsilon_i \,. \tag{6}$$

That the converse is also true has been recently proved in [9, Th. 6.1, Th. 6.4].

Theorem 3.5. *Let* $(\eta_1, \ldots, \eta_{m_1})$ *and* $(\varepsilon_1, \ldots, \varepsilon_{m_2})$ *be two lists of arbitrary non-negative integers that add up to the same sum:*

$$\sum_{j=1}^{m_1} \eta_j = \sum_{i=1}^{m_2} \varepsilon_i \,.$$

Then there always exist two matrix polynomials $N_1(\lambda) \in \mathbb{F}[\lambda]^{m_1\times n}$ *and* $N_2(\lambda) \in \mathbb{F}[\lambda]^{m_2\times n}$*, with* $n = m_1 + m_2$*, which are dual minimal bases and whose row degrees are, respectively,* $(\eta_1, \ldots, \eta_{m_1})$ *and* $(\varepsilon_1, \ldots, \varepsilon_{m_2})$*.*

In [9, Th. 6.1], a general explicit construction of a pair of dual minimal bases as stated in Theorem 3.5 is also provided. This construction proceeds by splitting the general problem into smaller subproblems and building up the pair of dual minimal bases as direct sums of the pairs of dual minimal bases that realize each of these smaller subproblems. The construction of the dual minimal bases of the smaller subproblems uses the so called *zigzag* and *dual zigzag* matrices [9, Def. 3.1, Def 3.21] via the simple algorithm presented in [9, Th. 5.1]. The nontrivial smaller subproblems satisfy, in addition to (6), the following assumptions: $\eta_j > 0$, $\varepsilon_i > 0$, for all $1 \leq j \leq m_1$ and $1 \leq i \leq m_2$, and $\sum_{j=1}^{\alpha} \eta_j \neq \sum_{i=1}^{\beta} \varepsilon_i$ whenever $(\alpha, \beta) \neq (m_1, m_2)$. The construction of dual "zigzag" minimal bases in [9, Th. 5.1] will be fundamental in Section 4, so we illustrate it in Example 3.2.

Example 3.2. Let (η_1, η_2, η_3) and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be two lists of positive integers such that

$$0 < \eta_1 < \varepsilon_1, \quad \eta_1 + \eta_2 > \varepsilon_1 + \varepsilon_2, \quad \text{and} \quad \eta_1 + \eta_2 + \eta_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Hence, the ordering of the partial sums of (η_1, η_2, η_3) and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is

$$0 < \eta_1 < \varepsilon_1 < \varepsilon_1 + \varepsilon_2 < \eta_1 + \eta_2 < \eta_1 + \eta_2 + \eta_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Now we set $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ for the differences of two consecutive terms in the previous ordering, that is:

$$\delta_1 = \eta_1, \quad \delta_2 = \varepsilon_1 - \eta_1, \quad \delta_3 = \varepsilon_2, \quad \delta_4 = \eta_1 + \eta_2 - (\varepsilon_1 + \varepsilon_2), \quad \delta_5 = \eta_3.$$

We construct a pair of dual minimal bases $N_1(\lambda), N_2(\lambda) \in \mathbb{F}[\lambda]^{3\times 6}$, where $N_1(\lambda)$ has $m_1 = 3$ row degrees η_1, η_2, η_3, and $N_2(\lambda)$ has $m_2 = 3$ row degrees $\varepsilon_1, \varepsilon_2, \varepsilon_3$. The construction is:

$$N_1(\lambda) = \begin{bmatrix} 1 & \lambda^{\delta_1} & & & & \\ & 1 & \lambda^{\delta_2} & \lambda^{\delta_2+\delta_3} & \lambda^{\delta_2+\delta_3+\delta_4} & \\ & & & & 1 & \lambda^{\delta_5} \end{bmatrix},$$

$$N_2(\lambda) = \begin{bmatrix} \lambda^{\delta_1+\delta_2} & -\lambda^{\delta_2} & 1 & & & \\ & & \lambda^{\delta_3} & -1 & & \\ & & & -\lambda^{\delta_4+\delta_5} & \lambda^{\delta_5} & -1 \end{bmatrix}.$$

Indeed, the row degrees of $N_1(\lambda)$ and $N_2(\lambda)$ are respectively given by $\eta_1 = \delta_1$, $\eta_2 = \delta_2 + \delta_3 + \delta_4$, $\eta_3 = \delta_5$, and $\varepsilon_1 = \delta_1 + \delta_2$, $\varepsilon_2 = \delta_3$, $\varepsilon_3 = \delta_4 + \delta_5$.

It is clear that these two matrix polynomials are row reduced and they have full row rank for all finite $\lambda \in \overline{\mathbb{F}}$, which implies that they are minimal bases. They satisfy $m_1 + m_2 = n = 6$, and it is easy to see from the particular *zigzag structure* of these matrices that they also satisfy $N_1(\lambda)N_2(\lambda)^T = 0$.

The zigzag construction of dual minimal bases illustrated in Example 3.2 allows for some flexibility regarding permutations of rows and columns in both $N_1(\lambda)$ and $N_2(\lambda)$. This flexibility is based on the following property, which is satisfied by *any* pair $N_1(\lambda)$ and $N_2(\lambda)$ of dual minimal bases: if Π, Π_1, and Π_2 are permutation matrices of appropriate sizes, then

$$\widetilde{N}_1(\lambda) = \Pi_1 N_1(\lambda)\Pi \qquad \text{and} \qquad \widetilde{N}_2(\lambda) = \Pi_2 N_2(\lambda)\Pi$$

are also dual minimal bases, since

$$\widetilde{N}_1(\lambda)\widetilde{N}_2(\lambda)^T = (\Pi_1 N_1(\lambda)\Pi)\left(\Pi^T N_2(\lambda)^T \Pi_2^T\right) = \Pi_1 N_1(\lambda) N_2(\lambda)^T \Pi_2^T = 0.$$

In addition, note that the roles of $N_1(\lambda)$ and $N_2(\lambda)$ can be exchanged, so that $N_2(\lambda)N_1(\lambda)^T = 0$.

Remark 3.6. A standard normalization of dual minimal bases, $N_1(\lambda), N_2(\lambda)$, is to perform an strict equivalence, i.e., to multiply by constant invertible matrices, R, R^{-T}, to get $N_1(\lambda)R$ and $N_2(\lambda)R^{-T}$, in order to make sure that the highest row degree coefficient matrices are

$$\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & I_{m_2} \end{bmatrix}.$$

For the bases in Example 3.2 (see [10, Lemma 3.6]), the matrix R is just a column permutation with the sign of the last column changed.

4. Constructing strong ℓ-ifications

In this section we focus on the construction of a strong ℓ-ification of a given (arbitrary) matrix polynomial $P(\lambda)$. We consider the general case where $P(\lambda)$ has size $m \times n$, i.e., allowing for rectangular matrix polynomials, normal rank r and degree d, so that $P(\lambda)$ is as in (2). As mentioned in Section 2, we are interested in the case $\ell < d$, and in constructing strong ℓ-ifications satisfying (4). Then, the ℓ-ification $L(\lambda)$ will have size $(\widehat{n} + m) \times (\widehat{n} + n)$, and normal rank $\rho = r + \widehat{n}$. In addition, we define the positive integer

$$\widehat{d} := d - \ell \tag{7}$$

which plays a relevant role in the construction of $L(\lambda)$.

A key point in our strategy is to look for solutions, $(L(\lambda), M(\lambda), N(\lambda))$, to (4) with the unimodular matrix $M(\lambda)$ constrained to be of the form

$$M(\lambda) = \begin{bmatrix} I_{\widehat{n}} & 0 \\ -X(\lambda) & I_m \end{bmatrix}, \qquad \text{with } \widehat{n} > 0, \tag{8}$$

for some matrix polynomial $X(\lambda)$. Our original motivation to use this block triangular form for $M(\lambda)$ is that such structure has been previously employed in the particular unimodular transformations used for the classical Frobenius linearizations (see for instance [8, Section 5.1]) and for the Frobenius-like strong ℓ-ifications introduced in [8, Section 5.2][1], when ℓ divides d. In fact, the developments in this section can be seen as a wide nontrivial generalization of the unimodular transformations and the structure of the strong ℓ-ifications studied in [8, Sections 5.1 and 5.2].

The block-triangular structure of $M(\lambda)$ in (8) allows us to split the construction introduced in this section into two steps, such that the construction corresponding to the first one has been already presented in [9, Th. 5.1, Th. 6.1]. These two steps are described in assumptions (a) and (b) of Theorem 4.1 in an abstract way, but their explicit realization will be addressed in detail in Section 4.1 for the case where ℓ divides nd or md. Let us briefly explain the key ideas of the construction of strong ℓ-ifications proposed in Theorem 4.1, and the reason why conditions (a) and (b), together with left transformations of the form (8) are enough to get (4). The starting point is a pair of dual minimal bases $\widehat{L}(\lambda)\widehat{N}(\lambda)^T = 0$ of appropriate degrees. Since $\widehat{N}(\lambda)$ is a minimal basis, $\widehat{N}(\lambda)^T$ can be completed, by adding some columns on the left, to a unimodular matrix $N(\lambda)^T$ in such a way that $\widehat{L}(\lambda)N(\lambda)^T = \begin{bmatrix} I & 0 \end{bmatrix}$, because the Smith form of $\widehat{L}(\lambda)$ is precisely $\begin{bmatrix} I & 0 \end{bmatrix}$ since $\widehat{L}(\lambda)$ is also a minimal basis. In addition, $\widehat{L}(\lambda)$ can be completed with $\widetilde{L}(\lambda)$ to get $L(\lambda)$ (see (10) below), so that $L(\lambda)N(\lambda)^T = \left[\begin{smallmatrix} I & 0 \\ X(\lambda) & P(\lambda) \end{smallmatrix}\right]$, for some $X(\lambda)$. Then $M(\lambda)$ as in (8) will be enough to get (4). We strongly encourage the reader to check that the unimodular transformations used in [8, Sections 5.1 and 5.2] for the first Frobenius companion linearization and the first Frobenius-like companion ℓ-ification, as well as the structure of such companion forms, are particular cases of the construction proposed in Theorem 4.1.

Theorem 4.1. *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree $d > 0$ and assume that there are two matrix polynomials $\widehat{L}(\lambda)$ and $\widetilde{L}(\lambda)$ of respective sizes $\widehat{n} \times (\widehat{n} + n)$ and $m \times (\widehat{n} + n)$, satisfying:*

(a) *$\widehat{L}(\lambda)$ is a minimal basis and has degree ℓ.*

(b) *$\widetilde{L}(\lambda)$ has degree less than or equal to ℓ and satisfies*

$$\widetilde{L}(\lambda)\widehat{N}(\lambda)^T = P(\lambda), \tag{9}$$

where $\widehat{N}(\lambda)$ is a minimal basis dual to $\widehat{L}(\lambda)$ (that is, it has size $n \times (\widehat{n} + n)$ and $\widehat{L}(\lambda)\widehat{N}(\lambda)^T = 0$).

Then:

[1] We remark that in [8, Eq. (5.19)] the ℓ-ification equation has as right-hand side $\mathrm{diag}(P(\lambda), I_{\widehat{n}})$, that is, with the order of the diagonal blocks reversed with respect to (4). Therefore the structure of $M(\lambda)$ in [8] is *block-upper* triangular.

(1) *The matrix polynomial of degree* ℓ

$$L(\lambda) := \left[\begin{array}{c} \widehat{L}(\lambda) \\ \hline \widetilde{L}(\lambda) \end{array} \right] \in \mathbb{F}[\lambda]^{(\widehat{n}+m)\times(\widehat{n}+n)} \tag{10}$$

is an ℓ*-ification of* $P(\lambda)$ *that satisfies* (4) *with* $M(\lambda)$ *being of the form* (8).

(2) *In addition, if the row degrees of* $\widehat{L}(\lambda)$ *are all equal to* ℓ *and the row degrees of* $\widehat{N}(\lambda)$ *are all equal to* $d-\ell$*, then* $L(\lambda)$ *is a strong* ℓ*-ification of* $P(\lambda)$.

Proof. Since $\widehat{N}(\lambda)^T$ has full column rank for all $\lambda \in \overline{\mathbb{F}}$, there exists a matrix polynomial $N_c(\lambda)$ of size $\widehat{n} \times (\widehat{n}+n)$ such that $\widetilde{N}(\lambda)^T := \left[\begin{array}{c|c} N_c(\lambda)^T & \widehat{N}(\lambda)^T \end{array} \right]$ is unimodular (see [10, Lemma 2.16]).

Now, since $\widehat{L}(\lambda)$ is a minimal basis, it has full row rank for all $\lambda \in \overline{\mathbb{F}}$, and, since $\widetilde{N}(\lambda)$ is unimodular, the product $\widehat{L}(\lambda)\widetilde{N}(\lambda)^T$ has full row rank for all $\lambda \in \overline{\mathbb{F}}$. Since, by construction,

$$\widehat{L}(\lambda)\widetilde{N}(\lambda)^T = \left[\begin{array}{c|c} \widehat{L}(\lambda)N_c(\lambda)^T & 0 \end{array} \right],$$

the $\widehat{n} \times \widehat{n}$ matrix $U(\lambda) := \widehat{L}(\lambda)N_c(\lambda)^T$ has full row rank for all $\lambda \in \overline{\mathbb{F}}$, so it is unimodular.

Then, the matrix

$$N(\lambda)^T := \widetilde{N}(\lambda)^T \begin{bmatrix} U(\lambda)^{-1} & \\ & I_n \end{bmatrix} = \left[\begin{array}{c|c} N_c(\lambda)^T U(\lambda)^{-1} & \widehat{N}(\lambda)^T \end{array} \right]$$

is unimodular, since it is the product of two unimodular matrices. Note that

$$\left[\begin{array}{c} \widehat{L}(\lambda) \\ \hline \widetilde{L}(\lambda) \end{array} \right] N(\lambda)^T = \left[\begin{array}{c} \widehat{L}(\lambda) \\ \hline \widetilde{L}(\lambda) \end{array} \right] \left[\begin{array}{c|c} N_c(\lambda)^T U(\lambda)^{-1} & \widehat{N}(\lambda)^T \end{array} \right] = \begin{bmatrix} I_{\widehat{n}} & 0 \\ X(\lambda) & P(\lambda) \end{bmatrix},$$

with $X(\lambda) := \widetilde{L}(\lambda)N_c(\lambda)^T U(\lambda)^{-1}$. Hence:

$$\begin{bmatrix} I_{\widehat{n}} & 0 \\ -X(\lambda) & I_m \end{bmatrix} \left[\begin{array}{c} \widehat{L}(\lambda) \\ \hline \widetilde{L}(\lambda) \end{array} \right] N(\lambda)^T = \begin{bmatrix} I_{\widehat{n}} & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

and this proves the first part of the statement.

Let us now prove the second part, namely that $L(\lambda)$ is a strong ℓ-ification of $P(\lambda)$ provided that the row degrees of $\widehat{L}(\lambda)$ are all equal to ℓ and the row degrees of $\widehat{N}(\lambda)$ are all equal to $d-\ell$. First recall that for any minimal basis, $B(\lambda)$, of size $p \times q$, with $p < q$ and

$$B(\lambda) = \begin{bmatrix} b_1(\lambda) \\ \vdots \\ b_p(\lambda) \end{bmatrix},$$

the matrix

$$\begin{bmatrix} \mathrm{rev} b_1(\lambda) \\ \vdots \\ \mathrm{rev} b_p(\lambda) \end{bmatrix},$$

obtained by taking the reversal of each row with respect the degree of that row is also a minimal basis (see [4, Th. 3.2] or [23, Th. 7.5]). Since all row degrees of $\widehat{L}(\lambda)$ are equal, then $\mathrm{rev}\widehat{L}(\lambda)$ is a minimal basis and has degree ℓ, because $\widehat{L}(0) \neq 0$. Analogously, $\mathrm{rev}\widehat{N}(\lambda)$ is a minimal basis with degree $d-\ell$. Moreover, the duality condition $\widehat{L}(\lambda)\widehat{N}(\lambda)^T = 0$ is equivalent to $\widehat{L}(1/\lambda)\widehat{N}(1/\lambda)^T = 0$, which implies $(\lambda^\ell\, \widehat{L}(1/\lambda))\,(\lambda^{d-\ell}\, \widehat{N}(1/\lambda)^T) = 0$, and from the definition of reversal we get

$$\mathrm{rev}\widehat{L}(\lambda)(\mathrm{rev}\widehat{N}(\lambda))^T = 0. \tag{11}$$

So $\mathrm{rev}\widehat{L}(\lambda)$ and $\mathrm{rev}\widehat{N}(\lambda)$ are dual minimal bases. In addition, equation (9) implies that the degree of $\widetilde{L}(\lambda)$ is exactly ℓ (since the degree of $\widehat{N}(\lambda)^T$ is $d-\ell$), and that $\widetilde{L}(1/\lambda)\widehat{N}(1/\lambda)^T = P(1/\lambda)$. Then $(\lambda^\ell \widetilde{L}(1/\lambda))(\lambda^{d-\ell}\widehat{N}(1/\lambda)^T) = \lambda^d P(1/\lambda)$, that is

$$\mathrm{rev}\widetilde{L}(\lambda)(\mathrm{rev}\widehat{N}(\lambda))^T = \mathrm{rev}P(\lambda). \tag{12}$$

Now, (11) and (12) together imply that $\mathrm{rev}\widehat{L}(\lambda), \mathrm{rev}\widehat{N}(\lambda), \mathrm{rev}\widetilde{L}(\lambda)$, and $\mathrm{rev}P(\lambda)$ satisfy conditions (a) and (b) in the statement. Therefore,

$$\mathrm{rev}L(\lambda) = \begin{bmatrix} \mathrm{rev}\widehat{L}(\lambda) \\ \mathrm{rev}\widetilde{L}(\lambda) \end{bmatrix}$$

is an ℓ-ification of $\mathrm{rev}P(\lambda)$ and this completes the proof. □

Looking at Theorem 4.1, a procedure to get a strong ℓ-ification, $L(\lambda)$, of $P(\lambda)$ as in (10) would be as follows:

Step 1: Choose $\widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}$ and $\widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ to be a pair of dual minimal bases, with $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ having row degrees all equal to ℓ and $d-\ell$, respectively.

Step 2: Solve for $\widetilde{L}(\lambda)$ in equation (9), to get $\widetilde{L}(\lambda)$ of degree at most ℓ.

We emphasize that Theorem 4.1 establishes that (strong) ℓ-ifications can be constructed through the dual minimal bases $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ and the matrix polynomial $\widetilde{L}(\lambda)$ associated with $P(\lambda)$, but Theorem 4.1 does not guarantee the existence of such matrix polynomials nor explains how to construct them. Next, we are going to show that, provided that ℓ divides nd, any pair of dual minimal bases (independent of $P(\lambda)$), $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ as in **Step 1** above, allow us to find a matrix polynomial $\widetilde{L}(\lambda)$ as in **Step 2** (in general non-unique), and so $L(\lambda)$ in Theorem 4.1 is a strong ℓ-ification of $P(\lambda)$. Moreover, $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ can be chosen to follow a general simple construction, valid for all matrix polynomials $P(\lambda)$ with fixed size $m\times n$ and degree d.

4.1. General construction for the case ℓ divides md or nd

We focus on the case ℓ divides nd. The case ℓ divides md is considered at the end of the section in Remark 4.3 as a corollary. The goal is to show how to construct, first, dual minimal bases $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ as in **Step 1** above and, then, to show that for each $\widehat{N}(\lambda)$ satisfying the conditions of **Step 1**, equation (9) has a solution $\widetilde{L}(\lambda)$ of degree ℓ, and characterize the set of all possible solutions of (9). Then, the matrix $L(\lambda)$ defined in (10) will be a strong ℓ-ification of $P(\lambda)$. Note that, given any $m\times n$ matrix polynomial $P(\lambda)$ of degree d and any integer ℓ such that $0<\ell<d$ and such that ℓ divides nd, we have that $k\ell = nd$, for some integer $k>n$. Therefore, there exists an integer $\widehat{n}>0$ such that $k=\widehat{n}+n$. Then:

$$(\widehat{n}+n)\ell = nd, \tag{13}$$

which is equivalent to

$$\widehat{n}\ell = n\widehat{d}, \tag{14}$$

with $\widehat{d}$ as in (7), where $\widehat{n}$ is going to be the size of the identity block in (4) or, in other words, the ℓ-ification $L(\lambda)$ is going to be of size $(\widehat{n}+m)\times(\widehat{n}+n)$.

Equation (14) above, together with Theorem 3.5 guarantee that there exist dual minimal bases $\widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}$ and $\widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ such that the row degrees of $\widehat{L}(\lambda)$ are all equal to ℓ and the row degrees of $\widehat{N}(\lambda)$ are all equal to $\widehat{d}$. Note that, conversely, given any $0<\ell<d$, the existence of such dual minimal bases implies, by (6), that (14) holds, and this in turn implies (13), so that ℓ divides nd. This means that the assumptions in Theorem 4.1(2) imply ℓ divides nd, which is not an additional assumption to those of Theorem 4.1.

Once the existence of $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ has been established, we discuss their construction in **Step 1**.

Step 1. (*Construction of dual minimal bases, $\widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}, \widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$, with all row degrees equal to ℓ and $\widehat{d}$, respectively, when ℓ divides nd and $\widehat{n}+n = (nd)/\ell$).*

In order to build up such dual minimal bases $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ we can follow the procedure devised in [9, Th. 5.1, Th. 6.1] (see also the alternative method in [9, Th. 5.3], which fits very well in our context). However, as explained in [9], there are infinitely many dual minimal bases $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ with the desired properties, although the ones constructed in [9, Th. 6.1] are particularly simple. We emphasize that for any of these infinitely many pairs of dual minimal bases $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ the construction we present below for the matrix polynomial $\widetilde{L}(\lambda)$ in (9) works, and $L(\lambda)$ in (10) is a strong ℓ-ification of $P(\lambda)$.

Step 2. (*Solutions, $\widetilde{L}(\lambda)$, of degree ℓ of $\widetilde{L}(\lambda)\widehat{N}(\lambda)^T = P(\lambda)$ for a given $\widehat{N}(\lambda)$ and $P(\lambda)$).*

We need to show that, given any $\widehat{N}(\lambda)$ constructed in **Step 1** above, there is a solution $\widetilde{L}(\lambda)$ to (9) with degree ℓ. For this, we set $\widetilde{L}(\lambda) = \lambda^\ell \widetilde{L}_\ell + \lambda^{\ell-1}\widetilde{L}_{\ell-1} + \cdots + \lambda \widetilde{L}_1 + \widetilde{L}_0$ and $\widehat{N}(\lambda) = \lambda^{\widehat{d}}\widehat{N}_{\widehat{d}} + \lambda^{\widehat{d}-1}\widehat{N}_{\widehat{d}-1} + \cdots + \lambda\widehat{N}_1 + \widehat{N}_0$, and write the convolution

$$\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} & \widetilde{L}_\ell \end{bmatrix} \begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & & \ddots & & \ddots & \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix} = \begin{bmatrix} P_0 & \dots & P_{d-1} & P_d \end{bmatrix}, \tag{15}$$

corresponding to (9), where:

$$\begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & & \ddots & & \ddots & \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix} \in \mathbb{F}^{(\widehat{n}+n)(\ell+1)\times n(d+1)}, \tag{16}$$

and

$$\begin{bmatrix} P_0 & \dots & P_{d-1} & P_d \end{bmatrix} \in \mathbb{F}^{m\times(d+1)n},$$

so that $\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} & \widetilde{L}_\ell \end{bmatrix}$ is a matrix of unknowns with size $m \times (\widehat{n}+n)(\ell+1)$.

Equation (15) does not have a unique solution, since the block Toeplitz matrix (16) has $\widehat{n}$ more rows than columns. However, the equation is consistent for any $P(\lambda)$ and the degrees of freedom are easy to describe. Indeed, one can solve first for $\widetilde{L}_\ell$ from

$$\widetilde{L}_\ell \widehat{N}_{\widehat{d}}^T = P_d\,, \tag{17}$$

which is always consistent since $\widehat{N}_{\widehat{d}}^T$ has full column rank, and we can move the last block row of the block Toeplitz matrix to the right hand side to obtain the reduced equation

$$\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} \end{bmatrix} \begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & & \ddots & & \ddots & \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix} = \tag{18}$$

$$\begin{bmatrix} P_0 & P_1 & \dots & P_{d-1} \end{bmatrix} - \widetilde{L}_\ell \begin{bmatrix} 0 \dots 0 & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}-1}^T \end{bmatrix},$$

where now the block Toeplitz matrix is $(\widehat{n}+n)\ell \times nd = nd \times nd$ and invertible. Indeed, if this square matrix was singular, it would have a left null vector. But that would correspond to a polynomial left null vector of $\widehat{N}(\lambda)^T$ that would have degree $\ell-1$, rather than ℓ, so that $\widehat{L}(\lambda)$ would not be a dual minimal basis to $\widehat{N}(\lambda)$. Therefore, (18) determines $\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} \end{bmatrix}$ uniquely for each possible choice of $\widetilde{L}_\ell$. We stress the fact that the arguments above proving the consistency of the equation (15), i.e., of (9), are heavily based on the fact that $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ are dual minimal bases, and not just submatrices of unimodular matrices.

We can also describe the degrees of freedom of $\widetilde{L}_\ell$. Following Remark 3.6, if we choose the dual bases such that their highest degree coefficients are

$$\widehat{L}_\ell = \begin{bmatrix} I_{\widehat{n}} & 0 \end{bmatrix}, \quad \widehat{N}_{\widehat{d}}^T = \begin{bmatrix} 0 \\ I_n \end{bmatrix},$$

respectively, then the general solutions for $\widetilde{L}_\ell$ and the coefficient L_ℓ of $L(\lambda)$ in Theorem 4.1 are given by

$$\widetilde{L}_\ell = \begin{bmatrix} X_\ell & P_d \end{bmatrix}, \quad \text{and} \quad L_\ell = \begin{bmatrix} I_{\widehat{n}} & 0 \\ X_\ell & P_d \end{bmatrix},$$

with $X_\ell \in \mathbb{F}^{m\times\widehat{n}}$ arbitrary. Clearly, the different solutions for L_ℓ (and therefore also $L(\lambda)$) are related by a constant left multiplication

$$\begin{bmatrix} I_{\widehat{n}} & 0 \\ X_\ell & I_m \end{bmatrix}$$

that we omit when choosing $\widetilde{L}_\ell = \begin{bmatrix} 0 & P_d \end{bmatrix}$.

Let us finish this section by summarizing part of the developments performed in **Step 2**.

Theorem 4.2. *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree $d > 0$ and let $\widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}$ and $\widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ be dual minimal bases with the row degrees of $\widehat{L}(\lambda)$ all equal to ℓ, for some $0 < \ell < d$, and the row degrees of $\widehat{N}(\lambda)$ all equal to $\widehat{d} = d - \ell$. Then the equation* (9) *for the unknown matrix polynomial $\widetilde{L}(\lambda)$ of degree at most ℓ has infinitely many solutions, all of them of degree exactly ℓ, and the set of solutions depends on $m\widehat{n}$ free variables. If, in addition, the highest degree coefficients of $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ are chosen to be $\begin{bmatrix} I_{\widehat{n}} & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & I_n \end{bmatrix}$, respectively, and $\widetilde{L}(\lambda)$ is a particular solution of* (9)*, then any other solution can be written as*

$$X_\ell \widehat{L}(\lambda) + \widetilde{L}(\lambda),$$

where $X_\ell \in \mathbb{F}^{m\times\widehat{n}}$ is an arbitrary matrix.

Remark 4.3. If the process developed in this section is applied to the matrix polynomial $P(\lambda)^T$, of size $n \times m$ and degree d, when ℓ divides md, then a strong ℓ-ification $L(\lambda)^T$ of $P(\lambda)^T$ is constructed, and this gives a strong ℓ-ification $L(\lambda)$ of $P(\lambda)$.

4.2. Recovery of minimal indices

Theorem 4.10 in [10] proves that any singular $m \times n$ matrix polynomial $P(\lambda)$, i. e., having nontrivial left or right null spaces, has strong ℓ-ifications for any value of $\ell > 0$. In addition, Theorem 4.10 in [10] characterizes all possible minimal indices of the strong ℓ-ifications of $P(\lambda)$ and shows that they may take a wide variety of values completely unrelated, in general, to the minimal indices of $P(\lambda)$. Surprisingly, all the infinitely many strong ℓ-ifications of $P(\lambda)$ constructed in Section 4.1 according to the two-step strategy presented in Section 4.1 have the same minimal indices and they are related in a very simple way with the ones of $P(\lambda)$. To prove this fact is the goal of this section. We focus first on the case ℓ divides nd and consider the case ℓ divides md as a corollary in Remark 4.6.

We will make use of the following lemma.

Lemma 4.4. *Let $A(\lambda)$ be an $n \times p$ matrix polynomial all whose columns have the same degree d, and let $B(\lambda)$ be another $p \times m$ matrix polynomial whose column degrees are $\varepsilon_1, \ldots, \varepsilon_m$. If both $A(\lambda)$ and $B(\lambda)$ are column reduced, then the product $A(\lambda)B(\lambda)$ is column reduced as well, and its column degrees are $\varepsilon_1 + d, \ldots, \varepsilon_m + d$.*

Proof. In the conditions of the statement we can write:

$$A(\lambda) = \lambda^d A_h + A_{\text{low}}(\lambda), \qquad B(\lambda) = B_h \begin{bmatrix} \lambda^{\varepsilon_1} & & \\ & \ddots & \\ & & \lambda^{\varepsilon_m} \end{bmatrix} + B_{\text{low}}(\lambda),$$

where A_h, B_h are the highest column degree coefficient matrices of $A(\lambda)$ and $B(\lambda)$, respectively. Then:

$$A(\lambda)B(\lambda) = A_h B_h \begin{bmatrix} \lambda^{d+\varepsilon_1} & & \\ & \ddots & \\ & & \lambda^{d+\varepsilon_m} \end{bmatrix} + (AB)_{\text{low}}(\lambda),$$

where the jth column of $(AB)_{\text{low}}(\lambda)$ has degree smaller than $d+\varepsilon_j$, for $j = 1, \dots, m$. Hence, the highest column degree coefficient matrix of $A(\lambda)B(\lambda)$ is $A_h B_h$. Since, by hypothesis, both A_h and B_h are of full (column) rank, their product $A_h B_h$ is of full column rank as well, so $A(\lambda)B(\lambda)$ is column reduced. $\square$

We are now in the position of proving the main result of this section.

Theorem 4.5. *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree $d > 0$ with left minimal indices equal to $\eta_1, \eta_2, \dots, \eta_q$ and right minimal indices $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p$. Let $L(\lambda) \in \mathbb{F}[\lambda]^{(\widehat{n}+m)\times(\widehat{n}+n)}$ be any strong ℓ-ification of $P(\lambda)$ as in part* (2) *of Theorem* 4.1*. Then:*

(a) *The right minimal indices of $L(\lambda)$ are $\varepsilon_1 + (d-\ell), \varepsilon_2 + (d-\ell), \dots, \varepsilon_p + (d-\ell)$.*

(b) *The left minimal indices of $L(\lambda)$ are $\eta_1, \eta_2, \dots, \eta_q$.*

Proof. We are going to see first that the right minimal indices of $L(\lambda)$ are the ones of $P(\lambda)$ all increased by $d-\ell$. For this, let the columns of $N_r(\lambda)$ form a minimal basis for the right null space of $P(\lambda)$. Then by Theorem 4.1 we have

$$L(\lambda)\widehat{N}(\lambda)^T N_r(\lambda) = \begin{bmatrix} 0 \\ P(\lambda) \end{bmatrix} N_r(\lambda) = 0.$$

Since $\widehat{N}(\lambda)^T$ is column reduced and, by construction, all its columns have the same degree $\widehat{d}$, and $N_r(\lambda)$ is column reduced as well, Lemma 4.4 guarantees that the product $\widehat{N}(\lambda)^T N_r(\lambda)$ is column reduced and its column degrees are the column degrees of $N_r(\lambda)$ all increased by $\widehat{d}$, with $\widehat{d}$ as in (7). Since both $\widehat{N}(\lambda)^T$ and $N_r(\lambda)$ are of full rank for all $\lambda \in \overline{\mathbb{F}}$, their product $\widehat{N}(\lambda)^T N_r(\lambda)$ has the same property. Moreover, $L(\lambda)$ and $P(\lambda)$ have the same number of right minimal indices by [8, Th. 4.1]. Hence, $\widehat{N}(\lambda)^T N_r(\lambda)$ is a right minimal basis of $L(\lambda)$ whose minimal indices are the ones of $P(\lambda)$ increased by $\widehat{d}$.

Now, we are going to prove that the left minimal indices of $L(\lambda)$ coincide with those of $P(\lambda)$. We already know that both $P(\lambda)$ and $L(\lambda)$ have the same number of left minimal indices (see [8, Th. 4.1]). Set $\eta_1(P), \dots, \eta_q(P)$ and $\eta_1(L), \dots, \eta_q(L)$ for the left minimal indices of $P(\lambda)$ and $L(\lambda)$, respectively. Let δ_{fin} be the sum of (the numbers in) all partial multiplicity sequences of $P(\lambda)$, and μ_∞ be the sum of (all numbers in) the infinite partial multiplicity sequence at ∞ of $P(\lambda)$. Since $L(\lambda)$ is a strong ℓ-ification of $P(\lambda)$, δ_{fin} and μ_∞ coincide with, respectively, the sum of (the numbers in) all partial multiplicity sequences and the sum of (all numbers in) the infinite partial multiplicity sequence of $L(\lambda)$ [8, Th. 4.1]. Then, by applying the Index Sum Theorem [8, Th. 6.5] to $P(\lambda)$ and $L(\lambda)$, respectively, we get:

$$\sum_{j=1}^{q} \eta_j(P) + \sum_{i=1}^{p} \varepsilon_i + \delta_{\text{fin}} + \mu_\infty = rd, \tag{19}$$

and

$$\sum_{j=1}^{q} \eta_j(L) + \sum_{i=1}^{p} (\varepsilon_i + \widehat{d}) + \delta_{\text{fin}} + \mu_\infty = \rho\ell, \tag{20}$$

where r is the normal rank of $P(\lambda)$ and ρ is the normal rank of $L(\lambda)$. By (4), we have $\rho = r + \widehat{n}$ and, since $p = n - r$, by subtracting (19) from (20), we get:

$$\sum_{j=1}^{q} \eta_j(L) - \sum_{j=1}^{q} \eta_j(P) + (n-r)\widehat{d} = (r+\widehat{n})\ell - rd,$$

so that, using (13),

$$\sum_{j=1}^{q}\eta_j(L)-\sum_{j=1}^{q}\eta_j(P)=nd-n\ell+r\ell-rd-(n-r)\widehat{d}=(n-r)(d-\ell)-(n-r)\widehat{d}=0,$$

hence

$$\sum_{j=1}^{q}\eta_j(L)=\sum_{j=1}^{q}\eta_j(P). \tag{21}$$

Now, we are going to see that $\eta_j(P)\le\eta_j(L)$, for $j=1,\dots,q$, which, together with (21) imply that $\eta_j(L)=\eta_j(P)$, for all $j=1,\dots,q$.

Let $v(\lambda)\in\mathbb{F}^{1\times(\widehat{n}+m)}[\lambda]$ be a vector polynomial in the left nullspace of $L(\lambda)$. Let us partition $v(\lambda)=\left[\begin{array}{c|c} v_1(\lambda) & v_2(\lambda)\end{array}\right]$, with $v_1(\lambda)\in\mathbb{F}^{1\times\widehat{n}}[\lambda]$ and $v_2(\lambda)\in\mathbb{F}^{1\times m}[\lambda]$. By Theorem 4.1 we know that $L(\lambda)$ satisfies (4) with $M(\lambda)$ of the form (8) and, therefore:

$$\begin{aligned} 0 &= v(\lambda)L(\lambda)N(\lambda)^T=\left[\begin{array}{c|c} v_1(\lambda) & v_2(\lambda)\end{array}\right]\begin{bmatrix} I_{\widehat{n}} & 0\\ X(\lambda) & P(\lambda)\end{bmatrix}\\ &=\left[\begin{array}{c|c} v_1(\lambda)+v_2(\lambda)X(\lambda) & v_2(\lambda)P(\lambda)\end{array}\right],\end{aligned}$$

so $v_2(\lambda)$ is a vector polynomial in the left nullspace of $P(\lambda)$ and $v_1(\lambda)=-v_2(\lambda)X(\lambda)$. As a consequence, any minimal basis of $L(\lambda)$ is of the form

$$\mathcal{B}_L=\left\{\left[\begin{array}{c|c} -Z_1(\lambda)X(\lambda) & Z_1(\lambda)\end{array}\right],\dots,\left[\begin{array}{c|c} -Z_q(\lambda)X(\lambda) & Z_q(\lambda)\end{array}\right]\right\},$$

for some $Z_1(\lambda),\dots,Z_q(\lambda)$, which are vector polynomials belonging to the left nullspace of $P(\lambda)$. In fact, $\{Z_1(\lambda),\dots,Z_q(\lambda)\}$ must be linearly independent, because otherwise $\mathcal{B}_L$ would not be linearly independent. So $\{Z_1(\lambda),\dots,Z_q(\lambda)\}$ is a basis of the left null space of $P(\lambda)$. If $\{Z_1(\lambda),\dots,Z_q(\lambda)\}$ is not a left minimal basis of $P(\lambda)$, then there would be a left minimal basis of $P(\lambda)$, $\{w_1(\lambda),\dots,w_q(\lambda)\}$, so that

$$\begin{aligned}\sum_{j=1}^{q}\eta_j(P) &= \deg w_1(\lambda)+\cdots+\deg w_q(\lambda)<\deg Z_1(\lambda)+\cdots+\deg Z_q(\lambda)\\ &\le \deg\left(\left[\begin{array}{c|c} -Z_1(\lambda)X(\lambda) & Z_1(\lambda)\end{array}\right]\right)+\cdots+\deg\left(\left[\begin{array}{c|c} -Z_q(\lambda)X(\lambda) & Z_q(\lambda)\end{array}\right]\right)\\ &=\sum_{j=1}^{q}\eta_j(L),\end{aligned}$$

which is in contradiction with (21). Hence $\{Z_1(\lambda),\dots,Z_q(\lambda)\}$ is a left minimal basis of $P(\lambda)$, and then $\eta_j(P)=\deg Z_j(\lambda)\le\deg\left(\left[\begin{array}{c|c} -Z_j(\lambda)X(\lambda) & Z_j(\lambda)\end{array}\right]\right)=\eta_j(L)$, for $j=1,\dots,q$, as wanted. □

Remark 4.6. In the case ℓ divides md any strong ℓ-ification of $P(\lambda)$, of size $m\times n$ and degree d, constructed according the procedure described in Remark 4.3, has the same right minimal indices as $P(\lambda)$, while the left minimal indices are those of $P(\lambda)$ shifted by $d-\ell$.

4.3. Main results and example

In this subsection we gather together the main results of this paper, which are immediate consequences of the arguments and developments carried out in Sections 4.1 and 4.2. In addition, we present an explicit example of constructing a strong ℓ-ification directly from the coefficients of the polynomial in a situation not covered by previous results in the literature, which are only valid for the case where ℓ divides d.

Theorem 4.7. *Let $P(\lambda)$ be an $m\times n$ matrix polynomial of degree d. Then for any ℓ such that $n\cdot d=\ell\cdot(\widehat{n}+n)$, with $\widehat{n}>0$, we can construct a strong ℓ-ification $L(\lambda)$ of $P(\lambda)$ with size $(\widehat{n}+m)\times(\widehat{n}+n)$ following* **Step 1** *and* **Step 2** *in Section* 4.1. *This strong ℓ-ification has the same left minimal indices as $P(\lambda)$, and the right minimal indices are those of $P(\lambda)$ all increased by $d-\ell$. More precisely:*

(i) If $\varepsilon_1,\dots,\varepsilon_p$ are the right minimal indices of $P(\lambda)$, then the right minimal indices of $L(\lambda)$ are $\varepsilon_1+(d-\ell),\dots,\varepsilon_p+(d-\ell)$.

(ii) If $\eta_1, \dots, \eta_q$ are the left minimal indices of $P(\lambda)$, then the left minimal indices of $L(\lambda)$ are $\eta_1, \dots, \eta_q$.

By considering the transpose of $P(\lambda)$, we can construct a strong ℓ-ification, $L(\lambda)^T$, of $P(\lambda)^T$, satisfying the statement of Theorem 4.7. Since the left (respectively, right) minimal indices of a matrix polynomial $P(\lambda)^T$ correspond to the right (resp., left) minimal indices of $P(\lambda)$, we get the analogue of Theorem 4.7.

Theorem 4.8. *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree d. Then for any ℓ such that $m \cdot d = \ell \cdot (\widehat{m} + m)$, we can construct a strong ℓ-ification $L(\lambda)$ of size $(\widehat{m} + m) \times (\widehat{m} + n)$, which has the same right minimal indices as $P(\lambda)$, and the left minimal indices are those of $P(\lambda)$ all increased by $d - \ell$. More precisely:*

(i) If $\varepsilon_1, \dots, \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then the right minimal indices of $L(\lambda)$ are $\varepsilon_1, \dots, \varepsilon_p$.

(ii) If $\eta_1, \dots, \eta_q$ are the left minimal indices of $P(\lambda)$, then the left minimal indices of $L(\lambda)$ are $\eta_1 + (d - \ell), \dots, \eta_q + (d - \ell)$.

Remark 4.9. Note that the size of the strong ℓ-ifications in the statement of Theorem 4.7 are $(\widehat{n} + m) \times (\widehat{n} + n)$, where $\widehat{n} = n(d - \ell)/\ell$. Similarly, the size of the strong ℓ-ifications in Theorem 4.8 is $(\widehat{m} + m) \times (\widehat{m} + n)$, with $\widehat{m} = m(d - \ell)/\ell$. For large values of m, n and ℓ, this quantity may be much smaller than the size of all families of strong linearizations of matrix polynomials with size $m \times n$ and degree d known so far. In particular, the size of the Fielder companion linearizations [7] is at least $((d-1)s + m) \times ((d-1)s + n)$, where $s = \min\{m, n\}$. This is the only family of companion linearizations known so far which is valid for rectangular matrix polynomials, and includes the classical first and second Frobenius companion forms, that have size $((d-1)n+m) \times dn$ and $dm \times ((d-1)m+n)$, respectively. Note also that for square $n \times n$ matrix polynomials of degree d the size of the strong ℓ-ifications in Theorems 4.7 and 4.8 is $(nd)/\ell \times (nd)/\ell$.

We warn the reader that the existence of strong ℓ-ifications as those in Theorems 4.7 and 4.8 follows immediately from [10, Th. 4.10] and that the main contribution of this work is to present an explicit procedure for their construction using only the coefficients of the given polynomial.

We want to emphasize that, though the strong ℓ-ifications $L(\lambda)$ constructed in Section 4.1 are not always companion forms, each of them is a general construction valid for all matrix polynomials in the conditions of the statements of Theorems 4.7 and 4.8. Example 4.1 illustrates one of these constructions for quadratifications (i.e., ℓ-ifications with $\ell = 2$) of cubic $m \times 2$ matrix polynomials. We remark that Example 4.1 is the first known concrete example of a strong quadratification of an arbitrary $m \times 2$ cubic matrix polynomial. In fact, Example 4.1 is the first concrete example of a strong ℓ-ification where ℓ does not divide d.

Example 4.1. Let $P(\lambda)$ be an $m \times 2$ cubic matrix polynomial. We explicitly describe how to construct a quadratification $L(\lambda)$ of $P(\lambda)$ following **Steps 1–2** in Section 4.1. Note that in this setting $n = 2, d = 3$, and $\ell = 2$, so that ℓ divides nd, but ℓ does not divide d. According to the notation above, we have $\widehat{n} = 1$ and $\widehat{d} = 1$.

Set $P(\lambda) = \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0$, as in (2). Following the zigzag construction in [9, Th. 5.1, Th. 6.1], the matrices $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ in **Step 1** in Section 4.1 can be taken as:

$$\widehat{L}(\lambda) = \begin{bmatrix} \lambda^2 & -\lambda & 1 \end{bmatrix}, \qquad \text{and} \qquad \widehat{N}(\lambda)^T = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

In particular,

$$\widehat{N}_{\widehat{d}}^T = \widehat{N}_1^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We look for a quadratic $m \times 3$ matrix polynomial $\widetilde{L}(\lambda) = \lambda^2 \widetilde{L}_2 + \lambda \widetilde{L}_1 + \widetilde{L}_0$ satisfying the convolution equation (15). We start by choosing $\widetilde{L}_2$ as a solution of (17), for instance:

$$\widetilde{L}_2 = \begin{bmatrix} 0_{m\times 1} & | & P_3 \end{bmatrix},$$

and then the convolution equation (18) for $\widetilde{L}_0, \widetilde{L}_1$ becomes

$$\left[\begin{array}{c|c} \widetilde{L}_0 & \widetilde{L}_1 \end{array}\right] \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c|c} P_0 & P_1 & P_2 \end{array}\right] - \left[\begin{array}{c|c} 0_{m\times 1} & P_3 \end{array}\right] \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The block Toeplitz matrix in the left hand side of this equation is invertible, so the equation has a unique solution, which is

$$\left[\begin{array}{c|c} \widetilde{L}_0 & \widetilde{L}_1 \end{array}\right] = \left[\begin{array}{cc|c} P_0 & P_1 & P_2 - P_3 e_1 e_2^T \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

that is:

$$\widetilde{L}_0 = \left[\begin{array}{cc} P_0 & P_1 e_2 - P_2 e_1 \end{array}\right], \quad \text{and} \quad \widetilde{L}_1 = \left[\begin{array}{cc} P_1 e_1 - P_0 e_2 & P_2 - P_3 e_1 e_2^T \end{array}\right],$$

where $\left[\begin{array}{cc} e_1 & e_2 \end{array}\right] = I_2$ is the 2×2 identity matrix.

With all this information, the quadratification of $P(\lambda)$ we obtain is:

$$L(\lambda) = \left[\begin{array}{c} \widehat{L}(\lambda) \\ \widetilde{L}(\lambda) \end{array}\right] = \lambda^2 \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & P_3 e_1 & P_3 e_2 \end{array}\right] + \lambda \left[\begin{array}{ccc} 0 & -1 & 0 \\ P_1 e_1 - P_0 e_2 & P_2 e_1 & P_2 e_2 - P_3 e_1 \end{array}\right] + \left[\begin{array}{ccc} 0 & 0 & 1 \\ P_0 e_1 & P_0 e_2 & P_1 e_2 - P_2 e_1 \end{array}\right].$$

4.4. The special case where ℓ divides d: Companion ℓ-ifications

The first known construction of companion ℓ-ifications was introduced in [8, p. 304, Th. 5.7, Th. 5.8] for the case where ℓ divides d. These companion strong ℓ-ifications where termed *Frobenius-like companion forms of degree ℓ*, because they resemble very much the first and second Frobenius companion linearizations. We are going to show that our construction allows us to get a companion form which is closely related to those.

Before proceeding, let us recall the basic features of companion forms of degree ℓ, or companion ℓ-ifications [8, Def. 5.1]. Companion ℓ-ifications are uniform templates for constructing matrix polynomials of degree ℓ, $\sum_{i=0}^{\ell} \lambda^i X_i$, which are strong ℓ-ifications for any matrix polynomial $P(\lambda) = \sum_{i=0}^{d} \lambda^i P_i$ of a fixed degree and size, and such that, for $i = 0, 1, \ldots, \ell$, each entry of the coefficient matrix X_i is a scalar-valued function of the entries of $[P_0, P_1, \ldots, P_d]$. These scalar-valued functions are either a constant or a constant multiple of just one of the entries of $[P_0, P_1, \ldots, P_d]$. In particular, note that sums of several different entries of $[P_0, P_1, \ldots, P_d]$ cannot appear in any coefficient X_i. As a consequence, the strong quadratification $L(\lambda)$ constructed in Example 4.1 is not a companion ℓ-ification, since some entries of the coefficients of the zero and the first degree terms of $L(\lambda)$ are sums of entries of $[P_0, P_1, \ldots, P_d]$.

The companion forms known in the literature include, among others, the family of Fiedler linearizations [5, 7] (which comprise the first and second Frobenius companion linearizations), and the companion ℓ-ifications introduced in [8]. It is worth emphasizing that all companion forms known so far share a common interesting feature, namely, their coefficients are block-partitioned matrices, whose nonzero blocks are either coefficients of the matrix polynomial or identity matrices, multiplied in some cases by -1.

When ℓ divides d we can write $d = k\ell$, with $k \geq 1$. It then follows from (13) that $\widehat{n} = n(k-1)$, so $\widehat{L}(\lambda)$ in **Step 1** in Section 4.1 has size $n(k-1) \times nk$ and $\widehat{N}(\lambda)$ has size $n \times nk$. The row degrees of $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ are, respectively, $\eta_j = \ell$, for $j = 1, \ldots, n(k-1)$, and $\varepsilon_i = \widehat{d} = (k-1)\ell$, for $i = 1, \ldots, n$. We can choose these dual minimal bases $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ as follows :

$$\widehat{L}(\lambda)=\left(\begin{bmatrix} \lambda^{\ell} & -1 & & \\ & \ddots & \ddots & \\ & & \lambda^{\ell} & -1 \end{bmatrix}_{(k-1)\times k}\right)\otimes I_n, \quad \text{and} \quad \widehat{N}(\lambda)^T=\begin{bmatrix} 1 \\ \lambda^{\ell} \\ \lambda^{2\ell} \\ \vdots \\ \lambda^{(k-1)\ell} \end{bmatrix}\otimes I_n. \tag{22}$$

Following the construction of Section 4.1, we take:

$$\widetilde{L}_{\ell}=\left[\begin{array}{cccc} 0 & \dots & 0 & P_d \end{array}\right]\in\mathbb{F}^{m\times nk}, \tag{23}$$

where each zero block in (23) has size $m\times n$, and the reduced (invertible) block-Toeplitz matrix of equation (18), namely

$$T_{red}:=\begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & & \ddots & & \ddots & \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix}\in\mathbb{F}^{nd\times nd},$$

is then just a permutation matrix, since $\widehat{N}_i^T=e_{(i/\ell)+1}\otimes I_n$ for $i=0,\ell,2\ell,\dots,(k-1)\ell=\widehat{d}$, and $\widehat{N}_i^T=0$ otherwise, so each row has exactly one entry equal to 1 and the remaining ones are zero. Here $e_j\in\mathbb{F}^{k\times 1}$ stands for the j-th canonical vector.

The resulting ℓ-ification follows a similar pattern to the one of the *first Frobenius-like companion ℓ-ification* $C_1^{\ell}(\lambda)$ introduced in [8, p. 304]. To see this, note first that the matrix T_{red} above is equal to:

$$T_{red}=\left[\begin{array}{cccc} I_{\ell}\otimes e_1 & I_{\ell}\otimes e_2 & \cdots & I_{\ell}\otimes e_k \end{array}\right]\otimes I_n,$$

which is a permutation matrix as mentioned before, so its inverse is equal to its transpose. Hence:

$$T_{red}^{-1}=\begin{bmatrix} I_{\ell}\otimes e_1^T \\ I_{\ell}\otimes e_2^T \\ \vdots \\ I_{\ell}\otimes e_k^T \end{bmatrix}\otimes I_n.$$

Moreover, since

$$\widehat{N}_i^T=\left[\begin{array}{c} * \\ \hline 0_{n\times n} \end{array}\right], \qquad \text{for } i=1,\dots,\widehat{d}-1,$$

where the block $*$ is not of relevance in the argument, from (23) we get

$$\widetilde{L}_{\ell}\left[\begin{array}{cccc} 0\ \dots\ 0 & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}-1}^T \end{array}\right]=0.$$

Then (18) reads

$$\left[\ \widetilde{L}_0 \mid \widetilde{L}_1 \mid \dots \mid \widetilde{L}_{\ell-1}\ \right] = \left[\ P_0 \quad P_1 \quad \dots \quad P_{d-1}\ \right] \left(\begin{bmatrix} I_\ell \otimes e_1^T \\ I_\ell \otimes e_2^T \\ \vdots \\ I_\ell \otimes e_k^T \end{bmatrix} \otimes I_n \right)$$

$$= \left[\ P_0\ P_1 \dots\ P_{\ell-1} \mid P_\ell\ P_{\ell+1}\ \dots\ P_{2\ell-1} \mid \dots \mid P_{(k-1)\ell}\ P_{(k-1)\ell+1}\ \dots P_{k\ell-1}\ \right]$$

$$\left[\begin{array}{c|c|c|c} e_1^T \otimes I_n & 0 & & 0 \\ 0 & e_1^T \otimes I_n & & \vdots \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & & e_1^T \otimes I_n \\ \hline e_2^T \otimes I_n & 0 & & 0 \\ 0 & e_2^T \otimes I_n & & \vdots \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & & e_2^T \otimes I_n \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline e_k^T \otimes I_n & 0 & & 0 \\ 0 & e_k^T \otimes I_n & & \vdots \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & & e_k^T \otimes I_n \end{array}\right]$$

$$\begin{aligned} &= \big[P_0(e_1^T \otimes I_n) + P_\ell(e_2^T \otimes I_n) + \dots + P_{(k-1)\ell}(e_k^T \otimes I_n)| \\ &\quad P_1(e_1^T \otimes I_n) + P_{\ell+1}(e_2^T \otimes I_n) + \dots + P_{(k-1)\ell+1}(e_k^T \otimes I_n)| \cdots \\ &\quad | \ P_{\ell-1}(e_1^T \otimes I_n) + P_{2\ell-1}(e_2^T \otimes I_n) + \dots + P_{k\ell-1}(e_k^T \otimes I_n)\big] \\ &= \left[\ P_0\ P_\ell\ \dots\ P_{(k-1)\ell} \ \mid\ P_1\ P_{\ell+1}\ \dots\ P_{(k-1)\ell+1}\ \mid \dots \mid\ P_{\ell-1}\ P_{2\ell-1}\ \dots\ P_{k\ell-1}\ \right]. \end{aligned}$$

Then, the strong ℓ-ification that we get is

$$L(\lambda) = \begin{bmatrix} \lambda^\ell I_n & -I_n & & \\ & \ddots & \ddots & \\ & & \lambda^\ell I_n & -I_n \\ B_0(\lambda) & \dots & B_{k-2}(\lambda) & B_{k-1}(\lambda) \end{bmatrix},$$

where $B_j(\lambda) = P_{j\ell} + \lambda P_{j\ell+1} + \dots + \lambda^{\ell-1} P_{(j+1)\ell-1}$, for $j = 0, 1, \dots, k-2$, and $B_{k-1}(\lambda) = P_{(k-1)\ell} + \lambda P_{(k-1)\ell+1} + \dots + \lambda^{\ell-1} P_{k\ell-1} + \lambda^\ell P_{k\ell}$.

Here we display the construction for $\ell = 2$ and $d = 6$:

$$L(\lambda) = \begin{bmatrix} \lambda^2 I_n & -I_n & \\ & \lambda^2 I_n & -I_n \\ P_0 + \lambda P_1 & P_2 + \lambda P_3 & P_4 + \lambda P_5 + \lambda^2 P_6 \end{bmatrix},$$

since

$$\left[\ \widetilde{L}_0 \mid \widetilde{L}_1\ \right] = \left[\ P_0 \quad P_2 \quad P_4 \mid P_1 \quad P_3 \quad P_5\ \right],$$

which is a block permutation of the coefficients of $P(\lambda)$.

The construction of the dual minimal bases $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ in (22) is, after exchanging the roles of $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$, a variation of the construction in [9, Th. 6.1] in terms of direct sums of dual zigzag matrices. More precisely, the construction in [9] gives a pair of dual minimal bases $N_1(\lambda) \in \mathbb{F}[\lambda]^{n \times nk}$ and

$N_2(\lambda) \in \mathbb{F}[\lambda]^{n(k-1)\times nk}$, with $N_2(\lambda)N_1(\lambda)^T = 0$, where:

$$N_2(\lambda) = I_n \otimes \left(\begin{bmatrix} \lambda^\ell & 1 & & \\ & \ddots & \ddots & \\ & & \lambda^\ell & 1 \end{bmatrix}_{(k-1)\times k} \Sigma_k \right), \quad \text{and} \quad N_1(\lambda)^T = I_n \otimes \begin{bmatrix} 1 \\ \lambda^\ell \\ \lambda^{2\ell} \\ \vdots \\ \lambda^{(k-1)\ell} \end{bmatrix},$$

where Σ_k is the alternating signs matrix of size $k \times k$, $\Sigma_k := \mathrm{diag}(1, -1, 1, \ldots, (-1)^{k-1})$. Now, since, first,

$$\begin{bmatrix} \lambda^\ell & -1 & & \\ & \ddots & \ddots & \\ & & \lambda^\ell & -1 \end{bmatrix}_{(k-1)\times k} = \Sigma_{k-1} \begin{bmatrix} \lambda^\ell & 1 & & \\ & \ddots & \ddots & \\ & & \lambda^\ell & 1 \end{bmatrix}_{(k-1)\times k} \Sigma_k$$

and, second, the Kronecker products $A \otimes B$ and $B \otimes A$ of two arbitrary matrices A, B are equivalent by permutations [16, Cor. 4.3.10], comparing with (22) and using that $(CD) \otimes I_n = (C \otimes I_n)(D \otimes I_n)$ (see [16, Lemma 4.2.10]), we get

$$\widehat{L}(\lambda) = (\Sigma_{k-1} \otimes I_n)\Pi_1 N_2(\lambda)\Pi, \quad \text{and} \quad \widehat{N}(\lambda)^T = \Pi^T N_1(\lambda)^T \Pi_2,$$

for some permutation matrices Π, Π_1, and Π_2 (see formula (4.3.11) in [16]). Hence, $\widehat{L}(\lambda)$ and $\widehat{N}(\lambda)$ are permutationally equivalent to the construction in [9], as explained in the paragraph right after Example 3.2.

5. Concluding remarks

We have presented a general construction of strong ℓ-ifications for arbitrary matrix polynomials with size $m \times n$ and degree d which is valid in the case where ℓ divides one of nd or md. The contribution of this construction relies not only on the fact that it is a general construction of strong ℓ-ifications using only simple operations on the coefficients of the matrix polynomial in a more general setting than the one considered so far (namely, when ℓ divides d), but also on the fact that there is a simple relationship between the left and right minimal indices of the ℓ-ification and the ones of the matrix polynomial. Hence, we can easily recover all the spectral information of the matrix polynomial from the spectral information of the ℓ-ification. More precisely, in the first case (that is, when ℓ divides nd), the left minimal indices of the ℓ-ification coincide with those of the polynomial, whereas the right minimal indices of the ℓ-ification are those of the polynomial increased by a fixed quantity equal to $d-\ell$ (each). In the case where ℓ divides md, the situation is the same one after exchanging the roles of the left and right minimal indices, namely, the right minimal indices of the ℓ-ification coincide with the ones of the polynomial, and the left minimal indices of the ℓ-ification are the ones of the polynomial increased by $d-\ell$ (each). Our construction allows for some flexibility that depends on the choice of some of the ingredients involved in the construction, so that it gives not only a single ℓ-ification, but a family of strong ℓ-ifications.

When particularizing to the case when ℓ divides d, a particular choice of the ingredients just mentioned above allows us to derive a companion ℓ-ification, which resembles the ones presented recently in [8]. That ones were the only companion ℓ-ifications known so far (for arbitrary ℓ dividing d). It is likely that other possible choices of these ingredients may give different companion forms. The analysis of this issue remains as an open question and will be the subject of further research

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