Perturbation theory of block LU factorization

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Setting the Problem

The perturbation theory of LU and Cholesky factorizations is well understood. Norm and componentwise bounds have been obtained by Barrlund, Sun, Stewart, Chang, Paige and others in the last years.

There are not perturbation results for block LU factorization. Block LU factorization arises in several applications. One of the most important is the factorization of Hermitian indefinite matrices, where the diagonal pivoting method leads to a block LDL^* factorization.

The diagonal pivoting method with partial pivoting (Bunch and Kaufman) is used in LAPACK for solving Hermitian indefinite linear systems. Bunch and Parlett's complete pivoting strategy has been used by several authors to get rank-revealing factorizations of Hermitian indefinite matrices. This is a previous step to compute accurate eigenvalues and eigenvectors of Hermitian matrices. The accuracy of the computed factorization may be estimated by combining backward error results and perturbation bounds.

Goal: Get norm and componentwise perturbation bounds for block LU and LDL^* factorizations.

Block notation

A block LU factorization of an $n \times n$ matrix A is as follows:

$$\underbrace{\begin{bmatrix} A_{11} & \cdots & A_{1p} \\ A_{21} & \cdots & A_{2p} \\ \vdots & \vdots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} I & & & \\ L_{21} & I & & \\ \vdots & & \ddots & \\ L_{p1} & \cdot & L_{p,p-1} & I \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} U_{11} & U_{12} & \cdot & U_{1p} \\ & U_{22} & & \cdot \\ & & \ddots & U_{p-1,p} \\ & & & U_{pp} \end{bmatrix}}_{U}$$

The matrices L_{ij}, U_{ij} and A_{ij} have dimensions $n_i \times n_j$, and $\sum_{i=1}^{p} n_i = n$. These dimensions remain fixed when the matrix A is perturbed. The matrices U_{ii} are not, in general, upper triangular.

The block strict lower and upper triangular parts of A are denoted by:

$$A_L = \begin{bmatrix} 0 & & & \\ A_{21} & 0 & & \\ \vdots & & \ddots & \\ A_{p1} & \cdots & A_{p,p-1} & 0 \end{bmatrix}; \quad A_U = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ & A_{22} & & \vdots \\ & & \ddots & A_{p-1,p} \\ & & & A_{pp} \end{bmatrix}$$

and the block diagonal part by:

$$A_D = \mathrm{diag}(A_{11}, \ldots, A_{pp}).$$

Finally, $\rho(B)$ denotes the spectral radius of a matrix B, and $|B|$ is the entrywise absolute value of B.

Perturbation bounds for block LU factorization

Theorem: Let the nonsingular $n \times n$ complex matrix A have the block LU factorization $A = LU$ appearing in the previous page. Let us consider the matrix $A + E$, and define $F = L^{-1}EU^{-1}$. If $\rho(|F|) < 1$ then

1. The matrix $A + E$ is nonsingular and has a unique block LU factorization, $A + E = \widetilde{L}\widetilde{U}$, with the same block dimensions as those in $A = LU$.

2. $$\left|\widetilde{L} - L\right| \leq |L| \left(|F|\,(I - |F|)^{-1}\right)_L,$$

   and

   $$\left|\widetilde{U} - U\right| \leq \left(|F|\,(I - |F|)^{-1}\right)_U |U|.$$

3. If moreover $\|\cdot\|$ is an absolute and consistent matrix norm and $\|F\| < 1$ then

   $$\max\left\{\frac{\|\widetilde{L} - L\|}{\|L\|}, \frac{\|\widetilde{U} - U\|}{\|U\|}\right\} \leq \frac{\|F\|}{1 - \|F\|}.$$

Remarks on the condition $\rho(|F|) < 1$

The condition $\rho(|F|) < 1$ implies that the block factorization of $A + E$ exists and is unique. Notice

$$A + E = L(I + L^{-1}EU^{-1})U$$

and

$$\rho(F(1:k, 1:k)) \leq \rho(|F(1:k, 1:k)|) \leq \rho(|F|).$$

Then the condition $\rho(|L^{-1}EU^{-1}|) < 1$ implies that all the leading principal submatrices of $(I + L^{-1}EU^{-1})$ are nonsingular. As a consequence $(I + L^{-1}EU^{-1})$ has a unique block LU factorization with the same block dimensions of $A = LU$. Therefore

$$A + E = L(I + L^{-1}EU^{-1})U = (L\mathcal{L})(\mathcal{U}U) \equiv \widetilde{L}\widetilde{U},$$

is the unique block LU factorization of $A + E$ with the same block dimensions of $A = LU$.

For an absolute and consistent matrix norm $\rho(|F|) \leq \|F\|$. Then the condition $\|F\| < 1$ implies $\rho(F) < 1$, but $\rho(F) < 1$ remains valid for a wider set of perturbations.

Remark on practical bounds

Assuming that $\| \, |L^{-1}| \, |E| \, |U^{-1}| \, \| < 1$, we can replace $|F|$ by $|L^{-1}| \, |E| \, |U^{-1}|$, and $\|F\|$ by $\|L^{-1}\| \, \|E\| \, \|U^{-1}\|$ in the bounds of the previous page. This is more useful in practice, when the only information on E is a bound on $|E|$ or $\|E\|$.

Block LDL^* factorization of Hermitian matrices

Given a Hermitian indefinite matrix B, the following block factorization is frequently used:

$$PBP^T = LDL^*,$$

where,

- P is a permutation matrix.
- L is unit lower triangular.
- D is block diagonal with diagonal blocks of dimension 1×1 or 2×2.
- The 2×2 diagonal blocks of D are Hermitian indefinite matrices.
- The diagonal blocks of L corresponding to the 2×2 blocks of D are 2×2 identity matrices.

This factorization method is usually called the diagonal pivoting method.

Three pivoting strategies to choose the permutation matrix P are available: complete pivoting (Bunch & Parlett), partial pivoting (Bunch & Kaufman, implemented in LAPACK), and rook pivoting (Ashcraft, Grimes, and Lewis).

Perturbation bounds for LDL^* factorization

Theorem: Let the nonsingular $n \times n$ Hermitian matrix A have the block factorization $A = LDL^*$. Let us consider the Hermitian matrix $A + E$, and define $F = L^{-1}EL^{-*}D^{-1}$. If $\rho(|F|) < 1$ then

1. The matrix $A + E$ is nonsingular and has a unique block LDL^* factorization, $A + E = \widetilde{L}\widetilde{D}\widetilde{L}^*$, with the same block dimensions as those in $A = LDL^*$. Moreover, let us denote the block diagonal matrices D and $\widetilde{D}$ by: $D = \mathrm{diag}(D_{11}, \ldots, D_{pp})$ and $\widetilde{D} = \mathrm{diag}(\widetilde{D}_{11}, \ldots, \widetilde{D}_{pp})$. Then, if D_{ii} and $\widetilde{D}_{ii}$ are 1×1 both have the same sign, and, if D_{ii} and $\widetilde{D}_{ii}$ are 2×2 both are Hermitian indefinite matrices.

2. $$\left|\widetilde{L} - L\right| \leq |L| \left(|F| \, (I - |F|)^{-1} \right)_L ,$$
   and
   $$\left|\widetilde{D} - D\right| \leq \left(|F| \, (I - |F|)^{-1} \right)_D |D|.$$

3. If moreover $\|\cdot\|$ is an absolute and consistent matrix norm and $\|F\| < 1$ then
   $$\max\left\{ \frac{\|\widetilde{L} - L\|}{\|L\|}, \frac{\|\widetilde{D} - D\|}{\|D\|} \right\} \leq \frac{\|F\|}{1 - \|F\|}.$$

How are these bounds proved?

They are proved by using series expansions. Consider again

$$A + E = L(I + \underbrace{L^{-1}EU^{-1}}_{F})U = (L\mathcal{L})(\mathcal{U}U) \equiv \widetilde{L}\widetilde{U}.$$

Theorem. Let F be an $n \times n$ matrix with $\rho(|F|) < 1$ then:

1. $I + F$ has a unique block LU factorization:
   $I + F = \mathcal{L}\mathcal{U}$.

2. $$\mathcal{L} = \sum_{k=0}^{\infty} \mathcal{L}_k \quad and \quad \mathcal{U} = \sum_{k=0}^{\infty} \mathcal{U}_k,$$
   with $\mathcal{L}_0 = I$, $\mathcal{U}_0 = I$, $\mathcal{L}_1 = F_L$, $\mathcal{U}_1 = F_U$ and for $k \geq 2$:
   $$\mathcal{L}_k = (-\mathcal{L}_1\mathcal{U}_{k-1} - \mathcal{L}_2\mathcal{U}_{k-2} \cdots - \mathcal{L}_{k-1}\mathcal{U}_1)_L,$$
   $$\mathcal{U}_k = (-\mathcal{L}_1\mathcal{U}_{k-1} - \mathcal{L}_2\mathcal{U}_{k-2} \cdots - \mathcal{L}_{k-1}\mathcal{U}_1)_U.$$

3. $$|\mathcal{L}_k + \mathcal{U}_k| \leq |F|^k \quad \text{for} \quad k \geq 1.$$

   Therefore $|\mathcal{L}_k| \leq \left(|F|^k\right)_L$ and $|\mathcal{U}_k| \leq \left(|F|^k\right)_U$.

The bound in the last item is the fundamental tool to get strict perturbation bounds.

A few explicit terms:

Explicit expressions for the terms in the series can be obtained

$$\begin{aligned}
\mathcal{L}_1 + \mathcal{U}_1 &= F \\
\mathcal{L}_2 + \mathcal{U}_2 &= -F_L F_U \\
\mathcal{L}_3 + \mathcal{U}_3 &= F_L (F_L F_U)_U + (F_L F_U)_L F_U \\
\mathcal{L}_4 + \mathcal{U}_4 &= -F_L (F_L (F_L F_U)_U)_U - F_L ((F_L F_U)_L F_U)_U \\
&\quad -(F_L F_U)_L (F_L F_U)_U - (F_L (F_L F_U)_U)_L F_U \\
&\quad -((F_L F_U)_L F_U)_L F_U
\end{aligned}$$

Proving the bounds:

The Theorem in previous page implies

$$\widetilde{L} = L\mathcal{L} = L \sum_{k=0}^{\infty} \mathcal{L}_k,$$

then

$$\begin{aligned}
|\widetilde{L} - L| \le |L| \sum_{k=1}^{\infty} |\mathcal{L}_k| &\le |L| \left(\sum_{k=1}^{\infty} |F|^k \right)_L \\
&= |L| \left(|F|(I - |F|)^{-1} \right)_L
\end{aligned}$$

$$\Longrightarrow$$