Fast and accurate computations with some classes of Quasiseparable Matrices

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Abstract

- There exists a long tradition connecting bidiagonal factorizations with fast algorithms for **solving linear systems** whose coefficient matrix has a particular structure.
- Some classic and new examples are:

Matrices	Algorithms	Error analysis
Vandermonde	Björck-Pereyra (1970)	Higham (1987)
Cauchy	Gohberg-Koltracht (1990)	Boros-Kailath-Olshevsky (1999)
Quasi-separable	Gemignani (2008)	This Talk

- For Vandermonde and Cauchy satisfactory error bounds are only obtained in **Totally Nonnegative (TN)** case. **Same for quasiseparable**.
- **We only deal with $(1,1)$-quasiseparable matrices** (Gemignani general case) and, in addition error analysis, we obtain simple characterizations of TN $(1,1)$-quasiseparable matrices.

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Outline

1. Introduction and goals
2. Neville elimination, TN and quasiseparable matrices
3. Totally Nonnegative (TN) quasiseparable matrices
4. Error analysis for quasiseparable linear systems
5. Other results in accurate quasiseparable computations
6. Conclusions and future work

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Quasiseparable matrices (I): Definition

Definition

A square matrix C is **quasiseparable of order** $(1,1)$ if

- every submatrix of C entirely located in the **strictly lower or upper triangular part** of C **have rank at most** 1, and
- at least one of these submatrices has rank equal to 1.

Remark

In this talk, for brevity, the simple term **quasiseparable** is used instead of $(1,1)$-quasiseparable.

It is necessary and sufficient that the following submatrices have rank at most 1:

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A square matrix G is **Green's quasiseparable of order** $(1,1)$ if

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Parametrization of quasiseparable matrices

Theorem (Eidelman and Gohberg (1999))

The set of $n \times n$ quasiseparable matrices can be parameterized in terms of $7n - 8$ independent parameters or generators.

Example (Every 5×5 quasiseparable matrix is of the form)

$$C = \left[\begin{array}{cc|c|cc} \boxed{d_1} & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & \boxed{d_2} & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ \hline p_3 a_2 q_1 & p_3 q_2 & \boxed{d_3} & g_3 h_4 & g_3 b_4 h_5 \\ \hline p_4 a_3 a_2 q_1 & p_4 a_3 q_2 & p_4 q_3 & \boxed{d_4} & g_4 h_5 \\ p_5 a_4 a_3 a_2 q_1 & p_5 a_4 a_3 q_2 & p_5 a_4 q_3 & p_5 q_4 & \boxed{d_5} \end{array}\right]$$

Parametrization of Green's quasiseparable

Theorem

The set of $n \times n$ Green's quasiseparable matrices can be parameterized in terms of $6n-2$ parameters with **the constraints $p_i q_i = g_i h_i$ for $i = 1 : n$.**

Example (Every 5×5 Green's quasiseparable matrix is of the form)

$$G = \left[\begin{array}{cc|c|cc} \boxed{p_1q_1} & g_1b_1h_2 & g_1b_1b_2h_3 & g_1b_1b_2b_3h_4 & g_1b_1b_2b_3b_4h_5 \\ p_2a_1q_1 & \boxed{p_2q_2} & g_2b_2h_3 & g_2b_2b_3h_4 & g_2b_2b_3b_4h_5 \\ \hline p_3a_2a_1q_1 & p_3a_2q_2 & \boxed{p_3q_3} & g_3b_3h_4 & g_3b_3b_4h_5 \\ \hline p_4a_3a_2a_1q_1 & p_4a_3a_2q_2 & p_4a_3q_3 & \boxed{p_4q_4} & g_4b_4h_5 \\ p_5a_4a_3a_2a_1q_1 & p_5a_4a_3a_2q_2 & p_5a_4a_3q_3 & p_5a_4q_4 & \boxed{p_5q_5} \end{array}\right]$$

Highlights on research on quasiseparable matrices

- A main line of research has been the **development of structured fast algorithms by using the low number of parameters defining this class**. There are many algorithms an their costs are:

Problem	**Cost of traditional algorithms**	**Cost of structured quasiseparable algs.**
systems of equations	$O(n^3)$	$O(n)$
eigenvalues	$O(n^3)$	$O(n^2)$
singular values	$O(n^3)$	$O(n^2)$

- **The stability of these algorithms is not guaranteed and**, as far as we know, **error analysis have not been developed** even for the most simple cases.

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Totally Nonnegative (TN) matrices

Definition

A matrix A is totally nonnegative if all its minors are nonnegative.

TN and accurate computations

TN matrices are a classical set of matrices that appear in many applications and are amenable for **guaranteed accurate computations** (Boros-Kailath-Olshevsky, Demmel, D., Koev, Higham, Peña,...)

An historical TN-quasiseparable connection

In *Oscillation Matrices (1941)* by **Gantmacher and Krein** a **particular class of symmetric Green's quasiseparable** matrices is considered. These are called **single-pair** matrices and are defined as

$$S = \begin{bmatrix} p_1 q_1 & q_1 p_2 & q_1 p_3 & \cdots \\ p_2 q_1 & p_2 q_2 & q_2 p_3 & \cdots \\ p_3 q_1 & p_3 q_2 & p_3 q_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \text{tril}(pq^T) + \text{strict-triu}(qp^T),$$

where all the numbers $p = [p_1, \ldots, p_n]^T, q = [q_1, \ldots, q_n]^T$ are nonzero.

These matrices are obtained from general Green's quasiseparable matrices by taking $a_i = b_i = 1$, $g_i = q_i$, and $h_i = p_i$.

Theorem (Gantmacher and Krein (1941))

S is **TN** *if and only if all the numbers $p_1, \ldots, p_n, q_1, \ldots, q_n$ have the same sign and*

$$\frac{q_1}{p_1} \le \frac{q_2}{p_2} \le \cdots \le \frac{q_n}{p_n}$$

Goals of the talk

- We initiate the study of **stability of fast algorithms for quasiseparable matrices**, by presenting **rounding error analysis** of the solution of **quasiseparable linear systems** by using a **bidiagonal factorization** followed by a **Björck-Pereyra** type algorithm.
- We prove that this algorithm is **componentwise backward stable in a strong sense** in the **TN-quasiseparable** case.
- For **TN-Green's quasiseparable matrices simple forward error bounds for this algorithm are presented and we show that it is frequently accurate**, independently of the traditional condition number of the matrix.
- We characterize **the set of nonsingular TN-quasiseparable** matrices through **the quasiseparable generators, the entries and the bidiagonal factorizations**. This extends Gantmacher and Krein's result.
- We briefly mention other results on accurate computations with quasiseparable matrices.

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Outline

Brief summary on Neville elimination (I)

- It is a classic **procedure to create zeros in a matrix by adding to a row (resp. column) a multiple of the previous row (resp. column)**.
- **Without interchanges**, it was carefully analyzed by **Gasca and Peña** in a series of seminal papers in the 90s. In particular, its **matricial description** in terms of **bidiagonal factorizations** and its fundamental relationship with **total nonnegativity** were established.

Theorem (Gasca and Peña (1994))

A nonsingular matrix A is TN if and only if complete Neville elimination can be performed on A without row or column exchanges, with nonnegative multipliers and positive diagonal pivots.

- Neville elimination with interchanges was generalized for rectangular and singular (TN) matrices by **Gassó and Torregrosa** (2002, 04, 08).

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Brief summary on Neville elimination (II)

- **Neville elimination without exchanges adapts very well to the quasiseparable structure** (Gemignani 2008).

- In this talk, **Neville elimination is never applied numerically**. **It is a theoretical way to get formulae, in terms of the generators, for the bidiagonal factors of the matrix**. These formulae are then used
  1. **to develop fast algorithms,**
  2. **to perform their error analysis,**
  3. and, **to establish simple characterization of TN-quasiseparable matrices.**

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Bidiagonal factorizations (I)

Theorem (Gasca and Peña (1996))

If complete Neville elimination runs without exchanges on a **nonsingular $n \times n$ square matrix A** *then*

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D\, U^{(n-1)} \cdots U^{(2)} U^{(1)},$$

where **D is diagonal** *and*

$$L^{(k)} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & \ell_1^{(k)} & 1 & & & \\ & & \ell_2^{(k)} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ell_k^{(k)} & 1 \end{bmatrix}$$

This is known as a **bidiagonal factorization** *of A and* **all the factors amount to n^2 nontrivial entries**.

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This is known as a **bidiagonal factorization** *of* A *and* **all the factors amount to** n^2 **nontrivial entries**.

Bidiagonal factorizations (II)

Example (Bidiagonal factorization of a 5×5 matrix)

$$A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} \mathbf{D}\, U^{(4)} U^{(3)} U^{(2)} U^{(1)},$$

$$\mathbf{D} = \begin{bmatrix} \times & & & & \\ & \times & & & \\ & & \times & & \\ & & & \times & \\ & & & & \times \end{bmatrix}$$

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Example (Bidiagonal factorization of a 5×5 matrix)

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$$\mathbf{L}^{(4)} = \begin{bmatrix} 1 & & & & \\ \times & 1 & & & \\ & \times & 1 & & \\ & & \times & 1 & \\ & & & \times & 1 \end{bmatrix} \qquad \mathbf{U}^{(4)} = \begin{bmatrix} 1 & \times & & & \\ & 1 & \times & & \\ & & 1 & \times & \\ & & & 1 & \times \\ & & & & 1 \end{bmatrix}$$

Bidiagonal factorizations (II)

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Bidiagonal factorizations (II)

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Bidiagonal factorizations (II)

Example (Bidiagonal factorization of a 5×5 matrix)

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Theorem (Gasca and Peña (1996))

A nonsingular matrix A is TN if and only if all the nontrivial entries of its bidiagonal factors are nonnegative ($D > 0$).

One more piece of standard notation...

$$E_i(\alpha) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \alpha & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix},$$

where α is in $(i, i-1)$ entry

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In the rest of the talk, we assume that **Neville elimination runs without exchanges**. This is not a restriction for nonsingular TN matrices.

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Neville on Green's quasiseparable: $E_5(-\ell_5)G$

$$\ell_5 := \begin{cases} \dfrac{p_5 a_4}{p_4} \left(= \dfrac{g_{51}}{g_{41}} \right) & \text{if } p_4 \neq 0 \\ 0 & \text{if } p_4 = 0 \end{cases} \quad \rightarrow \quad E_5(-\ell_5) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & -\ell_5 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} p_1q_1 & g_1b_1h_2 & g_1b_1b_2h_3 & g_1b_1b_2b_3h_4 & g_1b_1b_2b_3b_4h_5 \\ p_2a_1q_1 & p_2q_2 & g_2b_2h_3 & g_2b_2b_3h_4 & g_2b_2b_3b_4h_5 \\ p_3a_2a_1q_1 & p_3a_2q_2 & p_3q_3 & g_3b_3h_4 & g_3b_3b_4h_5 \\ p_4a_3a_2a_1q_1 & p_4a_3a_2q_2 & p_4a_3q_3 & p_4q_4 & g_4b_4h_5 \\ p_5a_4a_3a_2a_1q_1 & p_5a_4a_3a_2q_2 & p_5a_4a_3q_3 & p_5a_4q_4 & p_5q_5 \end{bmatrix}$$

rank one matrix $\Rightarrow \dfrac{g_{51}}{g_{41}} = \dfrac{g_{52}}{g_{42}} = \dfrac{g_{53}}{g_{43}} = \dfrac{g_{54}}{g_{44}}$

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$$p_5 q_5 = g_5 h_5 \Longrightarrow g_5' = g_5 - \ell_5 g_4 b_4$$

Bidiagonal factorization of Green's quasiseparable

Theorem

Complete Neville elimination runs without exchanges on a nonsingular $n \times n$ *Green's quasiseparable matrix* G *specified by its generators if and only if*

$$G = E_n(\ell_n) \cdots E_3(\ell_3)\, E_2(\ell_2)\, D\, E_2(u_2)^T\, E_3(u_3)^T \cdots E_n(u_n)^T,$$

where $D =$ *diag*$(\mathbf{d}_1, \ldots, \mathbf{d}_n)$, *and*

$$\ell_i := \begin{cases} \frac{p_i a_{i-1}}{p_{i-1}} & \text{if } p_{i-1} \neq 0 \\ 0 & \text{if } p_{i-1} = 0 \end{cases} \qquad u_i := \begin{cases} \frac{h_i b_{i-1}}{h_{i-1}} & \text{if } h_{i-1} \neq 0 \\ 0 & \text{if } h_{i-1} = 0 \end{cases}$$

$$\mathbf{d}_1 = p_1 q_1, \qquad \mathbf{d}_i = p_i q_i - \ell_i\, u_i\, p_{i-1} q_{i-1} \quad \text{for } i = 2 : n$$

Remarks

- The **bidiagonal factorization of G is sparse: $3n-2$ nontrivial entries**
- The bidiagonal factorization of G **can be computed through explicit formulae from generators** (also from entries) **in $O(n)$ flops**.

Bidiagonal factorization of Green's quasiseparable

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$$T = \begin{bmatrix} y_1 & z_2 & & & \\ x_2 & y_2 & z_3 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & y_{n-1} & z_n \\ & & & x_n & y_n \end{bmatrix},$$

has LDU factorization, where

Bidiagonal factorization of a quasiseparable matrix (II)

Theorem (continued)

$$x_2 = p_2 q_1, \;\; x_j = p_j\, q_{j-1} - \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$

$$y_1 = d_1, \;\; y_2 = d_2, \;\; y_j = d_j - \ell_j\, g_{j-1}\, h_j - u_j\, p_j\, q_{j-1} + u_j\, \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$

$$z_2 = g_1 h_2, \;\; z_j = g_{j-1}\, h_j - u_j d_{j-1} \quad \text{for } j = 3:n$$

Remarks

- **We can compute through formulae from the generators** (or entries)

$$C = E_n(\ell_n) \cdots E_4(\ell_4)\, E_3(\ell_3)\, T\, E_3(u_3)^T\, E_4(u_4)^T \cdots E_n(u_n)^T,$$

- but, **to get the bidiagonal factorization**, it remains to compute with **usual Gaussian (Neville) elimination on a tridiagonal** matrix,

$$T = L^{(n-1)} D\, U^{(n-1)}.$$

- Total cost is $O(n)$ **flops** and the **bidiagonal factorization of C is sparse: $5n-6$ nontrivial entries**

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Outline

TN quasiseparable and bidiagonal factorizations

Theorem

An $n \times n$ nonsingular matrix C **is TN and quasiseparable if and only if**

$$C = E_n(\ell_n) \cdots E_3(\ell_3)\, L^{(n-1)} D\, U^{(n-1)}\, E_3(u_3)^T \cdots E_n(u_n)^T,$$

with **all the bidiagonal factors nonnegative and the diagonal entries of D positive**.

Theorem

An $n \times n$ nonsingular matrix G **is TN and Green's quasiseparable if and only if**

$$G = E_n(\ell_n) \cdots E_2(\ell_2)\, D\, E_2(u_2)^T \cdots E_n(u_n)^T,$$

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TN quasiseparable and bidiagonal factorizations

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TN-Green's quasiseparable matrices (I)

Theorem (characterization in terms of the parameters)

Let G be an $n \times n$ Green's quasiseparable matrix specified by its generators. Then

$$G \text{ is nonsingular and TN} \iff \begin{cases} p_1\, q_1 > 0, \quad \text{and} \\ p_i\, q_i - \left(\dfrac{p_i a_{i-1}}{p_{i-1}}\right)\left(\dfrac{h_i b_{i-1}}{h_{i-1}}\right) p_{i-1}\, q_{i-1} > 0, \\ \dfrac{p_i a_{i-1}}{p_{i-1}} \geq 0, \quad \dfrac{h_i b_{i-1}}{h_{i-1}} \geq 0, \quad \text{for } 2 \leq i \leq n \end{cases}$$

These conditions can be checked in $O(n)$ flops.

TN-Green's quasiseparable matrices (II)

Theorem (characterization in terms of the entries)

Let G be an $n \times n$ Green's quasiseparable matrix. Then

$$G \text{ is nonsingular and TN} \iff \begin{cases} g_{ii} > 0, & 1 \leq i \leq n \\ g_{i,i-1} \geq 0, \quad g_{i-1,i} \geq 0, & 2 \leq i \leq n \\ \det G(i-1:i\,,\, i-1:i) > 0, & 2 \leq i \leq n \end{cases}$$

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$$C = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

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$$\ell_i := \begin{cases} \frac{p_i a_{i-1}}{p_{i-1}} & \text{if } p_{i-1} \neq 0 \\ 0 & \text{if } p_{i-1} = 0 \end{cases} \qquad u_i := \begin{cases} \frac{h_i b_{i-1}}{h_{i-1}} & \text{if } h_{i-1} \neq 0 \\ 0 & \text{if } h_{i-1} = 0 \end{cases} \qquad \text{for } i = 3:n,$$

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with

$$x_2 = p_2 q_1, \ \ x_j = p_j\, q_{j-1} - \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$

$$y_1 = d_1, \ \ y_2 = d_2, \ \ y_j = d_j - \ell_j\, g_{j-1}\, h_j - u_j\, p_j\, q_{j-1} + u_j\, \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$

$$z_2 = g_1 h_2, \ \ z_j = g_{j-1}\, h_j - u_j d_{j-1} \quad \text{for } j = 3:n$$

TN-quasiseparable matrices (II)

Theorem (continued)

Then, C **is nonsingular and TN** *if and only if*

- $\ell_i \geq 0$ *and* $u_i \geq 0$ *for* $i = 3:n$.
- *The* **tridiagonal matrix T is nonsingular and TN**.
- *If* $p_j\, q_{j-1} = 0$, *for some* j, *then* $C(j:n\,,\,1:j-1) = 0$.
- *If* $g_{j-1}\, h_j = 0$, *for some* j, *then* $C(1:j-1\,,\,j:n) = 0$.

These conditions can be checked in $O(n)$ *flops.*

$$C = \begin{bmatrix} \boxed{d_1} & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & \boxed{d_2} & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & \boxed{d_3} & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 = 0 & \boxed{d_4} & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & \boxed{d_5} \end{bmatrix}$$

Theorem (continued)

Then, C **is nonsingular and TN** *if and only if*

- $\ell_i \geq 0$ *and* $u_i \geq 0$ *for* $i = 3 : n$.
- *The* **tridiagonal matrix T is nonsingular and TN**.
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TN-quasiseparable matrices (II)

Theorem (continued)

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Outline

1. Introduction and goals
2. Neville elimination, TN and quasiseparable matrices
3. Totally Nonnegative (TN) quasiseparable matrices
4. **Error analysis for quasiseparable linear systems**
5. Other results in accurate quasiseparable computations
6. Conclusions and future work

Solving linear systems given a bidiagonal factorization...

Assume that for a **general matrix** A:

- We know a bidiagonal factorization

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D\, U^{(n-1)} \cdots U^{(2)} U^{(1)}.$$

- We want to solve $Ax = b$.

Then, we solve the sequence of systems

$$L^{(1)} x^{(1)} = b \to L^{(2)} x^{(2)} = x^{(1)} \to \cdots \to U^{(1)} x = x^{(2n-2)}$$

Observe that for quasiseparable matrices **many of the bidiagonal systems** are very simple and **can be solved in two flops**:

$$E_i(\alpha) z = y \iff z = E_i(-\alpha) y.$$

In fact, **all are of this type for Green's quasiseparable** matrices.

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The complete $O(n)$ quasiseparable algorithm

ALGORITHM 1

INPUT: **Generators** of C (resp. G) $n \times n$ quasiseparable (resp. Green's quasiseparable) matrix and vector b

OUTPUT: Solution of $Cx = b$ (resp. $Gx = b$)

- **Compute bidiagonal factorization with formulae** as in the first part of the talk:

  $$C = E_n(\ell_n) \cdots E_3(\ell_3)\, L^{(n-1)}\, D\, U^{(n-1)}\, E_3(u_3)^T \cdots E_n(u_n)^T,$$

  (resp. $G = E_n(\ell_n) \cdots E_2(\ell_2)\, D\, E_2(u_2)^T \cdots E_n(u_n)^T$)

- **Solve a sequence of bidiagonal systems** to get x as in the previous slide.

Backward errors for general quasiseparable matrix

Theorem

If Algorithm 1 is applied to solve $Cx = b$*, where* C **is** $n \times n$ **quasiseparable matrix***, and*

$$E_n(\widehat{\ell}_n), \ldots, E_3(\widehat{\ell}_3),\ \widehat{L}^{(n-1)},\ \widehat{D},\ \widehat{U}^{(n-1)},\ E_3(\widehat{u}_3)^T, \ldots, E_n(\widehat{u}_n)^T,$$

are the **computed bidiagonal factors of** C *with unit roundoff* ϵ*, then* **the computed solution** $\widehat{x}$ **satisfies**

$$(C+E)\widehat{x} = b,$$

where

- $(C+E)$ **is quasiseparable***, and*
-

$$|E| \leq \frac{27n\epsilon}{1-27n\epsilon} E_n(|\widehat{\ell}_n|) \cdots E_3(|\widehat{\ell}_3|)\, |\widehat{L}^{(n-1)}|\, |\widehat{D}|\, |\widehat{U}^{(n-1)}|\, E_3(|\widehat{u}_3|)^T \cdots E_n(|\widehat{u}_n|)^T$$

Comments on this backward error analysis

- Similar result for **Green's quasiseparable matrices**, preserving the Green's structure.
- It is very **tricky**.
- It requires a **delicate way to evaluate the formulae** for the bidiagonal/tridiagonal factors
- It combines
  1. **mixed backward-forward errors in terms of parameters and bidiagonal factors**, with
  2. **backward** errors **in terms of entries**.
- The **bound may be not satisfactory** if

$$E_n(|\widehat{\ell}_n|)\cdots E_3(|\widehat{\ell}_3|)\,|\widehat{L}^{(n-1)}|\,|\widehat{D}|\,|\widehat{U}^{(n-1)}|\,E_3(|\widehat{u}_3|)^T\cdots E_n(|\widehat{u}_n|)^T >> |C|$$

- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.

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Satisfactory backward errors for TN-quasiseparable matrices

Theorem

If Algorithm 1 is applied to solve $Cx = b$, *where* C **is** $n \times n$ **quasiseparable matrix**, *and all the* **computed bidiagonal factors of** C **are nonnegative** $(\mathrm{diag}\, \widehat{D} > 0)$, *then* **the computed solution** $\widehat{x}$ **satisfies**

$$(C + E)\widehat{x} = b,$$

where

- $(C + E)$ **is TN-quasiseparable**, *and*
- $$\boxed{|\mathbf{E}| \leq \frac{\mathbf{27n\epsilon}}{\mathbf{1 - 54n\epsilon}}|\mathbf{C}|}$$

Similar for Green's quasiseparable matrices.

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Similar for Green's quasiseparable matrices.

Forward errors for TN-Green's quasiseparable (I)

Theorem

Let G **be an** $n \times n$ **Green's quasiseparable matrix** *and define*

$$\kappa_{\mathbf{GQ}}(\mathbf{G}) = \max_{2 \leq i \leq n} \frac{|g_{i,i}\, g_{i-1,i-1}| + |g_{i,i-1}\, g_{i-1,i}|}{|g_{i,i}\, g_{i-1,i-1} - g_{i,i-1}\, g_{i-1,i}|}.$$

Assume that Algorithm 1 is applied to solve $Gx = b$ *with unit roundoff* ϵ *and that the* **computed bidiagonal factors of** G **are nonnegative** *(*$\widehat{D}$ *nonsingular). If*

$$\frac{9\epsilon}{1-9\epsilon}\kappa_{\mathbf{GQ}}(\mathbf{G}) < \frac{1}{2},$$

then

- G **is nonsingular and TN**.
- *The computed solution* $\widehat{\mathbf{x}}$ *satisfies*

$$|x - \widehat{\mathbf{x}}| \leq 2 \left(\frac{8n\epsilon}{1-8n\epsilon} + \kappa_{\mathbf{GQ}}(\mathbf{G}) \frac{9\epsilon}{1-9\epsilon} \right) |G^{-1}||b|$$

Forward errors for TN-Green's quasiseparable (II)

$$|x - \widehat{\mathbf{x}}| \leq 2 \left(\frac{8n\epsilon}{1 - 8n\epsilon} + \kappa_{\mathbf{GQ}}(\mathbf{G}) \frac{9\epsilon}{1 - 9\epsilon} \right) |G^{-1}||b|$$

- If $\epsilon\, \kappa_{\mathbf{GQ}}(\mathbf{G}) \ll 1$, this is a very satisfactory bound, because
- $\dfrac{\| \, |G^{-1}||b| \, \|_\infty}{\|x\|_\infty} = \dfrac{\| \, |G^{-1}||b| \, \|_\infty}{\|G^{-1}b\|_\infty}$ is moderate except for particular b's.
- $$\kappa_{\mathbf{GQ}}(\mathbf{G}) = \max_{2 \leq i \leq n} \frac{|g_{i,i}\, g_{i-1,i-1}| + |g_{i,i-1}\, g_{i-1,i}|}{|g_{i,i}\, g_{i-1,i-1} - g_{i,i-1}\, g_{i-1,i}|},$$

  is a **condition number for this problem**.
- Note that

  $$\det G(i-1:i\,,\, i-1:i) = g_{i,i}\, g_{i-1,i-1} - g_{i,i-1}\, g_{i-1,i}.$$

Forward errors for TN-Green's quasiseparable (II)

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Outline

Other results for Green's quasiseparable

- We know how to **compute in $O(n^2)$ flops eigenvalues and singular values of TN-Green's quasiseparable** matrices **with relative errors**

$$O(\epsilon\, \kappa_{\mathbf{GQ}}(\mathbf{G}))$$

- We have shown perfect componentwise backward stability in **solving linear systems through bidiagonalization** for **diagonally dominant** Green's quasiseparable matrices.
- We know how to compute **with cost $O(n^2)$ accurate eigenvalues of skew-symmetric real Green's quasiseparable matrices** and of symmetric Green's quasiseparable matrices with zero diagonal.

Outline

Conclusions and future work

- Bidiagonalization + solving a sequence of bidiagonal systems **is fast** on quasiseparable matrices **but not backward stable**.
- **It is fast and backward stable on TN-quasiseparable**.
- Simple **forward error bounds for TN Green's quasiseparable** matrices available.
- **Next step:** error analysis of quasiseparable structured algorithms for QR factorization and its use for solving linear systems.
- **Error analysis of structured rank algorithms for quasiseparable matrices is an open area of research full of challenging problems...**

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