Fast and accurate computations with Totally Nonnegative Quasiseparable Matrices

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Outline

1. Quasiseparable matrices
2. Goals of the talk
3. Neville elimination and quasiseparable matrices
4. Totally Nonnegative (TN) quasiseparable matrices
5. Solving quasiseparable linear systems
6. Error analysis for quasiseparable linear systems
7. Conclusions and future work

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Quasiseparable matrices (I): Definition

Definition

A square matrix C is **quasiseparable of order** (n_L, n_U) if

- every submatrix of C entirely located in the **strictly lower (resp. upper) triangular part** of C **have rank at most** n_L (resp. n_U), and
- at least one of these submatrices has rank equal to n_L (resp. n_U).

Remark

In this talk, we are interested only in the order $(1, 1)$ and for brevity the simple term **quasiseparable** is used instead of $(1, 1)$-quasiseparable.

It is necessary and sufficient that the following submatrices have rank at most 1:

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Quasiseparable matrices (II)

$$C = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

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Green's quasiseparable matrices (I)

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A square matrix G is **Green's quasiseparable of order** $(1,1)$ if

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Parametrization of quasiseparable matrices

Theorem (Eidelman and Gohberg (1999))

The set of $n \times n$ quasiseparable matrices can be parameterized in terms of $7n-8$ independent parameters or generators.

Example (Every 5×5 quasiseparable matrix is of the form)

$$C = \left[\begin{array}{cc|c|cc} \boxed{d_1} & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & \boxed{d_2} & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ \hline p_3a_2q_1 & p_3q_2 & \boxed{d_3} & g_3h_4 & g_3b_4h_5 \\ \hline p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & \boxed{d_4} & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & \boxed{d_5} \end{array}\right]$$

Remark

There are seven vectors (families) of parameters: $p_{2:n}$, $a_{2:n-1}$, $q_{1:n-1}$, $d_{1:n}$, $g_{1:n-1}$, $b_{2:n-1}$, $h_{2:n}$.

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Parametrization of Green's quasiseparable

Theorem

The set of $n \times n$ Green's quasiseparable matrices can be parameterized in terms of $6n-2$ parameters with **the constraints $p_i q_i = g_i h_i$ for $i = 1:n$**.

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There are six vectors of parameters: $p_{1:n}$, $a_{1:n-1}$, $q_{1:n}$, $g_{1:n}$, $b_{1:n-1}$, $h_{1:n}$.

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Highlights on research on quasiseparable matrices (I)

- During the last six years an intense research has been performed on quasiseparable matrices.
- Many researchers from different countries: Belgium, Check Republic, Italy, Israel, Russia, USA...
- Quasiseparable matrices appear in many applications: systems theory and signal processing, discretization of integral equations, covariance matrices in multivariate statistics, discretization of elliptic PDEs....
- **Quasiseparable matrices include many important classes of matrices:** companion matrices of polynomials, **tridiagonal matrices and their inverses (Green's quasiseparable)**, unitary Hessenberg, banded matrices (for order larger than $(1,1)$),.....
- Inverses of quasiseparable matrices are quasiseparable.

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- A main line of research has been the **development of structured fast algorithms by using the low number of parameters defining this class**. There are many algorithms an their costs are:

Problem	Cost of traditional algorithms	Cost of structured quasiseparable algs.
systems of equations	$O(n^3)$	$O(n)$
eigenvalues	$O(n^3)$	$O(n^2)$
singular values	$O(n^3)$	$O(n^2)$

- **The stability of these algorithms is not guaranteed and**, as far as we know, **error analysis have not been developed** even for the most simple cases.

Highlights on research on quasiseparable matrices (II)

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An historical TN-quasiseparable connection

In *Oscillation Matrices (1941)* by **Gantmacher and Krein** a **particular class of symmetric Green's quasiseparable** matrices is considered. These are called **single-pair** matrices and are defined as

$$S = \begin{bmatrix} p_1 q_1 & q_1 p_2 & q_1 p_3 & \cdots \\ p_2 q_1 & p_2 q_2 & q_2 p_3 & \cdots \\ p_3 q_1 & p_3 q_2 & p_3 q_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \text{tril}(pq^T) + \text{strict-triu}(qp^T),$$

where all the numbers $p = [p_1, \ldots, p_n]^T, q = [q_1, \ldots, q_n]^T$ are nonzero.

These matrices are obtained from general Green's quasiseparable matrices by taking $a_i = b_i = 1$, $g_i = q_i$, and $h_i = p_i$.

Theorem (Gantmacher and Krein (1941))

S is **TN** *if and only if all the numbers $p_1, \ldots, p_n, q_1, \ldots, q_n$ have the same sign and*

$$\frac{q_1}{p_1} \le \frac{q_2}{p_2} \le \cdots \le \frac{q_n}{p_n}$$

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Outline

Goals of the talk

- We initiate the study of **stability of fast algorithms for quasiseparable matrices**, by presenting **rounding errors analysis** of the solution of **quasiseparable linear systems** by using a **bidiagonal factorization** followed by a **Björck-Pereyra** type algorithm.
- We prove that this algorithm is **componentwise backward stable in a strong sense**, i.e., preserving the structure, in the **TN-quasiseparable** case.
- For **Green's quasiseparable matrices simple forward errors bounds for this algorithm are presented and we show that it is frequently accurate**, independently of the traditional condition number of the matrix.
- We characterize **the set of nonsingular TN-quasiseparable** matrices through **the quasiseparable generators, the entries and the bidiagonal factorizations**. This extends Gantmacher and Krein's result.
- We briefly mention other results on accurate computations with **Green's quasiseparable matrices**.

Goals of the talk

- We initiate the study of **stability of fast algorithms for quasiseparable matrices**, by presenting **rounding errors analysis** of the solution of **quasiseparable linear systems** by using a **bidiagonal factorization** followed by a **Björck-Pereyra** type algorithm.
- We prove that this algorithm is **componentwise backward stable in a strong sense**, i.e., preserving the structure, in the **TN-quasiseparable** case.
- For **Green's quasiseparable matrices simple forward errors bounds for this algorithm are presented and we show that it is frequently accurate**, independently of the traditional condition number of the matrix.
- We characterize **the set of nonsingular TN-quasiseparable** matrices through **the quasiseparable generators, the entries and the bidiagonal factorizations**. This extends Gantmacher and Krein's result.
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L. Gemignani, *Neville elimination for rank-structured matrices*, LAA 2008.

- We develop error analysis and Gemignani does not.
- Gemignani deals with general (n_L, n_U)-order quasiseparable matrix and we only with $(1,1)$-order.
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Outline

Brief summary on Neville elimination (I)

- It is a classic **procedure to create zeros in a matrix by adding to a row (resp. column) a multiple of the previous row (resp. column)**.
- **It is valid for any matrix, it is different than Gaussian eliminination** and it has different backward errors.
- **Without interchanges**, it was carefully analyzed by **Gasca and Peña** in a series of seminal papers in the 90s. In particular, its **matricial description** in terms of **bidiagonal factorizations** and its fundamental relationship with **total nonnegativity** were established.

Theorem (Gasca and Peña (1994))

A nonsingular matrix A is TN if and only if complete Neville elimination can be performed on A without row or column exchanges, with nonnegative multipliers and positive diagonal pivots.

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Brief summary on Neville elimination (II)

- **Neville elimination without exchanges adapts very well to the quasiseparable structure** (Gemignani 2008).
- The quasiseparable structure is destroyed by row or column exchanges.
- In this talk, **Neville elimination is never applied numerically**. **It is a theoretical way to get formulae, in terms of the generators, for the bidiagonal factors of the matrix**. These formulae are then used
  1. **to develop fast algorithms,**
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Bidiagonal factorizations (I)

Theorem (Gasca and Peña (1996))

If complete Neville elimination runs without exchanges on a **nonsingular** $n \times n$ **square matrix** A *then*

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D \, U^{(n-1)} \cdots U^{(2)} U^{(1)},$$

where D **is diagonal** *and*

$$L^{(k)} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & \ell_1^{(k)} & 1 & & & \\ & & \ell_2^{(k)} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ell_k^{(k)} & 1 \end{bmatrix}$$

This is known as a **bidiagonal factorization** *of* A *and* **all the factors amount to** n^2 **nontrivial entries**.

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Bidiagonal factorizations (II)

Example (Bidiagonal factorization of a 5×5 matrix)

$$A = L^{(1)} L^{(2)} L^{(3)} L^{(4)} \mathbf{D}\, U^{(4)} U^{(3)} U^{(2)} U^{(1)},$$

$$\mathbf{D} = \begin{bmatrix} \times & & & & \\ & \times & & & \\ & & \times & & \\ & & & \times & \\ & & & & \times \end{bmatrix}$$

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Simplifying assumptions

- The generators of a quasiseparable matrix **are not unique**.
- We will assume (not essential) that for any C quasiseparable matrix

$$C(i, 1:i-1) = 0 \Longrightarrow p_i = 0 \quad \text{and} \quad C(1:i-1, i) = 0 \Longrightarrow h_i = 0$$

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$$C = \left[\begin{array}{ccc|c|c} \boxed{d_1} & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & \boxed{d_2} & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ p_3 a_2 q_1 & p_3 q_2 & \boxed{d_3} & g_3 h_4 & g_3 b_4 h_5 \\ \hline p_4 a_3 a_2 q_1 & p_4 a_3 q_2 & p_4 q_3 & \boxed{d_4} & g_4 h_5 \\ \hline p_5 a_4 a_3 a_2 q_1 & p_5 a_4 a_3 q_2 & p_5 a_4 q_3 & p_5 q_4 & \boxed{d_5} \end{array}\right]$$

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One more piece of standard notation...

$$E_i(\alpha) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \alpha & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}, \qquad \text{where } \alpha \text{ is in } (i, i-1) \text{ entry}$$

Remark

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Neville on Green's quasiseparable: $E_5(-\ell_5)G = G_4$

$$\ell_5 := \begin{cases} \dfrac{p_5 a_4}{p_4} \left(= \dfrac{g_{51}}{g_{41}} \right) & \text{if } p_4 \neq 0 \\ 0 & \text{if } p_4 = 0 \end{cases} \quad \rightarrow \quad E_5(-\ell_5) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & -\ell_5 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} p_1 q_1 & g_1 b_1 h_2 & g_1 b_1 b_2 h_3 & g_1 b_1 b_2 b_3 h_4 & g_1 b_1 b_2 b_3 b_4 h_5 \\ p_2 a_1 q_1 & p_2 q_2 & g_2 b_2 h_3 & g_2 b_2 b_3 h_4 & g_2 b_2 b_3 b_4 h_5 \\ p_3 a_2 a_1 q_1 & p_3 a_2 q_2 & p_3 q_3 & g_3 b_3 h_4 & g_3 b_3 b_4 h_5 \\ p_4 a_3 a_2 a_1 q_1 & p_4 a_3 a_2 q_2 & p_4 a_3 q_3 & p_4 q_4 & g_4 b_4 h_5 \\ p_5 a_4 a_3 a_2 a_1 q_1 & p_5 a_4 a_3 a_2 q_2 & p_5 a_4 a_3 q_3 & p_5 a_4 q_4 & p_5 q_5 \end{bmatrix}$$

rank one matrix $\Rightarrow \dfrac{g_{51}}{g_{41}} = \dfrac{g_{52}}{g_{42}} = \dfrac{g_{53}}{g_{43}} = \dfrac{g_{54}}{g_{44}}$

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$$p_5q_5 = g_5h_5 \Longrightarrow g_5' = g_5 - \ell_5 g_4 b_4$$

Bidiagonal factorization of Green's quasiseparable

Theorem

Complete Neville elimination runs without exchanges on a nonsingular $n \times n$ *Green's quasiseparable matrix* G *specified by its generators if and only if*

$$G = E_n(\ell_n) \cdots E_3(\ell_3)\, E_2(\ell_2)\, D\, E_2(u_2)^T\, E_3(u_3)^T \cdots E_n(u_n)^T,$$

where $D = \text{diag}(\mathbf{d}_1, \ldots, \mathbf{d}_n)$, *and*

$$\ell_i := \begin{cases} \frac{p_i a_{i-1}}{p_{i-1}} & \text{if } p_{i-1} \neq 0 \\ 0 & \text{if } p_{i-1} = 0 \end{cases} \qquad u_i := \begin{cases} \frac{h_i b_{i-1}}{h_{i-1}} & \text{if } h_{i-1} \neq 0 \\ 0 & \text{if } h_{i-1} = 0 \end{cases}$$

$$\mathbf{d}_1 = p_1 q_1, \qquad \mathbf{d}_i = p_i q_i - \ell_i\, u_i\, p_{i-1} q_{i-1} \quad \text{for } i = 2 : n$$

Remarks

- The **bidiagonal factorization of G is sparse: $3n-2$ nontrivial entries**
- The bidiagonal factorization of G **can be computed through explicit formulae from generators** (also from entries) **in $O(n)$ flops**.

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where $D =$ diag$(\mathbf{d}_1, \ldots, \mathbf{d}_n)$, and

$$\ell_i := \begin{cases} \frac{p_i a_{i-1}}{p_{i-1}} & \text{if } p_{i-1} \neq 0 \\ 0 & \text{if } p_{i-1} = 0 \end{cases} \qquad u_i := \begin{cases} \frac{h_i b_{i-1}}{h_{i-1}} & \text{if } h_{i-1} \neq 0 \\ 0 & \text{if } h_{i-1} = 0 \end{cases}$$

$$\mathbf{d}_1 = p_1 q_1, \qquad \mathbf{d}_i = p_i q_i - \ell_i\, u_i\, p_{i-1} q_{i-1} \quad \text{for } i = 2 : n$$

Remarks

- The **bidiagonal factorization of G is sparse: $3n-2$ nontrivial entries**
- The bidiagonal factorization of G **can be computed through explicit formulae from generators** (also from entries) **in $O(n)$ flops**.

Neville on quasiseparable: $E_5(-\ell_5)C = C_4$

$$\ell_5 := \begin{cases} \dfrac{p_5 a_4}{p_4} \left(= \dfrac{c_{51}}{c_{41}} \right) & \text{if } p_4 \neq 0 \\ 0 & \text{if } p_4 = 0 \end{cases} \quad \rightarrow \quad E_5(-\ell_5) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & -\ell_5 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

rank one matrix $\Rightarrow \dfrac{c_{51}}{c_{41}} = \dfrac{c_{52}}{c_{42}} = \dfrac{c_{53}}{c_{43}}$

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Bidiagonal factorization of a quasiseparable matrix (I)

Theorem

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and

$$T = \begin{bmatrix} y_1 & z_2 & & & \\ x_2 & y_2 & z_3 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & y_{n-1} & z_n \\ & & & x_n & y_n \end{bmatrix},$$

has LDU factorization, where

Bidiagonal factorization of a quasiseparable matrix (II)

Theorem (continued)

$$x_2 = p_2 q_1, \;\; x_j = p_j\, q_{j-1} - \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$
$$y_1 = d_1, \;\; y_2 = d_2, \;\; y_j = d_j - \ell_j\, g_{j-1}\, h_j - u_j\, p_j\, q_{j-1} + u_j\, \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$
$$z_2 = g_1 h_2, \;\; z_j = g_{j-1}\, h_j - u_j d_{j-1} \quad \text{for } j = 3:n$$

Remarks

- **We can compute through formulae from the generators** (or entries)

$$C = E_n(\ell_n) \cdots E_4(\ell_4)\, E_3(\ell_3)\, T\, E_3(u_3)^T\, E_4(u_4)^T \cdots E_n(u_n)^T,$$

- but, **to get the bidiagonal factorization**, it remains to compute with **usual Gaussian (Neville) elimination on a tridiagonal** matrix,

$$T = L^{(n-1)}\, D\, U^{(n-1)}.$$

- Total cost is $O(n)$ **flops** and the **bidiagonal factorization of C is sparse: $5n-6$ nontrivial entries**

Bidiagonal factorization of a quasiseparable matrix (II)

Theorem (continued)

$$x_2 = p_2 q_1, \;\; x_j = p_j\, q_{j-1} - \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$

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Outline

TN quasiseparable and bidiagonal factorizations

Theorem

An $n \times n$ *nonsingular matrix* C **is TN and quasiseparable if and only if**

$$C = E_n(\ell_n) \cdots E_3(\ell_3)\, L^{(n-1)} D\, U^{(n-1)}\, E_3(u_3)^T \cdots E_n(u_n)^T,$$

with **all the bidiagonal factors nonnegative and the diagonal entries of** D **positive**.

Theorem

An $n \times n$ *nonsingular matrix* G **is TN and Green's quasiseparable if and only if**

$$G = E_n(\ell_n) \cdots E_2(\ell_2)\, D\, E_2(u_2)^T \cdots E_n(u_n)^T,$$

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TN-Green's quasiseparable matrices (I)

Theorem (characterization in terms of the parameters)

Let G be an $n \times n$ Green's quasiseparable matrix specified by its generators. Then

$$G \text{ is nonsingular and TN} \iff \begin{cases} p_1\, q_1 > 0, \quad \text{and} \\ p_i\, q_i - \left(\dfrac{p_i a_{i-1}}{p_{i-1}}\right)\left(\dfrac{h_i b_{i-1}}{h_{i-1}}\right) p_{i-1}\, q_{i-1} > 0, \\ \dfrac{p_i a_{i-1}}{p_{i-1}} \geq 0, \quad \dfrac{h_i b_{i-1}}{h_{i-1}} \geq 0, \quad \text{for } 2 \leq i \leq n \end{cases}$$

These conditions can be checked in $O(n)$ flops.

TN-Green's quasiseparable matrices (II)

Theorem (characterization in terms of the entries)

Let G be an $n \times n$ Green's quasiseparable matrix. Then

$$G \text{ is nonsingular and TN} \iff \begin{cases} g_{ii} > 0, & 1 \le i \le n \\ g_{i,i-1} \ge 0, \quad g_{i-1,i} \ge 0, & 2 \le i \le n \\ \det G(i-1:i\,,\, i-1:i) > 0, & 2 \le i \le n \end{cases}$$

These conditions can be checked in $O(n)$ flops.

$$C = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

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TN-quasiseparable matrices (I)

Theorem

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and

$$T = \begin{bmatrix} y_1 & z_2 & & & \\ x_2 & y_2 & z_3 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & y_{n-1} & z_n \\ & & & x_n & y_n \end{bmatrix},$$

with

$$x_2 = p_2 q_1, \;\; x_j = p_j\, q_{j-1} - \ell_j\, d_{j-1} \quad \text{for } j = 3:n$$
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TN-quasiseparable matrices (II)

Theorem (continued)

Then, C **is nonsingular and TN** *if and only if*

- $\ell_i \geq 0$ *and* $u_i \geq 0$ *for* $i = 3 : n$.
- *The* **tridiagonal matrix** T **is nonsingular and TN**.
- *If* $p_j\, q_{j-1} = 0$, *for some* j, *then* $C(j : n\,,\, 1 : j-1) = 0$.
- *If* $g_{j-1}\, h_j = 0$, *for some* j, *then* $C(1 : j-1\,,\, j : n) = 0$.

These conditions can be checked in $O(n)$ *flops.*

$$C = \begin{bmatrix} \boxed{d_1} & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & \boxed{d_2} & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ p_3 a_2 q_1 & p_3 q_2 & \boxed{d_3} & g_3 h_4 & g_3 b_4 h_5 \\ p_4 a_3 a_2 q_1 & p_4 a_3 q_2 & p_4 q_3 = 0 & \boxed{d_4} & g_4 h_5 \\ p_5 a_4 a_3 a_2 q_1 & p_5 a_4 a_3 q_2 & p_5 a_4 q_3 & p_5 q_4 & \boxed{d_5} \end{bmatrix}$$

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Outline

Given a bidiagonal factorization...

Assume that for a **general matrix** A:

- We know a bidiagonal factorization

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D\, U^{(n-1)} \cdots U^{(2)} U^{(1)}.$$

- We want to solve $Ax = b$.

Then

$$L^{(1)} x^{(1)} = b, \quad L^{(2)} x^{(2)} = x^{(1)}, \quad \cdots \quad L^{(n-1)} x^{(n-1)} = x^{(n-2)},$$
$$D x^{(n)} = x^{(n-1)},$$
$$U^{(n-1)} x^{(n+1)} = x^{(n)}, \quad U^{(n-2)} x^{(n+2)} = x^{(n+1)}, \quad \cdots \quad U^{(1)} x = x^{(2n-2)}$$

Observe that for quasiseparable matrices **many of the bidiagonal systems** are very simple and **can be solved in two flops**:

$$E_i(\alpha) z = y \Longleftrightarrow z = E_i(-\alpha) y.$$

In fact, **all are of this type for Green's quasiseparable** matrices.

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- We want to solve $Ax = b$.

Then

$$L^{(1)}x^{(1)} = b, \quad L^{(2)}x^{(2)} = x^{(1)}, \quad \cdots \quad L^{(n-1)}x^{(n-1)} = x^{(n-2)},$$
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Observe that for quasiseparable matrices **many of the bidiagonal systems** are very simple and **can be solved in two flops**:

$$E_i(\alpha)z = y \iff z = E_i(-\alpha)y.$$

In fact, **all are of this type for Green's quasiseparable** matrices.

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The complete $O(n)$ quasiseparable algorithm

ALGORITHM 1

INPUT: **Generators** of C (resp. G) $n \times n$ quasiseparable (resp. Green's quasiseparable) matrix and vector b
OUTPUT: Solution of $Cx = b$ (resp. $Gx = b$)

- **Compute bidiagonal factorization with formulae** as in the first part of the talk:

$$C = E_n(\ell_n) \cdots E_3(\ell_3)\, L^{(n-1)}\, D\, U^{(n-1)}\, E_3(u_3)^T \cdots E_n(u_n)^T,$$

(resp. $G = E_n(\ell_n) \cdots E_2(\ell_2)\, D\, E_2(u_2)^T \cdots E_n(u_n)^T$)

- **Solve a sequence of bidiagonal systems** to get x as in the previous slide.

Outline

Backward errors for general quasiseparable matrix

Theorem

If Algorithm 1 is applied to solve $Cx = b$, *where* C **is** $n \times n$ **quasiseparable matrix**, *and*

$$E_n(\widehat{\ell}_n), \ldots, E_3(\widehat{\ell}_3),\ \widehat{L}^{(n-1)},\ \widehat{D},\ \widehat{U}^{(n-1)},\ E_3(\widehat{u}_3)^T, \ldots, E_n(\widehat{u}_n)^T,$$

are the **computed bidiagonal factors of** C *with unit roundoff* ϵ, *then* **the computed solution** $\widehat{x}$ **satisfies**

$$(C + E)\widehat{x} = b,$$

where

- $(C + E)$ **is quasiseparable**, *and*
-

$$|E| \leq \frac{27n\epsilon}{1 - 27n\epsilon} E_n(|\widehat{\ell}_n|) \cdots E_3(|\widehat{\ell}_3|)\, |\widehat{L}^{(n-1)}|\, |\widehat{D}|\, |\widehat{U}^{(n-1)}|\, E_3(|\widehat{u}_3|)^T \cdots E_n(|\widehat{u}_n|)^T$$

Comments on this backward error analysis

- Similar result for **Green's quasiseparable matrices**, preserving the Green's structure.
- It is very **tricky**.
- It requires a **delicate way to evaluate the formulae** for the bidiagonal/tridiagonal factors
- It combines
  1. **mixed backward-forward errors in terms of parameters and bidiagonal factors**, with
  2. **backward** errors **in terms of entries**.
- The **bound may be not satisfactory** if

$$E_n(|\widehat{\ell}_n|)\cdots E_3(|\widehat{\ell}_3|)\,|\widehat{L}^{(n-1)}|\,|\widehat{D}|\,|\widehat{U}^{(n-1)}|\,E_3(|\widehat{u}_3|)^T\cdots E_n(|\widehat{u}_n|)^T >> |C|$$

- No way to incorporate pivoting to improve bounds because it destroys quasiseparable structure and it does not match well Neville elimination.

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Theorem

If Algorithm 1 is applied to solve $Cx = b$, *where* C **is** $n \times n$ **quasiseparable matrix**, *and all the* **computed bidiagonal factors of** C **are nonnegative** *(*$\mathrm{diag}\, \widehat{D} > 0$*), then* **the computed solution** $\widehat{x}$ **satisfies**

$$(C + E)\widehat{x} = b,$$

where

- $(C + E)$ **is TN-quasiseparable**, *and*
- $$|\mathbf{E}| \leq \frac{\mathbf{27n\epsilon}}{\mathbf{1 - 54n\epsilon}} |\mathbf{C}|$$

Similar for Green's quasiseparable matrices.

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Theorem

Let G **be an** $n \times n$ **Green's quasiseparable matrix** *and define*

$$\kappa_{\mathbf{GQ}}(\mathbf{G}) = \max_{2 \leq i \leq n} \frac{|g_{i,i}\, g_{i-1,i-1}| + |g_{i,i-1}\, g_{i-1,i}|}{|g_{i,i}\, g_{i-1,i-1} - g_{i,i-1}\, g_{i-1,i}|}.$$

Assume that Algorithm 1 is applied to solve $Gx = b$ *with unit roundoff* ϵ *and that the* **computed bidiagonal factors of** G **are nonnegative** *(*$\widehat{D}$ *nonsingular). If*

$$\frac{9\epsilon}{1-9\epsilon} \kappa_{\mathbf{GQ}}(\mathbf{G}) < \frac{1}{2},$$

then

- G **is nonsingular and TN**.
- *The computed solution* $\widehat{\mathbf{x}}$ *satisfies*

$$|x - \widehat{\mathbf{x}}| \leq 2 \left(\frac{8n\epsilon}{1-8n\epsilon} + \kappa_{\mathbf{GQ}}(\mathbf{G}) \frac{9\epsilon}{1-9\epsilon} \right) |G^{-1}||b|$$

Forward errors for TN-Green's quasiseparable (II)

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- $\frac{\| \, |G^{-1}||b| \, \|_\infty}{\|x\|_\infty} = \frac{\| \, |G^{-1}||b| \, \|_\infty}{\|G^{-1} b\|_\infty}$ is moderate except for particular b's.
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  is a **condition number for this problem**.
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- We know how to **compute in $O(n^2)$ flops eigenvalues and singular values of TN-Green's quasiseparable** matrices **with relative errors**

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Conclusions and future work

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- Simple **forward error bounds for TN Green's quasiseparable** available.
- **Next step:** error analysis of quasiseparable structured QR factorization for solving linear systems.
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- **Error analysis of structured rank algorithms is an open area full of challenging problems....**

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