Structured perturbation theory for diagonally dominant matrices

Froilán M. Dopico

ICMAT and Department of Mathematics,
Universidad Carlos III de Madrid,
Spain

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Outline

1. Introduction
2. Perturbation theory for the inverse
3. Perturbation theory for linear systems
4. Perturbation theory for LDU factorization
5. Perturbation theory for eigenvalues of symmetric matrices
6. Perturbation theory for singular values
7. Structured condition numbers for eigenvalues of nonsymmetric matrices
8. Conclusions and open problems

Motivation (I)

- Diagonally dominant matrices appear in many applications.
- Q. Ye, *Math. Comp. (2008)*, developed a very ingenuous algorithm for **computing accurately?** in $2n^3$ flops the LDU factorization with complete pivoting of row diagonally dominant **(rDD)** matrices
- that are parameterized in a particular way, but
- best error bounds that Q. Ye proved after a direct error analysis that requires considerable efforts are

$$\frac{\|L-\widehat{L}\|_\infty}{\|L\|_\infty} \leq 6\,n\,8^{(n-1)}\epsilon, \quad \frac{\|U-\widehat{U}\|_\infty}{\|U\|_\infty} \leq 6 \cdot 8^{(n-1)}\epsilon, \quad \frac{|d_{ii}-\widehat{d}_{ii}|}{|d_{ii}|} \leq 5 \cdot 8^{(n-1)}\epsilon,$$

  where $n \times n$ is the size of the matrix and ϵ the unit roundoff.
- $\epsilon = 2^{-53}$ in double precision, **so the bounds are** > 1 **for** $n > 20$...
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$$\frac{\|L-\widehat{L}\|_M}{\|L\|_M} \leq 14\, n^3 \epsilon, \quad \frac{\|U-\widehat{U}\|_M}{\|U\|_M} \leq 14\, n^3 \epsilon, \quad \frac{|d_{ii}-\widehat{d}_{ii}|}{|d_{ii}|} \leq 14\, n^3 \epsilon \ \ \forall i$$

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- **Fundamental consequences:** Q. Ye's algorithm + other existing implicit algorithms for factorized matrices allow **for DD matrices to compute with guaranteed high relative accuracy**

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- **To present a family of new perturbation bounds under structured perturbations** for several magnitudes corresponding to **DD matrices**: inverses, solutions of linear systems, LDU factorization, singular values, eigenvalues.
- Common key point in (almost all) these perturbation bounds: they are always tiny for tiny structured perturbations, even for extremely ill conditioned matrices (independent of traditional condition numbers).
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Parameterizing row diagonally dominant matrices (Q. Ye)

- We will assume that $A \in \mathbb{R}^{n \times n}$ satisfies $a_{ii} \geq 0$ for all i, unless otherwise stated (**no restriction** for linear systems, least squares, SVD, but **yes** for eigenvalues).
- Define diagonal dominances of A as $v = (v_1, v_2, \ldots, v_n)$ where

$$v_i := a_{ii} - \sum_{j \neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.
- $$A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$$
- The pair (A_D, v) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through pairs of this type. A matrix A parameterized in this way will be denoted as

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Example: Good perturbation properties of this parametrization (I)

Example: Two small relative componentwise perturbations of a row diag. dominant matrix A. **Both preserve the diag. dominant structure.**

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

$$\frac{\|A-B\|_2}{\|A\|_2} = 2.2 \cdot 10^{-4} \quad \text{and} \quad \frac{\|A-C\|_2}{\|A\|_2} = 4.6 \cdot 10^{-4}$$

For all $1 \leq i, j \leq 3$,

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Singular values of A, B and C

	A	B	C
σ_1	4.641	4.640	4.642
σ_2	2.910	2.909	2.910
σ_3	$6.663 \cdot 10^{-4}$	$3.332 \cdot 10^{-4}$	$6.673 \cdot 10^{-4}$

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Singular values of A, B and C

	A	B	C
σ_1	4.641	4.640	4.642
σ_2	2.910	2.909	2.910
σ_3	$6.663 \cdot 10^{-4}$	$3.332 \cdot 10^{-4}$	$6.673 \cdot 10^{-4}$

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Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ *be row diagonally dominant matrices such that*

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To be compared with

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Bounds for the inverse under structured perturbations

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Inverses of A, B and C:

$$A^{-1} = \begin{bmatrix} 500.56 & 500.00 & -500.33 \\ 500.00 & 500.00 & -500.00 \\ -500.44 & -500.00 & 500.67 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1000.6 & 1000.0 & -1000.3 \\ 1000.0 & 1000.0 & -1000.0 \\ -1000.4 & -1000.0 & 1000.7 \end{bmatrix}$$

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Consider the systems

$$A\,x = b \quad \text{and} \quad \widetilde{A}\,\widetilde{x} = \widetilde{b}$$

with $\|b - \widetilde{b}\|_\infty \leq \mu \|b\|_\infty$. *If* $2n\delta < 1$, *then*

$$\frac{\|\widetilde{x} - x\|_\infty}{\|x\|_\infty} \leq \left(\frac{n(3n-2)\delta}{1-2n\delta} + \mu + \frac{n(3n-2)\mu\,\delta}{1-2n\delta} \right) \frac{\|A^{-1}\|_\infty \|b\|_\infty}{\|x\|_\infty}$$

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For most vectors b, $\|A^{-1}\|_\infty \, \|b\|_\infty / \|x\|_\infty$ is a moderate number and for A ill-conditioned,

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Solutions of $Ax = b$, $Bx = b$ and $Cx = b$ for $b = [1, 1, -1]^T$:

	$Ax_A = b$	$Bx_B = b$	$Cx_C = b$
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Only one main point...

"Every lecture should make only one main point."

From Gian-Carlo Rota, *"Ten lessons I wish I had been taught"*, Notices of the AMS, 44 (1997) 22-25.

Outline

Theorem (D. and Koev, Numer. Math., 2011)

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$, $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ be row diag. dominant matrices, and $A = LDU$, $\widetilde{A} = \widetilde{L}\,\widetilde{D}\,\widetilde{U}$ be their factorizations. If

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with} \quad \delta < 1,$$

then

- $\widetilde{d}_{ii} = d_{ii} \dfrac{(1+\eta_1)\cdots(1+\eta_i)}{(1+\alpha_1)\cdots(1+\alpha_{i-1})} \qquad |\eta_k| \leq \delta,\ |\alpha_k| \leq \delta,$ for $i = 1:n$.
- $\dfrac{\|\widetilde{U} - U\|_M}{\|U\|_M} \leq 3n\delta$
- If the matrix A is ordered for complete (diagonal) pivoting, then

$$\frac{\|\widetilde{L} - L\|_M}{\|L\|_M} \leq 3\, n\delta + O(\delta^2)$$

Outline

Theorem (Q. Ye, SIMAX, 2009)

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ *be diagonally dominant symmetric matrices with nonnegative diagonal entries such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ *and* $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_n \geq 0$ *be, respectively, the eigenvalues of* $A = \mathcal{D}(A_D, v)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$ *. Then*

$$\boxed{|\widetilde{\lambda}_i - \lambda_i| \leq \delta\, |\lambda_i|, \qquad i = 1, \ldots, n.}$$

Parameterizing rDD matrices with diagonal entries of any sign

- Let $A \in \mathbb{R}^{n\times n}$.
- Define $v = (v_1, v_2, \dots, v_n)$ where

$$v_i := |a_{ii}| - \sum_{j\neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.
- $A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$
- Define $\boxed{S = \text{diag}(\text{sign}(a_{11}), \dots, \text{sign}(a_{nn}))}$ $(\text{sign}(0) := 1)$.
- The triplet (A_D, v, S) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through triplets of this type. Any matrix A parameterized is this way will be denoted as

$$\boxed{A = \mathcal{D}(A_D, v, S)}$$

Theorem

Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}, S) \in \mathbb{R}^{n\times n}$ *be diagonally dominant symmetric matrices such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

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Assume $n^3\delta < 1/5$ *and define* $\nu := \dfrac{4n^3\delta}{1 - n\delta}$.

Then

$$\begin{aligned} |\widetilde{\lambda}_i - \lambda_i| &\leq (2\nu + \nu^2)\, |\lambda_i| \\ &= (\, 8n^3\delta + O(\delta^2)\,)\, |\lambda_i|, \qquad i = 1, \ldots, n \end{aligned}$$

Outline

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ *be row diagonally dominant matrices with nonnegative diagonal entries such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ *and* $\widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_n \geq 0$ *be, respectively, the singular values of* $A = \mathcal{D}(A_D, v)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$ *.*
Define

$$\nu_1 := \frac{n^2\delta}{1-n\delta}\left(3 + \frac{2n\delta}{1-n\delta}\right)\|L^{-1}\|_2 \quad \text{and} \quad \nu_2 = \frac{5n^3\delta}{1-2n\delta},$$

where L *is the* LDU *factor corresponding to complete (diagonal) pivoting.*
If $n\delta < 1$ *and* $\nu := \max\{\nu_1\,,\,\nu_2\} < 1$*, then*

$$\boxed{|\widetilde{\sigma}_i - \sigma_i| \leq \ (2\nu + \nu^2)\ \sigma_i, \qquad i = 1, \ldots, n.}$$

Outline

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be a rDD matrix with nonnegative diagonal entries and λ a simple eigenvalue of A. Consider left and right eigenvectors

$$y^* A = \lambda y^* \qquad \text{and} \qquad Ax = \lambda x$$

and define

$$\operatorname{relcond}(\lambda; A_D, v) := \lim_{\delta\to 0} \sup \left\{ \frac{|\widetilde{\lambda} - \lambda|}{\delta\, |\lambda|} \; : \widetilde{\lambda} \text{ eigenvalue of } \widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}), \right.$$
$$\left. |\widetilde{v} - v| \le \delta\, v,\; |\widetilde{A}_D - A_D| \le \delta |A_D| \right\}.$$

If $s_{ij} = sign(a_{ij})$, then

$$\operatorname{relcond}(\lambda; A_D, v) = \frac{1}{|\lambda|\, |y^* x|} \sum_{i=1}^{n} |y_i| \left(v_i\, |x_i| + \sum_{j\neq i} |a_{ij}|\, |x_i + s_{ij} x_j| \right)$$

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *be a rDD matrix with nonnegative diagonal entries and* λ *a simple eigenvalue of* A*. Consider left and right eigenvectors*

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Example: Good perturbation properties of eigenvalues of nonsymmetric row diagonally dominant matrices

Two types of small ($\approx 10^{-3}$) relative componentwise perturbations of a rDD matrix A:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(B) = \begin{bmatrix} 0 \\ 0.001 \\ 0 \end{bmatrix}$$

$$v(C) = \begin{bmatrix} 0 \\ 0.002002 \\ 0 \end{bmatrix}, \quad c_{12} = -1.5015 \Longrightarrow C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}$$

Eigenvalues and condition numbers:

	A	B	C	$\frac{\|y\|^T \|A\| \|x\|}{\|\lambda\| \|y^* x\|}$	relcond$(\lambda; A_D, v)$
λ_1	$8.5686 \cdot 10^{-4}$	$4.2850 \cdot 10^{-4}$	$8.5803 \cdot 10^{-4}$	$6.75 \cdot 10^{3}$	2.14
λ_2	3.5011	3.5006	3.5023	1.44	1.00
λ_3	6.0000	6.0000	6.0003	1.05	1.00

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Conclusions and open problems

- We have presented structured perturbation results of rDD matrices for all basic problems in Numerical Linear Algebra, **except least squares problems**.

- Except in the case of eigenvalues of nonsymmetric matrices, **the perturbation bounds that we have obtained are rigorous and we have proved that are always tiny for tiny perturbations**.

- **Numerical methods to perform accurate and efficient dense Numerical Linear Algebra with parameterized rDD matrices are available**, **except in the case of eigenvalues of nonsymmetric matrices (open problem!!)**.

- We believe that some of the presented bounds can be improved, in particular the one for singular values.