

Matrix polynomials with completely prescribed eigenstructure

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Joint ALAMA-GAMM/ANLA 2014 Meeting
Facultat de Matemàtiques i Estadística
Universitat Politècnica de Catalunya
Barcelona, July 14-16, 2014

- 1 Preliminary concepts
- 2 Goal of the talk and previous works
- 3 The Index Sum Theorem
- 4 The main theorem
- 5 Existence, possible sizes and eigenstructures of ℓ -ifications
- 6 Conclusions

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The Smith normal form and the invariant polynomials

Definition

The **Smith normal form** of $P(\lambda) = \sum_{i=0}^d P_i \lambda^i \in \mathbb{F}[\lambda]^{m \times n}$ is the following diagonal matrix obtained under unimodular transformations $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \left[\begin{array}{cccc|c} p_1(\lambda) & 0 & \dots & 0 & \\ 0 & p_2(\lambda) & \ddots & \vdots & 0_{r \times (n-r)} \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \dots & 0 & p_r(\lambda) & \\ \hline & & & & 0_{(m-r) \times r} \\ & & & & 0_{(m-r) \times (n-r)} \end{array} \right].$$

- $p_1(\lambda), \dots, p_r(\lambda)$ are monic scalar polynomials in \mathbb{F} ,
- $p_j(\lambda)$ divides $p_{j+1}(\lambda)$, for $j = 1, \dots, r-1$,
- $p_1(\lambda), \dots, p_r(\lambda)$ are unique and are the **invariant polynomials** of $P(\lambda)$.
- $p_j(\lambda)$ is **trivial** if $p_j(\lambda) = 1$, otherwise is **non-trivial**.
- r is the **rank** of $P(\lambda)$.

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Finite eigenvalues and elementary divisors of $P(\lambda)$

- $\alpha \in \overline{\mathbb{F}}$ is a **finite eigenvalue** of $P(\lambda)$ if $p_j(\alpha) = 0$, for some $j = 1, \dots, r$.
- The **partial multiplicity sequence** of $P(\lambda)$ at $\alpha \in \overline{\mathbb{F}}$ is the sequence

$$0 \leq \delta_1 \leq \delta_2 \leq \dots \leq \delta_r, \quad \text{s. t.}$$

$$p_j(\lambda) = (\lambda - \alpha)^{\delta_j} q_j(\lambda), \quad \text{with } q_j(\alpha) \neq 0, \quad \text{for } j = 1, \dots, r$$

- The **elementary divisors** of $P(\lambda)$ **at** α are the collection of factors

$$(\lambda - \alpha)^{\delta_j} \quad \text{with } \delta_j > 0$$

This is a **local** definition at α that may drive us **out** of \mathbb{F} .

- For a **global** definition that keeps us **in** \mathbb{F} , we follow Gantmacher:

$$p_j(\lambda) = \phi_1(\lambda)^{\beta_{j1}} \phi_2(\lambda)^{\beta_{j2}} \dots \phi_s(\lambda)^{\beta_{js}}, \quad \text{for } j = 1, \dots, r,$$

where $\phi_1(\lambda), \dots, \phi_s(\lambda)$ are **monic scalar polynomials over \mathbb{F} irreducible in \mathbb{F} of degree at least 1.**

The (finite) **elementary divisors of $P(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ in \mathbb{F}** is the collection

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$$P(\lambda) = \begin{bmatrix} 1 & & & \\ & \lambda - 1 & & \\ & & (\lambda - 1)(\lambda^2 + 1) & \\ & & & 0 \end{bmatrix} \in \mathbb{R}[\lambda]^{4 \times 4}$$

- Invariant polynomials of $P(\lambda)$:

$$p_1(\lambda) = 1, p_2(\lambda) = (\lambda - 1), p_3(\lambda) = (\lambda - 1)(\lambda^2 + 1).$$

- Finite elementary divisors of $P(\lambda)$ (in \mathbb{R}): $(\lambda - 1)$, $(\lambda - 1)$, $(\lambda^2 + 1)$

- Eigenvalues of $P(\lambda)$: $\{1, 1, i, -i\}$

- Elementary divisors of $P(\lambda)$ at 1: $(\lambda - 1)$, $(\lambda - 1)$
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Let

$$P(\lambda) = P_d \lambda^d + P_{d-1} \lambda^{d-1} + \cdots + P_0, \quad P_d \neq 0$$

be a matrix polynomial of **degree** d , the **reversal** of $P(\lambda)$ is

$$\text{rev}P(\mu) := \mu^d P\left(\frac{1}{\mu}\right) = P_d + P_{d-1} \mu + \cdots + P_0 \mu^d.$$

Definition (Elementary divisors at ∞)

$\mu = \infty$ is an eigenvalue of $P(\lambda)$ if 0 is an eigenvalue of $\text{rev}P(\mu)$.

The partial multiplicity sequence of $P(\lambda)$ at ∞ is defined as that of 0 in $\text{rev}P(\mu)$.

The elementary divisors μ^{γ_j} , $\gamma_j > 0$, for 0 of $\text{rev}P(\mu)$ are the **elementary divisors for ∞** of $P(\lambda)$.

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Minimal bases

An $m \times n$ matrix polynomial $P(\lambda)$ whose rank r is smaller than m and/or n has non-trivial left and/or right null-spaces over the field $\mathbb{F}(\lambda)$ of rational functions:

$$\begin{aligned}\mathcal{N}_\ell(P) &:= \{y(\lambda)^T \in \mathbb{F}(\lambda)^{1 \times m} : y(\lambda)^T P(\lambda) \equiv 0^T\}, \\ \mathcal{N}_r(P) &:= \{x(\lambda) \in \mathbb{F}(\lambda)^{n \times 1} : P(\lambda)x(\lambda) \equiv 0\}.\end{aligned}$$

$\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$ have bases consisting entirely of vector polynomials.

Definition (Minimal bases)

A **right minimal basis** of $P(\lambda)$ is a basis of $\mathcal{N}_r(P)$

- 1 consisting of vector polynomials
- 2 whose sum of degrees is minimal among all bases of $\mathcal{N}_r(P)$ consisting of vector polynomials.

Analogous definition for **left minimal basis**.

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There are many (infinite) right minimal bases of $P(\lambda)$, but...

Theorem (Forney, 1975)

The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{N}_r(P)$ is always the same.

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These degrees are called the **right minimal indices** of $P(\lambda)$.

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Example of right minimal basis and minimal indices

$$P(\lambda) = \begin{bmatrix} 1 & -\lambda^3 & & & \\ & & 1 & -\lambda & \\ & & & 1 & -\lambda \\ & & & & \\ & & & & \end{bmatrix} \in \mathbb{R}[\lambda]^{3 \times 5}$$

$$\mathcal{N}_r(P) = \text{Span} \left\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{u_2} \right\} = \text{Span} \left\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ \lambda^3 \\ \lambda^2 \\ \lambda \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} \lambda^5 \\ \lambda^2 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{w_2} \right\}$$

Sum of degrees of $\{u_1, u_2\} = 3 + 2 = 5$ (right minimal bases of $P(\lambda)$)

Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

Right minimal indices of $P(\lambda) = \{2, 3\}$

Example of right minimal basis and minimal indices

$$P(\lambda) = \begin{bmatrix} 1 & -\lambda^3 & & & \\ & & 1 & -\lambda & \\ & & & 1 & -\lambda \\ & & & & \\ & & & & \end{bmatrix} \in \mathbb{R}[\lambda]^{3 \times 5}$$

$$\mathcal{N}_r(P) = \text{Span} \left\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{u_2} \right\} = \text{Span} \left\{ \underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ \lambda^3 \\ \lambda^2 \\ \lambda \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} \lambda^5 \\ \lambda^2 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{w_2} \right\}$$

Sum of degrees of $\{u_1, u_2\} = 3 + 2 = 5$ (right minimal bases of $P(\lambda)$)

Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

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Definition

Given an $m \times n$ matrix polynomial $P(\lambda)$ with rank r , the **eigenstructure** of $P(\lambda)$ consists of the following lists of scalar polynomials and non-negative integers:

- (i) the invariant polynomials $p_1(\lambda), \dots, p_r(\lambda)$, with degrees $\delta_1, \dots, \delta_r$ (**finite structure**),
- (ii) the partial multiplicity sequence at ∞ , $\gamma_1, \dots, \gamma_r$ (**infinite structure**),
- (iii) the right minimal indices $\varepsilon_1, \dots, \varepsilon_{n-r}$ (**right singular structure**), and
- (iv) the left minimal indices $\eta_1, \dots, \eta_{m-r}$ (**left singular structure**).

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- 2 Goal of the talk and previous works**
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- 6 Conclusions

If a **complete eigenstructure is prescribed** and a **degree d is also prescribed**, we want

- to find necessary and sufficient conditions for the existence of a matrix polynomial $P(\lambda)$ with precisely this eigenstructure and this degree,
- and to construct such $P(\lambda)$.

Remarks:

- If the degree is not prescribed, then the problem is trivial...one can realize any eigenstructure with degree one via the Kronecker canonical form of pencils.
- As far as we know, all previous results on inverse polynomial eigenvalue problems **do not consider prescribed simultaneously the complete eigenstructure and an arbitrary degree.**

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Previous works on prescribed degree inverse poly eigenproblems

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Theorem (Praagman 1991, De Terán & D & Mackey, 2014)

Let $P(\lambda)$ be an $m \times n$ matrix polynomial of **degree d and rank r** having the following regular and singular eigenstructure:

- r invariant polynomials $p_j(\lambda)$ of degrees δ_j , for $j = 1, \dots, r$,
- r infinite partial multiplicities $\gamma_1, \dots, \gamma_r$,
- $n - r$ right minimal indices $\varepsilon_1, \dots, \varepsilon_{n-r}$, and
- $m - r$ left minimal indices $\eta_1, \dots, \eta_{m-r}$,

where some of the degrees, partial multiplicities or indices can be zero, and/or one or both of the lists of minimal indices can be empty. Then

$$\sum_{j=1}^r \delta_j + \sum_{j=1}^r \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr.$$

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Definition

A list \mathcal{L} of elementary divisors over a field \mathbb{F} is the concatenation of **two lists**:

- a list \mathcal{L}_{fin} of positive integer powers of monic irreducible polynomials of degree at least 1 with coefficients in \mathbb{F} , and
- a list \mathcal{L}_{∞} of elementary divisors $\mu^{\alpha_1}, \mu^{\alpha_2}, \dots, \mu^{\alpha_{g_{\infty}}}$ at ∞ .

Definition

- The length of the longest sublist of \mathcal{L}_{fin} containing powers of the same irreducible polynomial is denoted by $g_{\text{fin}}(\mathcal{L})$.
- The length of \mathcal{L}_{∞} is denoted by $g_{\infty}(\mathcal{L})$.
- The sum of the degrees of the elements in \mathcal{L}_{fin} is denoted by $\delta_{\text{fin}}(\mathcal{L})$.
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Remark.

- In \mathbb{C} , $\mathcal{L}_{\text{fin}} = \{(\lambda - a_1)^{\delta_1}, (\lambda - a_2)^{\delta_2}, \dots, \}$, with $a_i \in \mathbb{C}$.

- In \mathbb{R} ,

$$\mathcal{L}_{\text{fin}} = \{(\lambda - a_1)^{\delta_1}, (\lambda - a_2)^{\delta_2}, \dots, (\lambda^2 + b_1\lambda + c_1)^{\beta_1}, (\lambda^2 + b_2\lambda + c_2)^{\beta_2}, \dots\},$$

with $a_i, b_i, c_i \in \mathbb{R}$ and the quadratic polys having complex roots.

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Theorem

Let \mathcal{L} be a list of elementary divisors over an infinite field \mathbb{F} ,
let $\mathcal{M}_r = \{\varepsilon_1, \dots, \varepsilon_p\}$ be a list of right minimal indices,
let $\mathcal{M}_l = \{\eta_1, \dots, \eta_q\}$ be a list of left minimal indices, and define

$$S(\mathcal{L}) = \delta_{\text{fin}}(\mathcal{L}) + \delta_{\infty}(\mathcal{L}) + \sum_{i=1}^q \eta_i + \sum_{i=1}^p \varepsilon_i.$$

Then, there exists a matrix polynomial $P(\lambda)$ with coefficients in \mathbb{F} of degree d and whose elementary divisors, right minimal indices, and left minimal indices are, respectively, those in the lists \mathcal{L} , \mathcal{M}_r , and \mathcal{M}_l , **if and only if**

- (i) d is a divisor of $S(\mathcal{L})$, (ii) $\frac{S(\mathcal{L})}{d} \geq g_{\text{fin}}(\mathcal{L})$, and (iii) $\frac{S(\mathcal{L})}{d} > g_{\infty}(\mathcal{L})$.

In this case,

- the rank of $P(\lambda)$ is $S(\mathcal{L})/d$, and
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The proof of sufficiency is long and involved, but necessity it is easy

- Necessity of (i) d is a divisor of $S(\mathcal{L})$:

If $P(\lambda)$ exists, then it must satisfy the **Index Sum Theorem**, i.e.,

$$d \operatorname{rank}(P) = S(\mathcal{L}) = \delta_{\text{fin}}(\mathcal{L}) + \delta_{\infty}(\mathcal{L}) + \sum_{i=1}^q \eta_i + \sum_{i=1}^p \varepsilon_i .$$

- Necessity of (ii) $\frac{S(\mathcal{L})}{d} \geq g_{\text{fin}}(\mathcal{L})$: If $P(\lambda)$ exists, then

$$\operatorname{rank}(P) = \frac{S(\mathcal{L})}{d} \geq g_{\text{fin}}(\mathcal{L}) = \text{largest geometric multiplicity finite e-values}$$

- Necessity of (iii) $\frac{S(\mathcal{L})}{d} > g_{\infty}(\mathcal{L})$

If $P(\lambda) = P_0 + \lambda P_1 + \cdots + \lambda^d P_d$ with $P_d \neq 0$ exists, then $0 \neq \operatorname{rev} P(0) = P_d$ and

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Definition (De Terán & D & Mackey, 2014)

A matrix polynomial $R(\lambda)$ of degree $\ell > 0$ is said to be an ℓ -ification of a given matrix polynomial $P(\lambda)$ if for some $r, s \geq 0$ there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda) \begin{bmatrix} R(\lambda) & \\ & I_s \end{bmatrix} V(\lambda) = \begin{bmatrix} P(\lambda) & \\ & I_r \end{bmatrix}.$$

If, in addition, $\text{rev } R(\lambda)$ is an ℓ -ification of $\text{rev } P(\lambda)$, then $R(\lambda)$ is said to be a **strong ℓ -ification** of $P(\lambda)$.

Remarks:

- (strong) ℓ -ifications generalize (strong) linearizations ($\ell = 1$).
- **Strong ℓ -ifications have the same finite and infinite elementary divisors as $P(\lambda)$.**
- We hope that ℓ -ifications will be useful in structured problems where structured linearizations do not exist.

Definition (De Terán & D & Mackey, 2014)

A matrix polynomial $R(\lambda)$ of degree $\ell > 0$ is said to be an ℓ -ification of a given matrix polynomial $P(\lambda)$ if for some $r, s \geq 0$ there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda) \begin{bmatrix} R(\lambda) & \\ & I_s \end{bmatrix} V(\lambda) = \begin{bmatrix} P(\lambda) & \\ & I_r \end{bmatrix}.$$

If, in addition, $\text{rev } R(\lambda)$ is an ℓ -ification of $\text{rev } P(\lambda)$, then $R(\lambda)$ is said to be a *strong* ℓ -ification of $P(\lambda)$.

Remarks:

- (strong) ℓ -ifications generalize (strong) linearizations ($\ell = 1$).
- **Strong ℓ -ifications have the same finite and infinite elementary divisors as $P(\lambda)$.**
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Strong ℓ -ifications indeed exist...(at least sometimes)

Let $P(\lambda) = \lambda^d A_d + \lambda^{d-1} A_{d-1} + \cdots + \lambda A_1 + A_0 \in \mathbb{F}[\lambda]^{m \times n}$, $d = \ell s$, and define the following matrix polynomials of degree $\leq \ell$:

$$B_1(\lambda) := \lambda^\ell A_\ell + \lambda^{\ell-1} A_{\ell-1} + \cdots + \lambda A_1 + A_0,$$

$$B_j(\lambda) := \lambda^\ell A_{\ell j} + \lambda^{\ell-1} A_{\ell(j-1)+1} + \cdots + \lambda A_{\ell(j-1)+1}, \quad \text{for } j = 2, \dots, s.$$

Then

$$C_1^\ell(\lambda) := \begin{bmatrix} B_s(\lambda) & B_{s-1}(\lambda) & B_{s-2}(\lambda) & \cdots & B_1(\lambda) \\ -I_n & \lambda^\ell I_n & 0 & \cdots & 0 \\ & -I_n & \lambda^\ell I_n & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & -I_n & \lambda^\ell I_n \end{bmatrix} \in \mathbb{F}[\lambda]^{(m+(s-1)n) \times sn}$$

is a **strong ℓ -ification** of $P(\lambda)$ (De Terán & D & Mackey, 2014).

The solution of the general inverse problem has allowed us

- 1 To prove that every **singular matrix polynomial** has a strong ℓ -ification for any $\ell > 0$.
- 2 To determine all possible sizes of strong ℓ -ifications of a **given singular matrix polynomial**.
- 3 To characterize all possible sets of minimal indices that strong ℓ -ifications of a **given singular matrix polynomial** may have.
- 4 To establish necessary and sufficient conditions for the existence of strong ℓ -ifications of a **given regular matrix polynomial** of size $n \times n$ and degree d
- 5 and to determine that in this **regular** case the size of any strong ℓ -ification is always

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- 1 Preliminary concepts
- 2 Goal of the talk and previous works
- 3 The Index Sum Theorem
- 4 The main theorem
- 5 Existence, possible sizes and eigenstructures of ℓ -ifications
- 6 Conclusions**

- We have solved an inverse matrix polynomial eigenstructure problem
- which is more complete than other inverse results available in the literature, since it includes simultaneously minimal indices and arbitrary degree.
- This inverse result has been used to determine all possible sizes and eigenstructures of the ℓ -ifications of a given poly $P(\lambda)$.
- Apart from trivial restrictions coming from the rank and geometric multiplicities, we can summarize our results in plain words as

“We can realize with a prescribed degree every eigenstructure allowed by the **Index Sum Theorem** and only these eigenstructures”

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