# The Inverse Complex Eigenvector Problem for Real Tridiagonal Matrices

**Froilán M. Dopico** (U. Carlos III, Spain)
**Beresford Parlett** (U. California-Berkeley, USA)
**Carla Ferreira** (U. Minho, Portugal)

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# Outline

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## General real tridiagonal matrices

We consider real tridiagonal matrices

$$C = \begin{bmatrix} a_1 & f_1 & & & & \\ e_1 & a_2 & f_2 & & & \\ & e_2 & a_3 & f_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & e_{n-2} & a_{n-1} & f_{n-1} \\ & & & & e_{n-1} & a_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

- $C$ is **unreduced** if $e_i \neq 0$ and $f_i \neq 0$, for all $i$.
- Otherwise $C$ is **reduced**.

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**Lemma**

*For any real* **unreduced** *tridiagonal matrix $C \in \mathbb{R}^{n\times n}$ there exists a diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that*

$$D^{-1}CD = S\,T,$$

*where*

$$S = \begin{bmatrix} \pm 1 & & & & \\ & \pm 1 & & & \\ & & \ddots & & \\ & & & \pm 1 & \\ & & & & \pm 1 \end{bmatrix}, \quad T = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

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$$T\boldsymbol{x} = \lambda S \boldsymbol{x}$$

# Tridiagonal matrices with unit superdiagonal: the $J$ form

## Lemma

*For any real* **unreduced** *tridiagonal matrix* $C \in \mathbb{R}^{n\times n}$ *there exists a diagonal matrix* $\tilde{D} \in \mathbb{R}^{n\times n}$ *such that*

$$\tilde{D}^{-1} C \tilde{D} = J,$$

*where*

$$J = \begin{bmatrix} a_1 & 1 & & & \\ c_1 & a_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-2} & a_{n-1} & 1 \\ & & & c_{n-1} & a_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

# $T$-$S$ symmetric vs. $J$ form: Advantages-disadvantages

- $J$-form allows us to use *dqds* algorithms for computing eigenvalues (Day (Ph. D. Thesis, Berkeley, 1995), Parlett (Acta Numerica, 1995), Ferreira & Parlett (*Real-3dqds*, submitted)).
- $T$-$S$ symmetric form is balanced and balanced matrices are often considered advantageous in eigenvalue computations.
- Left eigenvectors of $S\,T$ are very simply related to right eigenvectors:

$$S\,T\,x = \lambda\,x \Longleftrightarrow (x^T\,S)\,S\,T = \lambda\,(x^T\,S) \Longleftrightarrow y^*\,S\,T = \lambda\,y^*\,,$$

  with $y^* = (x^T\,S)$. So, **we only need to compute one of them**.
- Generalized tridiagonal symmetric indefinite eigenvalue problems

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- There are good and **"fast" ($O(n^2)$ cost)** algorithms for computing all eigenvalues of an $n \times n$ nonsymmetric tridiagonal matrix:
  1. Bini, Gemignani, Tisseur (SIMAX 2005) "Ehrlich-Aberth Method".
  2. Ferreira, Parlett (submitted), Real dqds (related to LR).
- But, **we cannot guarantee that they are "backward" stable**,
- since the **stable orthogonal QR-iteration does not preserve the tridiagonal structure** and leads to algorithm with $O(n^3)$ cost.
- In this scenario, to deliver a "bound" on the error of each computed eigenvalue is essential.

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- For that we need a "condition number", to compute a "backward error", and to get from them a "forward error".
- The usual "unstructured" approach is very pessimistic in many critical situations and different "structured approaches" behave very differently.
- Structured eigenvalue cond. numbers have been extensively studied in
  - Ferreira, Parlett, D, *Sensitivity of eigenvalues of an unsymmetric tridiagonal matrix*, Numer. Math., 2012.
- Among many other results, this reference proves that, if $J = \mathcal{L}\mathcal{U}$, then very often for tiny eigenvalues

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- We still need structured backward errors.
- We deduced methods to compute in $O(n)$ flops structured backward errors from approximated eigenpairs $(\tilde{\lambda}, \tilde{x})$ or eigentriples $(\tilde{\lambda}, \tilde{x}, \tilde{y})$.
- For instance, for the $J = \mathcal{L}\mathcal{U}$ form (used in dqds), we computed

$$\eta(\tilde{\lambda}, \tilde{x}) = \min\left\{\epsilon \,:\, (\mathcal{L} + \Delta\mathcal{L})(\mathcal{U} + \Delta\mathcal{U})\tilde{x} = \tilde{\lambda}\tilde{x},\ |\Delta\mathcal{L}| \leq \epsilon|\mathcal{L}|,\ |\Delta\mathcal{U}| \leq \epsilon|\mathcal{U}|\right\}$$

- We tested our method to compute $\eta(\tilde{\lambda}, \tilde{x})$ on many tridiagonal matrices, with eigenvalues/vectors reliably computed by MATLAB, and
- we were happy, since we got almost always tiny $\eta(\tilde{\lambda}, \tilde{x})$.
- But, we asked for more: If $J$ **is real** and $(\tilde{\lambda}, \tilde{x})$ **are complex**, then the backward errors $\Delta\mathcal{L}$ **and** $\Delta\mathcal{U}$ **should be real**.

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# Tridiagonal eigenvalue sensitivity and backward error issues (III)

- We worked hard to compute in a least squares sense $\eta_{\mathbb{R}}(\tilde{\lambda}, \tilde{x})$ and....
- **Disaster:** often $\eta_{\mathbb{R}}(\tilde{\lambda}, \tilde{x})$ **was too large** and **sometimes huge**.
- We were puzzled for a period, but the reason is clear.

$$J = \begin{bmatrix} a_1 & 1 & & \\ c_1 & a_2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & c_{n-1} & a_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$ **depends on** $2n-1$ **real parameters**

- so to look for structured $\Delta J$ such that $(J + \Delta J)\tilde{x} = \tilde{\lambda}\tilde{x}$ leads to

$2n$ real equations for the $2n-1$ real unknowns in $\Delta J$,

- and the system has not solution in general.
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# The questions

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- **When given complex vectors are (right and/or left) eigenvectors of real tridiagonal matrices?**
- **How to construct the corresponding matrices?**

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**Theorem**

*Let* $A \in \mathbb{R}^{n \times n}$ *and let* $\lambda$ *be a nonreal number. If* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ *satisfy*

$$A\,\boldsymbol{x} = \lambda\,\boldsymbol{x} \quad \text{and} \quad \boldsymbol{y}^*\,A = \lambda\,\boldsymbol{y}^*\,,$$

*then* $\quad \boldsymbol{y}^T\,\boldsymbol{x} = 0\,.$

**Remark**

In contrast with $\boldsymbol{y}^*\,\boldsymbol{x} \neq 0$ for simple eigenvalues.

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*Let $C \in \mathbb{R}^{n\times n}$ be tridiagonal and let $\lambda$ be a nonreal eigenvalue of $C$ with geometric multiplicity $1$. If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$ satisfy*

$$C\,\boldsymbol{u} = \lambda\,\boldsymbol{u} \quad \text{and} \quad \boldsymbol{v}^*\,C = \lambda\,\boldsymbol{v}^*\,,$$

*then there exists $0 \neq \alpha \in \mathbb{C}$ such that* $\alpha\,u_k\,v_k \in \mathbb{R}$ *for* $k = 1, 2, \ldots, n.$

**In plain words:**

A pair of complex left-right eigenvectors of a real tridiagonal matrix can always be normalized so that $u_k v_k$ is real for all $k$.

**Remark 2**

- This property is specific of real tridiagonal matrices.

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$$C\,\boldsymbol{u} = \lambda\,\boldsymbol{u} \quad \text{and} \quad \boldsymbol{v}^*\,C = \lambda\,\boldsymbol{v}^*\,,$$

*then there exists $0 \neq \alpha \in \mathbb{C}$ such that* $\alpha\,u_k\,v_k \in \mathbb{R}$ *for* $k = 1, 2, \ldots, n.$

## In plain words:

A pair of complex left-right eigenvectors of a real tridiagonal matrix can always be normalized so that $u_k v_k$ is real for all $k$.

## Remark 2

- This property is specific of real tridiagonal matrices.

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As a consequence of previous slides, for solving the

**Inverse Complex Eigenvector Problem for real tridiagonals**

Given nonzero $u, v \in \mathbb{C}^n$,

- to determine necessary and sufficient conditions under which they are a pair of right-left eigenvectors of a real tridiagonal matrix, and
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we will assume in all our results

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# One additional hypothesis (not essential)

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## Remarks

- Most vectors that satisfy the basic hypotheses $\boldsymbol{v}^T\boldsymbol{u} = 0$ and $u_k v_k \in \mathbb{R}$ for all $k$, are left-right eigenvectors of real tridiagonals.
- The conditions $\mathcal{I}m(v_k\,u_{k+1}) \neq 0$ are surprisingly simple, taking into account that given $v$, $u$, and $\lambda$ one has $4n$ real linear equations for the the $3n-2$ real unknown entries of $C$.

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# Construction of $C$ in $O(n)$ flops

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*Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$ have no zero entries and satisfy $\boldsymbol{v}^T \boldsymbol{u} = 0$, $u_k v_k \in \mathbb{R}$ for $k = 1:n$, and $\mathcal{I}m(v_k\, u_{k+1}) \neq 0$ for $k = 1:n-1$. Choose any $\lambda \in \mathbb{C}$ and construct the following sequences of real numbers:*

- $f_k = \dfrac{\mathcal{I}m(\lambda) \sum_{i=1}^{k} u_i v_i}{\mathcal{I}m(v_k\, u_{k+1})}$, for $k = 1:n-1$,
- $e_k = f_k \dfrac{|v_k|^2\, |u_{k+1}|^2}{(u_k v_k)\,(u_{k+1} v_{k+1})}$, for $k = 1:n-1$,
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*Then*

$$C = \begin{bmatrix} a_1 & f_1 & & \\ e_1 & a_2 & \ddots & \\ & \ddots & \ddots & f_{n-1} \\ & & e_{n-1} & a_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

*is the unique real tridiagonal matrix that satisfies $C\,\boldsymbol{u} = \lambda\, \boldsymbol{u}$ and $\boldsymbol{v}^*\, C = \lambda\, \boldsymbol{v}^*$.*

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*Then,*

- $C = \mathcal{R}e(\lambda)\, I_n + \mathcal{I}m(\lambda)\, C^{(\mathrm{i})}$ *is the unique real tridiagonal matrix such that $C\, \boldsymbol{u} = \lambda\, \boldsymbol{u}$ and $\boldsymbol{v}^*\, C = \lambda\, \boldsymbol{v}^*$.*
- $\mathcal{W} = \mathrm{Span}_{\mathbb{R}}\{I_n\, ,\, C^{(\mathrm{i})}\}$ *is the family of all real tridiagonal matrices with $(\boldsymbol{u}, \boldsymbol{v})$ as a pair of right-left eigenvectors.*

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# Just existence of $C$

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# For completeness: if only one vector is prescribed?

- It is natural to wonder what happens if only $\boldsymbol{u} \in \mathbb{C}^n$ (or $\boldsymbol{v}$) is prescribed.
- This problem does not seem very challenging since $C\boldsymbol{u} = \lambda \boldsymbol{u}$ gives $2n$ real equations for the $3n-2$ unknown entries of $C \in \mathbb{R}^{n\times n}$.

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# Outline

# Three key points

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Recall,

$$S\,T\,x = \lambda\,x \Longleftrightarrow (x^T\,S)\,S\,T = \lambda\,(x^T\,S) \Longleftrightarrow y^*\,S\,T = \lambda\,y^*, \quad \text{i.e.,} \quad y = S\,\overline{x}$$

Therefore in the inverse problem:

1. **Only one vector should be prescribed** if $S$ is prescribed.
2. **The first basic hypothesis** $y^T x = 0$ reduces to $x^* S x = 0$.
3. **We do not need the second basic hypothesis** $x_k y_k \in \mathbb{R}$ for all $k$, **since it is automatically guaranteed by the structure** of the problem:

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**Theorem**

*Let $S$ be an indefinite signature matrix and let $\boldsymbol{x} \in \mathbb{C}^n$ have no zero entries and satisfy $\boldsymbol{x}^* S \boldsymbol{x} = 0$. For each nonreal $\lambda \in \mathbb{C}$ there exists a unique symmetric real tridiagonal matrix $T \in \mathbb{R}^{n \times n}$ such that*

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# Construction of $T$ in $O(n)$ flops

## Theorem

*Let $S$ be an indefinite signature matrix and let $\boldsymbol{x} \in \mathbb{C}^n$ have no zero entries, satisfy $\boldsymbol{x}^* S \boldsymbol{x} = 0$, and $\mathcal{I}m(\overline{x_k}\, x_{k+1}) \neq 0$ for $k = 1 : n-1$. Choose any $\lambda \in \mathbb{C}$ and construct the following sequences of real numbers:*

- $$b_k = \frac{\mathcal{I}m(\lambda)\ \sum_{i=1}^{k} s_i\, |x_i|^2}{\mathcal{I}m(\overline{x_k}\, x_{k+1})}, \qquad \text{for} \quad k = 1 : n-1,$$
- $$a_k = s_k\, \mathcal{R}e(\lambda) - \frac{b_{k-1} \mathcal{R}e(x_{k-1}\, \overline{x_k}) + b_k \mathcal{R}e(\overline{x_k} x_{k+1})}{|x_k|^2}, \qquad \text{for} \quad k = 1 : n.$$

*Then*

$$T = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

*is the unique real symmetric tridiagonal matrix that satisfies $T\, \boldsymbol{x} = \lambda\, S\, \boldsymbol{x}$.*

**Theorem**

*Let $S$ be an indefinite signature matrix and let $\boldsymbol{x} \in \mathbb{C}^n$ have no zero entries and satisfy $\boldsymbol{x}^* S \boldsymbol{x} = 0$. For each nonreal $\lambda \in \mathbb{C}$ there exists a symmetric real tridiagonal matrix $T \in \mathbb{R}^{n \times n}$ such that*

$$T\boldsymbol{x} = S\boldsymbol{x}\lambda$$

*if, and only if,*

$$\mathcal{I}m(\overline{x_k}\, x_{k+1}) = 0 \Longrightarrow \sum_{i=1}^{k} s_i |x_i|^2 = 0, \quad \text{for} \quad k = 1, \dots, n-1.$$

# Outline

- We have solved two inverse problems:
  1. A pair of potential right-left eigenvectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$ is given.
  2. Only one potential right eigenvector $\boldsymbol{u} \in \mathbb{C}^n$ is given.
- **Bottom line:** The inverse problems for the $J$-form **are rather different** that for general tridiagonals and for the T-S symmetric form, since the eigenvalue $\lambda$ has to be particularly related to the pair $(\boldsymbol{u}, \boldsymbol{v})$ or to $\boldsymbol{u}$.

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# Outline

- The simplicity of reconstruction expressions as, for instance,

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  in the $T$-$S$ symmetric form,
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## Refining $\mathcal{R}e(\lambda)$ and $\mathcal{I}m(\lambda)$ in the $T$-$S$ framework

- Assume $T$ and $S$ are given, we have computed an approximate nonreal $\widetilde{\lambda}$, and from it an approximate eigenvector $\widetilde{x}$.
- Assume $\widetilde{x}$ satisfies conditions for being e-vector (or we force it) and let $T^{(\mathrm{i})}\,\widetilde{x} = \mathrm{i}\,S\,\widetilde{x}$, then

$$T^{(\lambda)} = \mathcal{R}e(\lambda)\,S + \mathcal{I}m(\lambda)\,T^{(\mathrm{i})}$$

  is the unique real symmetric tridiagonal matrix such that $T^{(\lambda)}\,\widetilde{x} = \lambda\,S\,\widetilde{x}$.
- The solution of

$$\min_{\lambda\in\mathbb{C}} \|T^{(\lambda)} - T\|_F$$

  can be obtained in $24n$ flops solving a standard real least squares problem for $\mathcal{R}e(\lambda)$ and $\mathcal{I}m(\lambda)$,
- **just by vectorizing the nontrivial diagonals of $T$, $S$, and $T^{(\mathrm{i})}$.**
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