Recent Advances on Inverse Problems for Matrix Polynomials: The Inverse Row-Degree Problem for Dual Minimal Bases

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joint work with **Fernando De Terán** (UC3M),
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Definition

The complete eigenstructure of an $m \times n$ matrix polynomial $P(\lambda)$ of **rank** r is given by:

- r invariant polynomials $p_1(\lambda), \dots, p_r(\lambda)$,

  (equivalently the finite eigenvalues of $P(\lambda)$ and their Jordan structures),
- r infinite partial multiplicities $\gamma_1, \dots, \gamma_r$,

  (equivalently the Jordan structure of the infinite eigenvalue of $P(\lambda)$),
- $n - r$ right minimal indices $\varepsilon_1, \dots, \varepsilon_{n-r}$, and
- $m - r$ left minimal indices $\eta_1, \dots, \eta_{m-r}$.

Remark

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The complete inverse eigenstructure problem of matrix polynomials

If a **complete eigenstructure** and **a degree** d are **prescribed**, one wants

1. to find necessary and sufficient conditions for the existence of a matrix polynomial $P(\lambda)$ with precisely this eigenstructure and this degree,
2. to construct such $P(\lambda)$,
3. and, ideally, to construct $P(\lambda)$ in such a way that reveals "as simply as possible" the realized complete eigenstructure or a significant part of it.

Remarks:

- If the degree is not prescribed, the problem is trivial: $d = 1$ via the KCF.
- Goals 1 and 2 achieved in De Terán, D, Van Dooren, SIMAX, 2015 and the solution is heavily based on **dual minimal bases**.
- Goal 3 still under development $\longrightarrow$ **see key advances in Van Dooren's talk** and more to come soon.

Fundamental tool in Goal 3: Polynomial Zigzag Matrices for solving the inverse row-degree problem for dual minimal bases

(De Terán, D, Mackey, Van Dooren, LAA, in press)

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- Marques de Sá, LAA 1979: regular polys with nonsingular leading coefficient, arbitrary degree.
- Gohberg, Lancaster, Rodman, book, 1982: regular monic polys, arbitrary degree.
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Outline

1. Preliminary concepts
2. The inverse row-degree problem for dual minimal bases
3. Polynomial Zigzag Matrices
4. Solving the inverse row-degree problem for dual minimal bases
5. Conclusions

Outline

Definition (Minimal Indices of a Matrix Pencil)

Let $A - \lambda B$ be a matrix pencil with Kronecker Canonical Form

$$U(A-\lambda B)V = L_{\varepsilon_1} \oplus \cdots \oplus L_{\varepsilon_p} \oplus L_{\eta_1}^T \oplus \cdots \oplus L_{\eta_q}^T \oplus J_{k_1}(\lambda - \lambda_1) \oplus \cdots \oplus J_{k_f}(\lambda - \lambda_f) \oplus N_{\ell_1}(\lambda) \oplus \cdots \oplus N_{\ell_s}(\lambda),$$

where

$$L_\varepsilon = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}_{\varepsilon \times (\varepsilon+1)}, \quad L_\eta^T = \begin{bmatrix} 1 & & \\ \lambda & \ddots & \\ & \ddots & 1 \\ & & \lambda \end{bmatrix}_{(\eta+1)\times\eta}.$$

Then $\varepsilon_1, \ldots, \varepsilon_p$ are the **right minimal indices** of $A - \lambda B$ and $\eta_1, \ldots, \eta_q$ are the **left minimal indices** of $A - \lambda B$.

To extend the notion of minimal indices to matrix polynomials of arbitrary degree requires some additional concepts.

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In this talk:

- $\mathbb{F}$ is an arbitrary field and
- $\mathbb{F}[\lambda]$ is the ring of polynomials with coefficients in $\mathbb{F}$.
- In addition, $\mathbb{F}(\lambda)$ is the field of rational functions over $\mathbb{F}$ and
- $\mathbb{F}(\lambda)^n$ is the vector space over $\mathbb{F}(\lambda)$ of n-tuples with entries in $\mathbb{F}(\lambda)$.
- **Example:**

$$\begin{bmatrix} \dfrac{\lambda+2}{\lambda^2} \\ \dfrac{1}{(\lambda+1)^3} \end{bmatrix} \in \mathbb{R}(\lambda)^2$$

- $\mathbb{F}(\lambda)^n$ **is said to be a rational vector space and its subspaces are rational vector subspaces.** (Wolovich-1974, Forney-1975)

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$$\begin{bmatrix} \frac{\lambda+2}{\lambda^2} \\ \frac{1}{(\lambda+1)^3} \end{bmatrix} \in \mathcal{V} \Longrightarrow \lambda^2(\lambda+1)^3 \begin{bmatrix} \frac{\lambda+2}{\lambda^2} \\ \frac{1}{(\lambda+1)^3} \end{bmatrix} = \begin{bmatrix} (\lambda+2)(\lambda+1)^3 \\ \lambda^2 \end{bmatrix} \in \mathcal{V}$$

Definition (Minimal basis)

A **minimal basis** of the rational subspace $\mathcal{V} \in \mathbb{F}(\lambda)^n$ is a basis

1. consisting of vector polynomials
2. whose sum of degrees is minimal among all bases of $\mathcal{V}$ consisting of vector polynomials.

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Minimal indices of rational vector subspaces

There are infinitely many minimal bases of a rational subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$, but...

Theorem (Forney, 1975. Gantmacher, 1959...probably known before)

The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$ is always the same.

Definition

These degrees are called the **minimal indices** of $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$.

Minimal bases and indices were introduced by Plemelj-1908, Muskhelishvili and Vekua-1943, but **Forney-1975 made this concept very important in Multivariable Linear System Theory**, then appeared in the book by Kailath-1980, ...

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An $m \times n$ matrix polynomial $P(\lambda)$ whose rank r is smaller than m and/or n has non-trivial left and/or right rational null-spaces (over the field $\mathbb{F}(\lambda)$ of rational functions):

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

Definition (Right minimal bases and indices of $P(\lambda)$)

The **right minimal bases and indices** of $P(\lambda)$ are those of $\mathcal{N}_r(P)$.

Analogous definitions for **left minimal** bases and indices.

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Example of right minimal basis and indices of a matrix polynomial

$$P(\lambda) = \begin{bmatrix} 1 & -\lambda^3 & & & \\ & & 1 & -\lambda & \\ & & & 1 & -\lambda \end{bmatrix} \in \mathbb{R}[\lambda]^{3\times 5}$$

$$\mathcal{N}_r(P) = \text{Span}\{\underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{u_2}\} = \text{Span}\{\underbrace{\begin{bmatrix} \lambda^3 \\ 1 \\ \lambda^3 \\ \lambda^2 \\ \lambda \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} \lambda^5 \\ \lambda^2 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}}_{w_2}\}$$

Sum of degrees of $\{u_1, u_2\} = 3 + 2 = 5$ (right minimal bases of $P(\lambda)$)

Sum of degrees of $\{w_1, w_2\} = 3 + 5 = 8$

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REMARK: We often arrange minimal bases as the rows of matrices and call "basis" to the matrix.

Theorem (Forney 1975...probably known before)

The rows of a matrix polynomial $N(\lambda)$ over a field $\mathbb{F}$ are a minimal basis of the subspace they span if and only if

(a) $N(\lambda_0)$ *has full row rank for all* $\lambda_0 \in \overline{\mathbb{F}}$*, and*

(b) *the highest-row-degree coefficient matrix of* $N(\lambda)$ *has also full row rank.*

Example (of minimal basis)

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Outline

Definition (Dual Minimal Bases. (Forney, 1975))

Matrix polynomials $M(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ are said to be **dual minimal bases** if

(a) both are minimal bases,

(b) $m + k = n$,

(c) and $M(\lambda)\, N(\lambda)^T = 0$.

Remark

- Dual minimal bases have classical applications in Linear System Theory for constructing left and right coprime factorizations of transfer functions,
- and modern applications for constructing strong linearizations and ℓ-ifications of matrix polynomials, solving inverse eigenstructure problems for matrix polynomials, and performing backward error-analyses of polynomial eigenvalue problems solved by linearizations...

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Example ($M(\lambda)N(\lambda)^T = 0$)

$$M(\lambda) = \begin{bmatrix} 1 & \lambda & & \\ & 1 & \lambda & \\ & & 1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{3\times 4}$$

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Remarks

In general, for dual minimal bases $M(\lambda)N(\lambda)^T = 0$:

- $M(\lambda)$ is a left minimal basis of $N(\lambda)^T$ (so, the row-degrees of $M(\lambda)$ are the left minimal indices of $N(\lambda)^T$) and
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Dual Minimal Bases: Example and comments

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Forney's theorem and the inverse row-degree problem

Theorem (Forney 1975, but probably known before)

Let $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *and* $N(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ *be dual minimal bases with row degrees* $(\eta_1, \ldots, \eta_m)$ *and* $(\varepsilon_1, \ldots, \varepsilon_k)$*, respectively. Then*

$$\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j \, .$$

GOAL OF THE TALK: Solve the corresponding INVERSE PROBLEM

Given any two lists of nonnegative integers $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$ that have the same sum:

- do there exist dual minimal bases having these numbers as their row degrees?
- can we explicitly construct such dual minimal bases?
- can we do it in such a way that $M(\lambda)$ reveals transparently $(\varepsilon_1, \ldots, \varepsilon_k)$ and vice versa?

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Outline

Example of forward Zigzag matrix:

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{bmatrix}$$

Definition

$Z(\lambda) \in \mathbb{F}^{m \times n}$ with $m < n$ is a forward-zigzag matrix, if

(a) each row of $Z(\lambda)$ is of the form

$$\Big[\underbrace{0 \;\; \dots \;\; 0}_{\text{Maybe none}} \;\; 1 \;\; \lambda^{p_1} \;\; \lambda^{p_2} \;\; \dots \;\; \lambda^{p_k} \;\; \underbrace{0 \;\; \dots \;\; 0}_{\text{Maybe none}}\Big],$$

with $0 < p_1 < p_2 < \cdots < p_k$ and $k \geq 1$.

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- **Every zigzag matrix is a minimal basis.**
- Zigzag matrices generalize, in a nontrivial way, to degrees larger than 1 right singular blocks of the KCF of pencils, which are the unique zigzag matrices that have all the row-degrees equal to 1.

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- **Unit column sequence of a forward-zigzag matrix.** In the example

  U,N,N,U,U,N,N,N,U,N,N

- **Degree-gap sequence of a forward-zigzag matrix.** In the example

  $$2, 5, 1, 3, 1, 3, 4, 7, 2, 1$$

- **Structure sequence of a forward-zigzag matrix.** In the example

$$\left[\ _{\mathrm{U}}\ 2\ _{\mathrm{N}}\ 5\ _{\mathrm{N}}\ 1\ _{\mathrm{U}}\ 3\ _{\mathrm{U}}\ 1\ _{\mathrm{N}}\ 3\ _{\mathrm{N}}\ 4\ _{\mathrm{N}}\ 7\ _{\mathrm{U}}\ 2\ _{\mathrm{N}}\ 1\ _{\mathrm{N}}\ \right]$$

From the structure sequence we can construct $Z(\lambda)$.

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Tools for describing Forward-Zigzag Matrices

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- **Structure sequence of a forward-zigzag matrix.** In the example

$$\left[\ \mathrm{U}\quad 2\quad \mathrm{N}\quad 5\quad \mathrm{N}\quad 1\quad \mathrm{U}\quad 3\quad \mathrm{U}\quad 1\quad \mathrm{N}\quad 3\quad \mathrm{N}\quad 4\quad \mathrm{N}\quad 7\quad \mathrm{U}\quad 2\quad \mathrm{N}\quad 1\quad \mathrm{N}\ \right]$$

From the structure sequence we can construct $Z(\lambda)$.

Example of backward Zigzag matrix:

$$\widehat{Z}(\lambda) = \begin{bmatrix} \lambda^2 & 1 & & & & & & & & & \\ & \lambda^5 & 1 & & & & & & & & \\ & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\ & & & & & \lambda^3 & 1 & & & & \\ & & & & & & \lambda^4 & 1 & & & \\ & & & & & & & \lambda^9 & \lambda^2 & 1 & \\ & & & & & & & & & \lambda & 1 \end{bmatrix}$$

- Backward Zigzag matrices are obtained by reversing the order of rows and columns of forward Zigzag matrices.
- Similar properties and tools as for forward Zigzag matrices.
- In the example, the structure sequence is

$$[\ _{\mathrm{N}}\ 2\ _{\mathrm{U}}\ 5\ _{\mathrm{U}}\ 1\ _{\mathrm{N}}\ 3\ _{\mathrm{N}}\ 1\ _{\mathrm{U}}\ 3\ _{\mathrm{U}}\ 4\ _{\mathrm{U}}\ 7\ _{\mathrm{N}}\ 2\ _{\mathrm{U}}\ 1\ _{\mathrm{U}}\ ].$$

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\end{bmatrix}
$$

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$$

Definition

Suppose

1. $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is a forward-zigzag matrix and
2. $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{k\times n}$ is a backward-zigzag matrix

with the same number of columns. Then $Z(\lambda)$ and $Z^{\diamondsuit}(\lambda)$ are said to be dual zigzag matrices, if they have

(a) the same degree-gap sequence, but

(b) complementary unit column sequences, where U and N are each other's complement.

Example of Dual Zigzag Matrices

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{bmatrix}$$

$$Z^{\diamondsuit}(\lambda) = \begin{bmatrix} \lambda^2 & 1 & & & & & & & & & \\ & \lambda^5 & 1 & & & & & & & & \\ & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\ & & & & & \lambda^3 & 1 & & & & \\ & & & & & & \lambda^4 & 1 & & & \\ & & & & & & & \lambda^9 & \lambda^2 & 1 & \\ & & & & & & & & & \lambda & 1 \end{bmatrix}$$

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$$Z(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\
 & & & 1 & \lambda^3 & & & & & & \\
 & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\
 & & & & & & & & 1 & \lambda^2 & \lambda^3
\end{array}\right]$$

$$Z^{\diamondsuit}(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\lambda^2 & 1 & & & & & & & & & \\
 & \lambda^5 & 1 & & & & & & & & \\
 & & \lambda^5 & \lambda^4 & \lambda & 1 & & & & & \\
 & & & & & \lambda^3 & 1 & & & & \\
 & & & & & & \lambda^4 & 1 & & & \\
 & & & & & & & \lambda^9 & \lambda^2 & 1 & \\
 & & & & & & & & & \lambda & 1
\end{array}\right]$$

Both have the same degree-gap sequence:

$$2, 5, 1, 3, 1, 3, 4, 7, 2, 1$$

$$Z(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
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 & & & 1 & \lambda^3 & & & & & & \\
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\end{array}\right]$$

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First Key result on Dual Zigzag Matrices: they "are" Dual Minimal Bases

Theorem (from dual Zigzag to dual minimal bases)

Suppose

- $Z(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *and* $Z^{\diamondsuit}(\lambda) \in \mathbb{F}[\lambda]^{(n-m)\times n}$ *are dual Zigzag matrices, and*
- $\Sigma_n := \operatorname{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1})$.

Then $Z(\lambda)$ **and** $(Z^{\diamondsuit}(\lambda) \cdot \Sigma_n)$ **are dual minimal bases.**

$$Z(\lambda) = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & \\
 & & & 1 & \lambda^3 & & & & & & \\
 & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & \\
 & & & & & & & & 1 & \lambda^2 & \lambda^3
\end{array}\right]$$

$$Z^{\diamondsuit}(\lambda) \cdot \Sigma_n = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\lambda^2 & -1 & & & & & & & & & \\
 & -\lambda^5 & 1 & & & & & & & & \\
 & & \lambda^5 & -\lambda^4 & \lambda & -1 & & & & & \\
 & & & & & -\lambda^3 & 1 & & & & \\
 & & & & & & \lambda^4 & -1 & & & \\
 & & & & & & & -\lambda^9 & \lambda^2 & -1 & \\
 & & & & & & & & & -\lambda & 1
\end{array}\right]$$

Both have the same degree-gap sequence:

$$2, 5, 1, 3, 1, 3, 4, 7, 2, 1$$

Second Key result on Dual Zigzag Matrices: $Z(\lambda)$ reveals transparently the row degrees of its dual

, i.e., $Z(\lambda)$ reveals transparently its right minimal indices.

Lemma

Suppose $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ is a forward-zigzag matrix with structure sequence

$$\mathcal{S} = \begin{bmatrix} \mathrm{s}_1 & \delta_1 & \mathrm{s}_2 & \delta_2 & \dots & \mathrm{s}_{n-1} & \delta_{n-1} & \mathrm{s}_n \end{bmatrix}.$$

Then its dual, $Z^{\diamondsuit}(\lambda)$, has row degrees equal to the partial sums of degree gaps before the first N and between any two consecutive N's.

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Example of forward Zigzag matrix:

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 & & & & & & & & \\ & & & 1 & \lambda^3 & & & & & & & \\ & & & & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & & & \\ & & & & & & & & 1 & \lambda^2 & \lambda^3 \end{bmatrix}$$

$$\mathcal{S} = \left[\text{U} \quad 2 \quad \text{N} \quad 5 \quad \text{N} \quad 1 \quad \text{U} \quad 3 \quad \text{U} \quad 1 \quad \text{N} \quad 3 \quad \text{N} \quad 4 \quad \text{N} \quad 7 \quad \text{U} \quad 2 \quad \text{N} \quad 1 \quad \text{N} \right]$$

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$$\mathcal{S} = [\, \text{U} \; 2 \; \text{N} \; 5 \; \text{N} \; 1 \; \text{U} \; 3 \; \text{U} \; 1 \; \text{N} \; 3 \; \text{N} \; 4 \; \text{N} \; 7 \; \text{U} \; 2 \; \text{N} \; 1 \; \text{N} \,]$$

Row degrees $Z[\lambda]$

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Row degrees $Z^{\diamond}[\lambda]$ = Right Minimal Indices of $Z[\lambda]$

$$(2,5,1+3+1,3,4,7+2,1) = (2,5,5,3,4,9,1)\,.$$

Outline

Solution of the inverse row-degree problem for dual minimal bases

- The two key properties of dual Zigzag matrices and the simplicity of Zigzag matrices allow us

to solve the inverse row-degree problem

Given any two lists of nonnegative integers $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$ that have the same sum:

- do there exist dual minimal bases $M(\lambda)$ and $N(\lambda)$ having these numbers as their row degrees? **Yes**.
- can we explicitly construct such dual minimal bases? **Yes**.
- can we do it in such a way that $M(\lambda)$ reveals transparently $(\varepsilon_1, \ldots, \varepsilon_k)$ and vice versa? **Yes**.

- easily in such a way that $M(\lambda)$ and $N(\lambda)$ are constructed as (direct sums of) dual Zigzag matrices via a simple algorithm.

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Solving the inverse problem for dual Zigzag matrices: Construction

Example: $(\eta_1, \eta_2, \eta_3, \eta_4) = (8, 3, 15, 3)$, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7) = (2, 5, 5, 3, 4, 9, 1)$.

(1) Define the partial sums $\ell_0 := 0$,

$$\ell_\alpha := \sum_{i=1}^{\alpha} \eta_i, \quad \alpha = 1, 2, 3, \quad \text{and} \quad r_\beta := \sum_{i=1}^{\beta} \varepsilon_i, \quad \beta = 1, \ldots, 7.$$

(2) Order them in two lists

$$\begin{bmatrix} \ell_0 & \ell_1 & \ell_2 & \ell_3 \\ 0 & 8 & 11 & 26 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 \\ 2 & 7 & 12 & 15 & 19 & 28 & 29 \end{bmatrix}.$$

(3) Merge both lists in one ordered list

$$\begin{bmatrix} \ell_0 & r_1 & r_2 & \ell_1 & \ell_2 & r_3 & r_4 & r_5 & \ell_3 & r_6 & r_7 \\ 0 & 2 & 7 & 8 & 11 & 12 & 15 & 19 & 26 & 28 & 29 \end{bmatrix}.$$

(4) Replacements $\ell_i \to$ U, $r_j \to$ N gives unit column sequence of $Z(\lambda)$:

U,N,N,U,U,N,N,N,U,N,N

(5) Differences of consecutive terms gives the degree gap sequence of $Z(\lambda)$:

$$2, 5, 1, 3, 1, 3, 4, 7, 2, 1$$

Outline

Conclusions

- We have found an explicit simple solution of the inverse row degree problem for dual minimal bases via the new class of Zigzag matrices.
- This solution has been used (or is being used) by us and others (Lawrence, Pérez, Van Barel, ...) for:
- constructing strong linearizations and ℓ-ifications of matrix polynomials with certain desired properties,
- in backward error analyses of numerical algorithms for solving polynomial eigenvalue problems via linearizations, and
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