Strong linearizations of rational matrices: definition, explicit constructions, and associated recovery procedures

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joint work with **Agurtzane Amparan** (U. País Vasco, Spain),
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Setting (I): Rational eigenvalue problems (REPs)

- Given a nonsingular rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ (in practice $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) the rational eigenvalue problem (REP) consists in computing numbers $\lambda_0 \in \bar{\mathbb{F}}$ and vectors $x_0 \in \bar{\mathbb{F}}^p$ such that

$$G(\lambda_0)x_0 = 0.$$

- REPs appear in different applications. Examples can be found for instance in
  1. Mehrmann & Voss. GAMM-Reports, 2004,
  2. Su & Bai. SIMAX, 2011,
  3. Mohammadi & Voss, submitted, 2017,
  4. Karl Meerbergen's talk: REPs as approximations of other NLEPs.
- Example from Mehrmann & Voss, 2004: Damped vibration of a structure.

$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{\sigma_i}{\lambda + \sigma_i} L_i L_i^T,$$

$M, K \in \mathbb{R}^{p\times p}$ symmetric positive definite, $L_i \in \mathbb{R}^{p\times r_i}$, $r_i \ll p$ (rational parts with low rank are common in applications), $\sigma_i > 0$.

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Setting (II): Solution via Linearization + QZ (or other method)
Su & Bai, SIMAX 2011

1. Su & Bai (2011) write (via easy manipulations) rational matrices as the one in previous slide as
$$G(\lambda) = D_q\lambda^q + D_{q-1}\lambda^{q-1} + \cdots + D_0 + C(\lambda E - A)^{-1}B \ \in \mathbb{F}(\lambda)^{p\times p},$$
with $E \in \mathbb{F}^{n\times n}$ nonsingular, which is possible for any rational matrix (Rosenbrock, 1970-REALIZATIONS!!).

2. Then, they construct
$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda E - A & 0 & 0 & \dots & 0 & B \\ \hline -C & \lambda D_q + D_{q-1} & D_{q-2} & \cdots & D_1 & D_0 \\ 0 & -I_p & \lambda I_p & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & -I_p & \lambda I_p \end{array}\right],$$

3. and compute the eigenvalues of $G(\lambda)$ as the eigenvalues of the pencil $L(\lambda)$. They can also recover eigenvectors.

4. In large scale problems this allows to extend TOAR or CORK for PEPs to get memory efficient algorithms (Dopico & González-Pizarro, 2017).

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Setting (III): Open problems suggested by Su & Bai's paper (SIMAX 2011)

- Su & Bai's paper is a pioneer contribution that introduces a new, robust, and clear way to compute eigenvalues of REPs, but
- the provided theory is not complete (although is enough in most practical scenarios). More precisely:
- due to the lack of a key technical assumption on $C(\lambda E - A)^{-1}B$, it is not guaranteed that all (finite) eigenpairs of the rational matrix $G(\lambda)$ can be obtained from the (finite) eigenpairs of the linearization $L(\lambda)$;
- in case of multiple eigenvalues, it is not proved that they have the same partial multiplicities in the rational matrix $G(\lambda)$ and in the linearization $L(\lambda)$;
- only linearizations without eigenvalues at ∞ are considered, and no relation is established with the structure at infinite of the rational matrix $G(\lambda)$;
- no explicit definition is provided for "linearization" of a rational matrix and/or the properties it must satisfy;
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Setting (IV): Contribution by Alam & Behera (SIMAX 2016) (Behera's PhD Thesis 2014)

- These authors take care of many of the open problems suggested by Su & Bai's paper.
- They provide a clear definition of when a pencil, i.e., a linear matrix polynomial, is a linearization of a square rational matrix that may be regular or singular.
- Their definition guarantees that the complete structures of finite zeros and finite poles of the rational matrix are inside the linearization, which allows us to get from the linearization the finite eigenvalues (those finite zeros that are not poles) including partial multiplicities.
- They provide many other examples of Fiedler-like linearizations of rational matrices, apart from the unique one presented by Su & Bai.

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Setting (V): Some fundamental issues remain unsolved

Despite the very important advances made by Alam & Behera some fundamental issues remained unsolved:

1. No connection is established at all between the structure at infinity of the rational matrix and the one of the linearizations proposed so far, and the available definition does not seem amenable for getting this.
2. Rectangular rational matrices have not been considered.
3. The available definition does not guarantee that the transfer function of the linearization is "equivalent" to the original rational matrix. So, though the eigenvalues are in the linearization, other interesting properties can be missed.

In this scenario, **our goals are...**

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Goals of the talk

- To provide a definition of strong linearization of an arbitrary rational matrix that guarantees that the complete structures of finite and **infinite** zeros and poles of the rational matrix are inside the linearization.
- To emphasize that such definition guarantees that the "transfer" function of any strong linearization is "equivalent" (finite and at infinity) to the given rational matrix.
- To present infinitely many examples of such strong linearizations immediately constructible if the rational matrix is given in the form mentioned before, i.e.,

$$G(\lambda) = D_q\lambda^q + D_{q-1}\lambda^{q-1} + \cdots + D_0 + C(\lambda E - A)^{-1}B \ \in \mathbb{F}(\lambda)^{p\times m},$$

  or even if the polynomial part is expressed in some other important different bases

$$G(\lambda) = D_q b_q(\lambda) + D_{q-1}b_{q-1}(\lambda) + \cdots + D_0 + C(\lambda E - A)^{-1}B,$$

  whenever $C(\lambda E - A)^{-1}B$ is a **minimal order state-space realization**.
- To provide simple recovery rules for the eigenvectors from all these linearizations.

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- Very active area of research in the last decade: closely related to numerical algorithms for polynomial eigenproblems,
- even in the large-scale setting via Krylov methods for such problems: SOAR (Bai & Su, 2005), Q-Arnoldi (Meerbergen, 2008), TOAR (Su & Bai & Lu, 2008, 2016), Chebyshev basis (Kressner & Roman, 2014), CORK (Van Beeumen & Meerbergen & Michiels, 2015), Parallel-Symmetric versions (Campos & Roman, 2016)...
- A linearization for $D(\lambda) = D_d \lambda^d + \cdots + D_1 \lambda + D_0$ is a **matrix pencil** $\mathcal{L}(\lambda)$, such that,

$$U(\lambda)\,\mathcal{L}(\lambda)\,V(\lambda) = \begin{bmatrix} I_s & \\ & D(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

- $\mathcal{L}(\lambda)$ is a "strong linearization" if, **in addition**, $\operatorname{rev} \mathcal{L}(\lambda)$ is a linearization for $\operatorname{rev} P(\lambda)$, where $\operatorname{rev} D(\lambda) := D_0 \lambda^d + \cdots + D_{d-1} \lambda + D_d = \lambda^d D(1/\lambda)$.

$D(\lambda)$ and $\mathcal{L}(\lambda)$ have the same finite and infinite elementary divisors.

- Our definition of strong linearization for rational matrices is motivated by and collapses to the one for polynomial matrices.

Always in my mind: strong linearizations of polynomial matrices

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- even in the large-scale setting via Krylov methods for such problems: SOAR (Bai & Su, 2005), Q-Arnoldi (Meerbergen, 2008), TOAR (Su & Bai & Lu, 2008, 2016), Chebyshev basis (Kressner & Roman, 2014), CORK (Van Beeumen & Meerbergen & Michiels, 2015), Parallel-Symmetric versions (Campos & Roman, 2016)...
- A linearization for $D(\lambda) = D_d\,\lambda^d + \cdots + D_1\lambda + D_0$ is a **matrix pencil** $\mathcal{L}(\lambda)$, such that,

$$U(\lambda)\,\mathcal{L}(\lambda)\,V(\lambda) = \begin{bmatrix} I_s & \\ & D(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

- $\mathcal{L}(\lambda)$ is a "strong linearization" if, **in addition**, $\mathrm{rev}\,\mathcal{L}(\lambda)$ is a linearization for $\mathrm{rev}\,P(\lambda)$, where $\mathrm{rev}\,D(\lambda) := D_0\,\lambda^d + \cdots + D_{d-1}\,\lambda + D_d = \lambda^d\,D(1/\lambda)$.

$D(\lambda)$ and $\mathcal{L}(\lambda)$ have the same finite and infinite elementary divisors.

- Our definition of strong linearization for rational matrices is motivated by and collapses to the one for polynomial matrices.

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Outline

1. Basics on rational matrices with emphasis on structure at infinity
2. Definition of strong linearizations of rational matrices
3. Explicit constructions of many strong linearizations
4. Recovery of eigenvectors

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2. Definition of strong linearizations of rational matrices
3. Explicit constructions of many strong linearizations
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Polynomial and strictly proper parts of a rational matrix

- **Any rational matrix $G(\lambda)$ can be uniquely expressed as**

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda),$$

where

1. $D(\lambda)$ is a polynomial matrix (polynomial part), and
2. the rational matrix $G_{sp}(\lambda)$ is **strictly proper** (strictly proper part), i.e., $\lim_{\lambda\to\infty} G_{sp}(\lambda) = 0$.

- This decomposition is often immediately available in applications (Merhmann & Voss, 2004):

$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{\sigma_i}{\lambda + \sigma_i} L_i L_i^T.$$

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Definition (finite zeros, finite poles, finite eigenvalues)

Given the **Smith-McMillan form** of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$:

$$U(\lambda)G(\lambda)V(\lambda) = \text{diag}\left(\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{(p-r)\times(m-r)}\right).$$

- The **finite zeros** of $G(\lambda)$ are the roots of the numerators and the **finite poles** of $G(\lambda)$ are the roots of the denominators.
- The **finite eigenvalues** of $G(\lambda)$ are the finite zeros that are not poles.

Definition (structural indices)

Given any $c \in \overline{\mathbb{F}}$, one can write for each $i = 1, \dots, r$,

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with } \widetilde{\varepsilon}_i(c) \neq 0, \widetilde{\psi}_i(c) \neq 0.$$

Then, the sequence of structural indices of $G(\lambda)$ at c is

$$S(G, c) = (\sigma_1(c) \leq \sigma_2(c) \leq \cdots \leq \sigma_r(c)).$$

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Finite zeros, finite poles, and finite eigenvalues of a Rational Matrix

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The matrix

$$G(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

has the Smith-McMillan form

$$G(\lambda) \sim \begin{bmatrix} \frac{1}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & (\lambda-1)^2\lambda & 0 \end{bmatrix},$$

and the sequences of structural indices are ($\operatorname{rank}(G) = 5$)

- $S(G,1) = (-1,-1,0,0,2)$, **(pole and zero, NOT eigenvalue)**
- $S(G,0) = (0,0,0,0,1)$ **(NOT pole, zero, it is an eigenvalue)**.

Example: sequences of structural indices at finite values

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Poles and zeros at infinity of a Rational Matrix

Definition

The sequence of structural indices of $G(\lambda)$ at $\lambda = \infty$ is the sequence of structural indices of $G(1/\lambda)$ at $\lambda = 0$.

Proposition

The **smallest structural index at infinity** of $G(\lambda)$ is

1. $-$degree of its polynomial part if this part exists,
2. positive, otherwise.

KEY Remark

This has an **important impact on how to define strong linearizations of rational matrices** since **rational matrices with polynomial parts of different degrees do not have the same structure at infinity**.

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Definition (Biproper matrices)

A square rational matrix is biproper if

- for all its entries, the degree of the numerator is smaller than or equal to the degree of the denominator (that is, the entries are proper rational functions), and
- its determinant is a nonzero rational function whose numerator and denominator have the same degree.

Theorem (Vardulakis, 1991; Amparan, Marcaica, Zaballa, 2015)

Let $G(\lambda), R(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ be two rational matrices. Then the following statements are equivalent:

1. *$G(\lambda)$ and $R(\lambda)$ have the same structural indices at ∞.*
2. *There exist two biproper matrices $B_1(\lambda)$ and $B_2(\lambda)$ such that*

$$G(\lambda) = B_1(\lambda)\, R(\lambda)\, B_2(\lambda)\,.$$

The change of variable $G(\lambda) \to G(1/\lambda)$ is not needed: biproper matrices

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Outline

Definition

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, let

$$g = \begin{cases} -\text{degree of polynomial part of } G(\lambda), \\ 0 \ \text{ if } G(\lambda) \text{ has not polynomial part,} \end{cases}$$

and let

$$n = \text{least order of strictly proper part of } G(\lambda).$$

A **strong linearization of $G(\lambda)$ is a matrix pencil**

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

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Definition (continuation)

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such that the following conditions hold:

(a) if $n > 0$ then $\det(A_1\lambda + A_0) \neq 0$, and

(b) if $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ and $\widehat{g}$ is the corresponding quantity of $\widehat{G}(\lambda)$ then:

- **(i)** there exist unimodular matrices $U_1(\lambda)$, $U_2(\lambda)$ such that

$$U_1(\lambda)\ \mathrm{diag}(G(\lambda), I_s)\, U_2(\lambda) = \widehat{G}(\lambda), \quad \text{and}$$

- **(ii)** there exist biproper matrices $B_1(\lambda)$, $B_2(\lambda)$ such that

$$B_1(\lambda)\ \mathrm{diag}(\lambda^g\, G(\lambda), I_s)\, B_2(\lambda) = \lambda^{\widehat{g}}\, \widehat{G}(\lambda).$$

Comment on condition (ii) of previous definition

A completely equivalent definition is obtained if condition **(ii)** in previous slide is replaced by

(equivalent definition)

(ii)' there exist unimodular matrices $W_1(\lambda)$, $W_2(\lambda)$ such that

$$W_1(\lambda)\ \mathrm{diag}\left(\frac{1}{\lambda^g} G\left(\frac{1}{\lambda}\right), I_s\right) W_2(\lambda) = \frac{1}{\lambda^{\widehat{g}}}\, \widehat{G}\left(\frac{1}{\lambda}\right),$$

which most of the times can be written, if $G(\lambda)$ has a polynomial part $D(\lambda) \neq 0$ as

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This resembles the definition of strong linearizations of rational matrices through “reversals”.

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Strong linearizations contain the complete zero/pole structure

Theorem (Spectral characterization of strong linearizations)

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and* n *be the least order of the strictly proper part of* $G(\lambda)$*. Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))},$$

with A_1 *invertible. Then* $L(\lambda)$ **is a strong linearization of** $G(\lambda)$ **if and only if** *the following two conditions hold:*

(I) $G(\lambda)$ *and* $L(\lambda)$ *have the same number of left and the same number of right minimal indices, and*

(II) $L(\lambda)$ *preserves the finite and infinite structures of poles and zeros of* $G(\lambda)$*.*

Outline

(1) Polynomial ($D(\lambda)$) and strictly proper parts ($G_{sp}(\lambda)$) of the rational matrix.

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p\times m}\,.$$

Given in many applications of REPs.

(2) A minimal order state-space realization of $G_{sp}(\lambda)$:

$$G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B\,.$$

"Almost" given in many applications of REPs where $n \ll \min\{p, m\}$ and $\operatorname{rank} B = n$ and $\operatorname{rank} C = n$. (If not, use algorithms: Rosenbrock's method (1970) stabilized by Van Dooren (1979, 1981) implemented in SLICOT (1999). There are more...)

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(3) A strong block minimal bases linearization of the polynomial part $D(\lambda)$ (D., Lawrence, Pérez, Van Dooren, 2016)

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

There are infinitely many very easily constructible: Paul Van Dooren's Talk, Robol & Vandebril & Van Dooren (2016), Lawrence & Pérez (2016), Fassbender & Pérez & Shayanfar (2016), Fassbender & Saltenberger (2016), Bueno et al (2016)...

Some "easy" constant matrices $\widehat{K}_1$ and $\widehat{K}_2$ related to $\mathcal{L}(\lambda)$ are also needed.

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Main result on constructing strong linearizations

Theorem

With the notation and hypotheses of previous slides, for any nonsingular constant matrices $X, Y \in \mathbb{F}^{n\times n}$ the linear polynomial matrix

$$L(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CY & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right]$$

is a strong linearization of $G(\lambda)$.

Example 1. Strong linearization based on Frobenius companion linearization for polynomials

- Given rational matrix:

$$G(\lambda) = D_d\lambda^d + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B \in \mathbb{F}(\lambda)^{p\times m}.$$

- Strong linearization (Su & Bai (SIMAX, 2011) with minimal order state-space requirement):

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right]$$

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- Strong linearization (Su & Bai (SIMAX, 2011) with minimal order state-space requirement):

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right]$$

Example 2. Strong linearization based on Chebyshev colleague linearization for polynomials

- Given rational matrix:

$$G(\lambda) = D_d\, U_d(\lambda) + \cdots + D_1\, U_1(\lambda) + D_0 + C(\lambda I_n - A)^{-1} B \;\in \mathbb{F}(\lambda)^{p\times m},$$

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Example 3. Strong linearization based on another block Kronecker pencil

- Given rational matrix:

$$G(\lambda) = \lambda^5 D_5 + \lambda^4 D_4 + \lambda^3 D_3 + \lambda^2 D_2 + \lambda D_1 + D_0 + C(\lambda I_n - A)^{-1} B \in \mathbb{F}(\lambda)^{p \times m}$$

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Corollary of main result on constructing strong linearizations

Corollary

Let $G(\lambda) = D(\lambda) + C(\lambda I_n - A)^{-1}B \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix and consider any linearization of its polynomial part $D(\lambda)$ strictly equivalent to a strong block minimal bases linearization of $D(\lambda)$, i.e.,

$$\widetilde{\mathcal{L}}(\lambda) = W \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} Z,$$

with W and Z nonsingular, then for any nonsingular constant matrices $X, Y \in \mathbb{F}^{n\times n}$ the linear polynomial matrix

$$\widetilde{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & \begin{bmatrix} XB\widehat{K}_1 & 0 \end{bmatrix} Z \\ \hline W \begin{bmatrix} -\widehat{K}_2^T C Y \\ 0 \end{bmatrix} & W \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} Z \end{array}\right]$$

***is a strong linearization of $G(\lambda)$**. This includes the famous vector spaces of linearizations of $D(\lambda)$ (Mackey, Mackey, Mehl, Mehrmann-2006, Fassbender, Saltenberger-2017).*

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Outline

Almost same recovery rules as for linearizations of its polynomial part

- Let $G(\lambda) \in \mathbb{F}[\lambda]^{p \times p}$ be square and regular .
- All the strong linearizations of $G(\lambda)$ considered in the previous section have the structure

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_0 \\ -C_0 & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(p+s))},$$

- where $D_1\lambda + D_0$ is a strong linearization of the polynomial part of $G(\lambda)$.
- **Recovery of eigenvectors**:
  1. If $\left(\lambda_0, \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))}\right)$ is a right eigenpair of $L(\lambda)$,
  2. then the eigenvector x_0 of $G(\lambda)$ is recovered from z_0 following the rule given by the linearization of the polynomial part,
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