Paul Van Dooren's Index Sum Theorem and the solution of the inverse rational eigenvalue problem

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What is this theorem? Where can we find it?

INT. J. CONTROL, 1979, VOL. 30, NO. 2, 235–243

Properties of the system matrix of a generalized state-space system†

G. VERGHESE‡, P. VAN DOOREN§ and T. KAILATH‡

For an irreducible polynomial system matrix $P(s)=\begin{bmatrix} T(s) & -U(s) \\ V(s) & W(s) \end{bmatrix}$, Rosenbrock (1970, p. 111) has shown that the *polar* structure of the associated transfer function $R(s)=V(s)T^{-1}(s)U(s)$ at any finite frequency is isomorphic to the zero structure of $T(s)$ at that frequency, while the *zero* structure of $R(s)$ at any finite frequency is isomorphic to that of $P(s)$ at the same frequency. In this paper we obtain the appropriate extensions for the structure *at infinite frequencies* in the particular case of systems for which $T(s)=sE-A$ (with E possibly singular), $U(s)=B$, $V(s)=C$, and $W(s)=D$, under a strengthened irreducibility condition. We term such systems *generalized state-space systems*, and note that *any* rational $R(s)$ may be realized in this form. We also demonstrate in this case that a minimal basis (in the sense of Forney (1975) for the left or right null space of $P(s)$ directly generates one with the same minimal indices for the corresponding null space of $R(s)$, and vice versa. These results also enable us to identify the pole-zero excess of $R(s)$ as being equal to the sum of the minimal indices of its null spaces. Connections with Kronecker's theory of matrix pencils are made.

The following theorem†, whose proof we merely outline for lack of space, demonstrates an important consequence of the preceding two theorems.

Theorem 3

Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite *and* infinite) respectively of an arbitrary rational matrix $R(s)$, and let $\alpha(R)$ denote the sum of the minimal indices of the left and right null spaces of $R(s)$. Then

$$\delta_p(R) = \delta_z(R) + \alpha(R) \tag{21}$$

...

† First obtained, in a slightly different way, by Van Dooren, in earlier unpublished research.

G. Verghese, P. Van Dooren, T. Kailath, Int. J. Control, 1979, Vol 30, page 241.

Paul’s Index Sum Theorem is also in his PhD Thesis

Proposition 5.10

The polar and zero degree of a rational matrix and the minimal orders of its left and right null spaces satisfy the equality

$$\boxed{\delta_p(R) = \delta_z(R) + \hat{\epsilon}(R) + \hat{\eta}(R)}$$

Proof

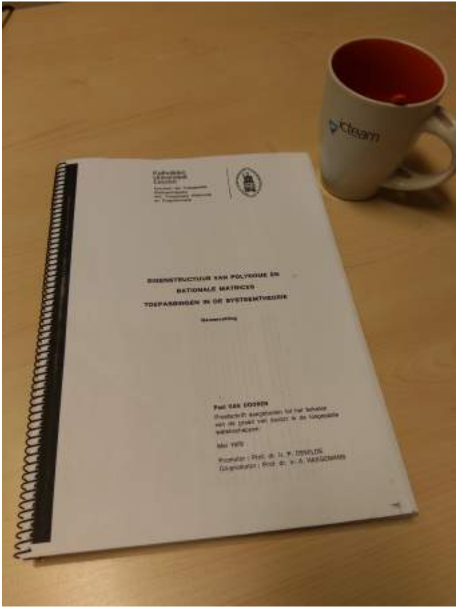
From the above remarks and theorem 3.8 it follows that $(\lambda E - A$ being regular) :

$$\delta_p(R) = \delta_z(\lambda E - A) = \delta_p(\lambda E - A)$$

$$\delta_z(R) + \hat{\epsilon}(R) + \hat{\eta}(R) = \delta_z(\lambda\hat{E} - \hat{A}) + \hat{\epsilon}(\lambda\hat{E} - \hat{A}) + \hat{\eta}(\lambda\hat{E} - \hat{A}) = \delta_p(\lambda\hat{E} - \hat{A})$$

Since $\delta_p(\lambda E - A) = \text{rank } E$ and $\delta_p(\lambda\hat{E} - \hat{A}) = \text{rank } \hat{E}$ (see theorem 3.8), we have that $\delta_p(\lambda E - A) = \delta_p(\lambda\hat{E} - \hat{A})$, which completes the proof.

∇∇∇



The proof of "Paul's Index Sum Theorem" is not simple

- **Step 1 (easy).** Paul's proves the result for pencils $\lambda B - A$.
- **Step 2 (difficult).** Paul proves that any rational matrix has a "strongly irreducible generalized state-space polynomial system pencil" and that such pencils contain the complete structures of poles and zeros (finite and infinite) of the rational matrix, as well as its minimal indices.

- at least up to my limited knowledge.
- In fact, I conjecture that it remained dormant (forgotten??) even in Paul's mind.
- In 1991 the index sum theorem appears again but **only for matrix polynomials** and written in such a form that **nobody established a connection between Paul's general result** for arbitrary rational matrices and the "new theorem" by
- C. Praagman. *Invariants of polynomial matrices*. Proceedings of the First European Control Conference, Grenoble 1991. (I. Landau, Ed.) INRIA, 1274-1277, 1991.
- W. H. L. Neven and C. Praagman. *Column reduction of polynomial matrices*. Linear Algebra Appl., 188/189:569–589, 1993.

INVARIANTS OF POLYNOMIAL MATRICES

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March 6, 1991

1 Introduction

2 Preliminaries

Theorem 3 *Let $P \in \mathbb{R}^{m\times n}[s]$ be a polynomial matrix of rank r and degree d. Then the sum of its structure indices equals rd.*

Proof. It can be deduced immediately from the Kronecker normal form displayed above that the theorem holds for matrix polynomials of degree 1. The rank of L^P equals $m(d-1)+r$, hence its number of left minimal indices (and that of P) is $m-r$.
From theorem 2 we conclude that the sum of the structure indices of P equals the sum of the structure indices of L^P minus $(m-r)(d-1)$, hence equals
$md - m + r - md + m + rd - r = rd$.

- **It looks very different than Paul's result**, since the rank and the degree do not appear at all in Paul's original statement.
- Connections with Paul's result are not mentioned.

Journal of Mathematical Sciences, Vol. 96, No. 3, 1999

METHODS AND ALGORITHMS OF SOLVING SPECTRAL PROBLEMS FOR POLYNOMIAL AND RATIONAL MATRICES

V. N. Kublanovskaya

UDC 519

Dedicated to the memory of my son Alexander

- It is an almost forgotten 203-pages long survey paper (almost a book),
- which includes among many other results

The following balance relations connecting scalar spectral characteristics of a λ-matrix hold:

(a)

$$\gamma_p[R] = \gamma_z[R] + \varepsilon[R] + \eta[R] \tag{1.1.21}$$

for a rational $m \times n$ matrix $R(\lambda)$ of rank ρ;

(b)

$$\beta_c[D] + \beta_\infty[D] + \varepsilon[D] + \eta[D] = \rho s \tag{1.1.22}$$

for a polynomial $m \times n$ matrix $D(\lambda)$ of rank ρ and degree s.

Here, $\gamma_p[R]$ is the sum of negative structural indices of all singular points of $R(\lambda)$; $\gamma_z[R]$ is the sum of positive structural indices of $R(\lambda)$; $\beta_c[D]$ is the sum of all finite elementary divisors of $D(\lambda)$; $\beta_\infty[D]$ is the sum of all infinite elementary divisors of $D(\lambda)$; $\varepsilon[R]$ and $\varepsilon[D]$ are the sums of all right minimal indices of the matrices $R(\lambda)$ and $D(\lambda)$, respectively; $\eta[R]$ and $\eta[D]$ are the sums of all left minimal indices of the matrices $R(\lambda)$ and $D(\lambda)$, respectively.

- Both versions of the Index Sum Theorem are stated one after the other: Paul's 1979 for arbitrary rational matrices and Praagman's 1991 for polynomial matrices,
- but **they are stated as independent results, without establishing any connection between them!!, and without proofs** **(surprising)**.
- The references provided are: Khazanov's PhD Thesis (1983) and Paul's PhD Thesis (1979).

I do not have more news to tell about Index Sum Theorems until...

- Fernando De Terán, Steve Mackey, and myself rediscovered (and baptized) Praagman's index sum theorem for polynomial matrices in Madrid in June 2009,
- while working on a different problem.
- At that time, we did not know Praagman's result and even less Paul's result,
- but, fortunately, we delayed the publication since we were solving other problems.
- In the meanwhile, we presented talks on related results in the ILAS Conferences in Pisa (2010) and Braunschweig (2011), and
- Stavros Vologiannidis recommended us to read one of his papers, which led us to Praagman's papers, but NOT to Paul's result.
- Finally, we published, **without any connection to Paul's result**...

The following very long paper in LAA (2014)

Linear Algebra and its Applications 459 (2014) 264–333

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Spectral equivalence of matrix polynomials and the Index Sum Theorem

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Theorem 6.5 *(Index Sum Theorem for Matrix Polynomials). Suppose $P(\lambda)$ is an arbitrary $m \times n$ matrix polynomial over an arbitrary field. Then*

$$\delta_{\text{fin}}(P) + \delta_{\infty}(P) + \mu(P) = \text{grade}(P) \cdot \text{rank}(P). \qquad (6.4)$$

The connection was finally established in 2015 (2014) in...

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MATRIX POLYNOMIALS WITH COMPLETELY PRESCRIBED EIGENSTRUCTURE*

FERNANDO DE TERÁN†, FROILÁN M. DOPICO†, AND PAUL VAN DOOREN‡

THEOREM 3.1 (index sum theorem). *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree d and rank r having the following eigenstructure:*

- *r invariant polynomials $p_j(\lambda)$ of degrees δ_j, for $j = 1, \ldots, r$,*
- *r infinite partial multiplicities $\gamma_1, \ldots, \gamma_r$,*
- *$n - r$ right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$, and*
- *$m - r$ left minimal indices $\eta_1, \ldots, \eta_{m-r}$,*

where some of the degrees, partial multiplicities, or indices can be zero, and/or one or both of the lists of minimal indices can be empty. Then

$$\sum_{j=1}^{r} \delta_j + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr. \tag{3.1}$$

Remark 3.2. A very interesting remark pointed out by an anonymous referee is that the index sum theorem for matrix polynomials can be obtained as an easy corollary of a more general result valid for arbitrary rational matrices, which is much older than reference [28]. This result is [36, Theorem 3], which can also be found in [18, Theorem 6.5-11]. Using the notion of structural indices at α introduced in

[28] is Praagman's 1991 paper; [36] Verghese, Van Dooren, Kailath's 1979 paper; [18] Kailath's 1980 book.

In the rest of the talk:

1. We will prove that Paul's Index Sum Theorem for Rational Matrices (1979) implies "easily" the Index Sum Theorem for Polynomial Matrices (1991).
2. We will emphasize why the connection between both results remained hidden for such a long time.
3. We will prove that the Index Sum Theorem for Polynomial Matrices (1991) implies "easily" Paul's Index Sum Theorem for Rational Matrices (1979).
4. We will prove that Paul's Index Sum Theorem for Rational Matrices is the unique **necessary and sufficient condition** for solving the most general form of inverse rational "eigenstructure" problem.
5. **Notation:** a few times "IST = Index Sum Theorem".

Outline

1. Basic concepts on rational matrices
2. From Paul's Rational to Polynomial Index Sum Theorem
3. From Polynomial to Paul's Rational Index Sum Theorem
4. The rational inverse "eigenstructure" problem

Outline

1. **Basic concepts on rational matrices**
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Rational matrices and polynomial matrices

- A rational matrix $R(\lambda)$ is a matrix whose entries are rational functions with coefficients in $\mathbb{C}$.
- A polynomial matrix $P(\lambda)$ is a matrix whose entries are polynomials with coefficients in $\mathbb{C}$.
- **Any rational matrix $R(\lambda)$ can be uniquely expressed as**

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda), \quad \text{where}$$

  - $P(\lambda)$ is a polynomial matrix (polynomial part), and
  - the rational matrix $R_{sp}(\lambda)$ is **strictly proper** (strictly proper part), i.e., $\lim_{\lambda\to\infty} R_{sp}(\lambda) = 0$.
- **Unimodular matrices** are square polynomial matrices with constant nonzero determinant.

Definition

The **Smith-McMillan form** of a rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ is the following **diagonal matrix** obtained under unimodular transformations $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda)R(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- $\varepsilon_1(\lambda), \cdots, \varepsilon_r(\lambda), \psi_1(\lambda), \cdots, \psi_r(\lambda)$ are monic polynomials,
- the fractions $\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)}$ are irreducible (invariant fractions),
- $\varepsilon_j(\lambda)$ divides $\varepsilon_{j+1}(\lambda)$ and $\psi_{j+1}(\lambda)$ divides $\psi_j(\lambda)$, for $j = 1, \ldots, r-1$,
- $r = \text{rank } G(\lambda)$.

Definition (finite zeros and finite poles)

Given the **Smith-McMillan form** of a rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$:

$$U(\lambda)R(\lambda)V(\lambda) = \text{diag}\left(\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{(m-r)\times(n-r)}\right)$$

The **finite zeros** of $R(\lambda)$ are the roots of the numerators and the **finite poles** are the roots of the denominators.

Remark

Given any $c \in \mathbb{C}$, one can write for each $i = 1, \dots, r$,

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with } \widetilde{\varepsilon}_i(c) \neq 0, \ \widetilde{\psi}_i(c) \neq 0.$$

Definition (Structural indices at $c \in \mathbb{C}$)

The structural indices of $R(\lambda)$ at c are

$$S(R, c) = (\sigma_1(c) \leq \sigma_2(c) \leq \cdots \leq \sigma_r(c)).$$

The matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

has the Smith-McMillan form

$$R(\lambda) \sim \begin{bmatrix} \frac{1}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & (\lambda-1)^2\lambda & 0 \end{bmatrix},$$

and the structural indices (rank$(R) = 5$)

- $S(R, 1) = (-1, -1, 0, 0, 2)$ (pole and zero),
- $S(R, 0) = (0, 0, 0, 0, 1)$ (zero).

Definition

The structural indices of $R(\lambda)$ at ∞ are the structural indices of $R(1/\lambda)$ at $\lambda = 0$, which are also known as the invariant orders at infinity of $R(\lambda)$.

Proposition: The smallest structural index at infinity (Amparan, Marcaida & Zaballa, ELA, 2015)

Let us express the rational matrix $R(\lambda)$ as

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda), \quad \text{where}$$

$P(\lambda)$ is its polynomial part and $R_{sp}(\lambda)$ is its strictly proper part. Then, the smallest structural index of $R(\lambda)$ at infinity is

1. $-\deg(P)$, if $P(\lambda) \neq 0$,
2. positive, otherwise.

Consider again the matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

and

$$\widetilde{R}(\lambda) := R(1/\lambda) = \begin{bmatrix} \frac{1}{1-\lambda} & & & & & \\ & \frac{\lambda}{1-\lambda} & & & & \\ & & \frac{(\lambda-1)^2}{\lambda^2} & & & \\ & & & 1 & \frac{1}{\lambda^2} & \\ & & & & 1 & \frac{1}{\lambda^7} \end{bmatrix},$$

whose Smith-McMillan form is

$$\widetilde{R}(\lambda) := R(1/\lambda) \sim \begin{bmatrix} \frac{1}{\lambda^7(\lambda-1)} & & & & & \\ & \frac{1}{\lambda^2(\lambda-1)} & & & & \\ & & \frac{1}{\lambda^2} & & & \\ & & & 1 & & \\ & & & & \lambda(\lambda-1)^2 & 0 \end{bmatrix}$$

Thus, the structural indices at infinity of $R(\lambda)$ are

$$S(R,\infty) = S(\widetilde{R},0) = (-7,-2,-2,0,1).$$

Note

$$R(\lambda) = \underbrace{\begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix}}_{P(\lambda)} + \underbrace{\begin{bmatrix} \frac{1}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & 0 & & & \\ & & & 0 & 0 & \\ & & & & 0 & 0 \end{bmatrix}}_{R_{sp}(\lambda)}$$

Theorem (Paul's Index Sum Theorem for Rational Matrices)

Let $\delta_p(R)$ and $\delta_z(R)$ denote the **total number of poles and zeros** *(finite and infinite) respectively of an arbitrary rational matrix $R(\lambda)$, and let $\alpha(R)$ denote the sum of its left and right minimal indices. Then*

$$\delta_p(R) = \delta_z(R) + \alpha(R) \,.$$

Definition (total numbers of poles and zeros)

- The total number of poles of $R(\lambda)$ is minus the sum of all negative structural indices of $R(\lambda)$ (including those at ∞).
- The total number of zeros of $R(\lambda)$ is the sum of all positive structural indices of $R(\lambda)$ (including those at ∞).

Consider again the matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

$$S(R,1) = (-1,-1,0,0,2),$$
$$S(R,0) = (0,0,0,0,1),$$
$$S(R,\infty) = (-7,-2,-2,0,1).$$

- $\delta_p(R) = 13$ **(total number of poles)**.
- $\delta_z(R) = 4$ **(total number of zeros)**.

Minimal bases and minimal indices of a singular Rational Matrix

1. They are defined in the same way as for polynomial matrices.
2. We do not have time to present the definitions.
3. They characterize the structure of the rational null spaces:

$$\begin{aligned} \mathcal{N}_\ell(R) &:= \left\{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} \, : \, y(\lambda)^T R(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(R) &:= \left\{ x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} \, : \, R(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

Example (continued): minimal indices and checking Paul's IST

Consider again the matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

$$\text{rank}(R) = 5 \implies \dim \mathcal{N}_\ell(R) = 0 \quad \text{and} \quad \dim \mathcal{N}_r(R) = 1$$

- $\{[0,0,0,\lambda^9,-\lambda^7,1]^T\}$ right minimal basis of $R(\lambda)$, which has only one right minimal index equal to 9.
- Sum of all minimal indices of $R(\lambda)$ is $\alpha(R) = 9$.
- $\delta_p(R) = 13$ (total number of poles) and $\delta_z(R) = 4$ (total number of zeros),
- $$\delta_p(R) = \delta_z(R) + \alpha(R)\,,$$

  which is Paul's Index Sum Theorem in action.

Polynomial matrices are particular instances of rational matrices

- The Smith-McMillan form reduces to the Smith form.
- Therefore, polynomial matrices do not have finite poles, so
- the structural indices at finite points coincide with the partial multiplicities at finite points
- (whose nonzero values are the degrees of the elementary divisors).
- The finite zeros are called in the polynomial context finite eigenvalues.
- However, **a polynomial matrix $P(\lambda)$ of degree d has always at least one pole of order d at infinity when seen as a rational matrix**, i.e.,
- if $\text{rank}(P) = r$, then the structural indices at infinity are

$$S(P, \infty) = (-d \leq s_2 \leq \cdots \leq s_r),$$

  which are the structural indices at 0 of $P(1/\lambda)$.
- **But**, in the "community of polynomial matrices", the structure at infinity is usually defined in a different way as follows.

Definition (Reversal polynomial)

Let $P(\lambda) = P_d\lambda^d + P_{d-1}\lambda^{d-1} + \cdots + P_0$ be a polynomial matrix of **degree** d. The **reversal** of $P(\lambda)$ is

$$\operatorname{rev}P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_d + P_{d-1}\,\lambda + \cdots + P_0\,\lambda^d\,.$$

Definition (Eigenvalues at ∞ of a polynomial matrix)

$P(\lambda)$ **has an eigenvalue at ∞ if 0 is an eigenvalue of** $\operatorname{rev}P(\lambda)$ and the partial multiplicity sequence of $P(\lambda)$ at ∞ is the same as that of 0 in $\operatorname{rev}P(\lambda)$.

Proposition

If $P(\lambda)$ is a polynomial matrix of degree d with structural indices at ∞

$$S(P,\infty) = (-d \leq s_2 \leq \cdots \leq s_r)\,.$$

Then, the partial multiplicity sequence of $P(\lambda)$ at infinity is

$$S(P,\infty) + d := (0 \leq s_2 + d \leq \cdots \leq s_r + d)\,.$$

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3. From Polynomial to Paul's Rational Index Sum Theorem
4. The rational inverse "eigenstructure" problem

Theorem (Paul's Index Sum Theorem for Rational Matrices)

Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite and infinite) respectively of an arbitrary rational matrix $R(\lambda)$, and let $\alpha(R)$ denote the sum of its left and right minimal indices. Then

$$\delta_p(R) = \delta_z(R) + \alpha(R)\,.$$

If $R(\lambda) = P(\lambda)$ is a polynomial matrix of degree d, $r = \text{rank}(P)$, and with structural indices at ∞ given by

$$S(P,\infty) = (-d \leq s_2 \leq \cdots \leq s_k < 0 \leq s_{k+1} \leq \cdots \leq s_r),$$

then, since $P(\lambda)$ has poles only at infinity,

$$\delta_p(P) = -\left(-d + \sum_{i=2}^{k} s_i\right) \quad \text{and} \quad \delta_z(P) = \sum_{i=k+1}^{r} s_i + \delta_z^{finite}(P)\,.$$

Therefore,

$$\delta_p(P) = \delta_z(P) + \alpha(P) \Longrightarrow -\left(-d + \sum_{i=2}^{k} s_i\right) = \sum_{i=k+1}^{r} s_i + \delta_z^{finite}(P) + \alpha(P)$$

$$\Longrightarrow 0 = \left(-d + \sum_{i=2}^{r} s_i\right) + \delta_z^{finite}(P) + \alpha(P)$$

$$\Longrightarrow d\,r = \left(0 + \sum_{i=2}^{r}(s_i + d)\right) + \delta_z^{finite}(P) \; + \; \alpha(P) \Longrightarrow d\,r =$$

Sum

We have obtained easily

Theorem (Index Sum Theorem for Polynomial Matrices)

Let $\delta(P)$ be the sum of the degrees of all the elementary divisors (finite and infinite) of an arbitrary polynomial matrix $P(\lambda)$, and let $\alpha(P)$ denote the sum of its left and right minimal indices. Then

$$\delta(P) + \alpha(P) = \textit{degree}(P) \cdot \textit{rank}(P).$$

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- This implication seems surprising since rational matrices are not a particular case of polynomial matrices.
- Given a rational matrix $R(\lambda)$, the key point is to apply the Polynomial IST to the polynomial matrix

$$P(\lambda) = \psi_1(\lambda)\, R(\lambda),$$

where $\psi_1(\lambda)$ is the first denominator in the Smith-McMillan form of $R(\lambda)$, taking into account that:

- the minimal indices of $P(\lambda)$ and $R(\lambda)$ are equal, the Smith form of $P(\lambda)$ is trivially obtained from the Smith-McMillan form of $R(\lambda)$, and
- rev$P(\lambda) = ($rev$\psi_1(\lambda))\, \lambda^{\deg(P)-\deg(\psi_1)}\, R\left(\frac{1}{\lambda}\right)$, which implies
- “partial multiplicities at ∞ of $P(\lambda)$”

$$= \text{“structural indices at } \infty \text{ of } R(\lambda)\text{”} + \deg(P) - \deg(\psi_1),$$

since rev$\psi_1(0) \neq 0$.

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2 From Paul's Rational to Polynomial Index Sum Theorem

3 From Polynomial to Paul's Rational Index Sum Theorem

4 **The rational inverse "eigenstructure" problem**

One application of IST for Polynomial Matrices is "the fundamental realization theorem for polynomial matrices" (Steve Mackey's name)

Theorem (De Terán, D, Van Dooren, SIMAX, (2015))

Consider that the following data

- m, n, d, *and* $r \leq \min\{m, n\}$ *positive integers,*
- r *scalar monic polynomials such that* $p_1(\lambda)|p_2(\lambda)|\cdots|p_r(\lambda)$,
- $0 = \gamma_1 \leq \cdots \leq \gamma_r$ *integers,*
- $0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-r}$ *and* $0 \leq \eta_1 \leq \cdots \leq \eta_{m-r}$ *integers*

are prescribed. Then, there exists an $m \times n$ polynomial matrix, with rank r, with degree d, with invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$, with partial multiplicities at infinity $\gamma_1, \cdots, \gamma_r$, and with right and left minimal indices equal to $\alpha_1, \cdots, \alpha_{n-r}$ and $\eta_1, \cdots, \eta_{m-r}$, respectively, if and only if

$$\sum_{j=1}^{r} \text{degree}(p_j) + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \alpha_j + \sum_{j=1}^{m-r} \eta_j = dr,$$

i.e., **if and only if the prescribed data satisfy the IST for poly matrices.**

Theorem (Anguas, D, Hollister, Mackey, in preparation, (2017))

Consider that the following data

- m, n, *and* $r \leq \min\{m, n\}$ *positive integers,*
- r *(monic) irreducible fractions* $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$*, such that* $\varepsilon_1(\lambda) | \cdots | \varepsilon_r(\lambda)$ *and* $\psi_r(\lambda) | \cdots | \psi_1(\lambda)$*,*
- $\gamma_1 \leq \cdots \leq \gamma_r$ *integers (sequence of potential structural indices at infinity),*
- $0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-r}$ *and* $0 \leq \eta_1 \leq \cdots \leq \eta_{m-r}$ *integers*

are prescribed. Then, there exists an $m \times n$ *rational matrix, with rank* r*, with finite invariant rational functions* $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$*, with structural indices at infinity* $\gamma_1, \cdots, \gamma_r$*, and with right and left minimal indices equal to* $\alpha_1, \cdots, \alpha_{n-r}$ *and* $\eta_1, \cdots, \eta_{m-r}$*, respectively,*

if and only if

the prescribed data satisfy Paul Van Dooren's Rational Index Sum Theorem.

- **Step 1.** Get from the prescribed data satisfying Paul's IST the polynomial data:
  - m, n, and $r \leq \min\{m, n\}$ positive integers,
  - r monic polys $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}\psi_1(\lambda) \,|\, \cdots \,|\, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}\psi_1(\lambda)$,
  - $0 \leq \gamma_2 - \gamma_1 \leq \cdots \leq \gamma_r - \gamma_1$ integers (sequence of multiplicities at ∞),
  - $0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-r}$ and $0 \leq \eta_1 \leq \cdots \leq \eta_{m-r}$ integers.
- **Step 2.** These data guarantee, through the "polynomial realization theorem", the existence of an $m \times n$ polynomial matrix $P(\lambda)$ of rank r and degree $\deg(\psi_1) - \gamma_1$ with the structural data in Step 1.
- **Step 3.** Prove that the rational matrix

$$R(\lambda) = \frac{1}{\psi_1(\lambda)} P(\lambda)$$

has the desired structural data.