Uniqueness of solution of a generalized $\star$-Sylvester equation

Fernando de Terán

Departamento de Matemáticas
Universidad Carlos III de Madrid

ALAMA2016, León
June 20–22, 2016

Joint work with **B. Iannazzo**

Generalized $\star$-Sylvester equation

Given $A, B, C, D, E \in \mathbb{C}^{n\times n}$

Goal: Find necessary and sufficient conditions for the equation

$$AXB + CX^{\star}D = E$$

generalized $\star$-Sylvester equation

to have a **unique solution**.

($X \in \mathbb{C}^{n\times n}$, unknown)

($\star = \top$ or $*$)

Motivation

- Natural extension of $AX + X^{\star}D = E$.
  - Numerical methods for palindromic eigenvalue problems [Byers-Kressner'06], [Kressner-Schröder-Watkins'09], [Dmytryshyn-Kågstöm'15]
  - Congruence orbits ($D = A, E = 0$) [D.-Dopico'11]

- Closely related to $AXB + CXD = E$ [Chu'87]

- Iterative algorithms for solving $\sum_{i=1}^{r} A_i X B_i + \sum_{j=1}^{s} C_j X^{\top} D_j = E$
  [Wang-Cheng-Wei'07], [Xie-Ding-Ding'09], [Li-Wang-Zhou-Duan'10], [Song-Chen'11], [Song-Chen-Zhao'11], [Song-Feng-Whang-Zhao'14],...

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Which kind of characterization are we looking for?

$\Lambda(A-\lambda B)$ = Spectrum of $A-\lambda B$

Theorem (Uniqueness of solution for generalized Sylvester) [Chu'87]

The equation $AXB-CXD=E$ has a **unique solution** iff $A-\lambda C$ and $D-\lambda B$ are **regular** and $\Lambda(A-\lambda C)\cap\Lambda(D-\lambda B)=\emptyset$.

$(A,C\in\mathbb{R}^{m\times m};\quad B,D\in\mathbb{R}^{n\times n})$

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- $\boxed{\star=*}$: If $\lambda\in\Lambda(A-\lambda D^*)$, then $(1/\overline{\lambda})\notin\Lambda(A-\lambda D^*)$.
- $\boxed{\star=\top}$: If $1\neq\lambda\in\Lambda(A-\lambda D^\top)$, then $(1/\lambda)\notin\Lambda(A-\lambda D^\top)$, and $m_1(A-\lambda D^\top)\leq 1$.

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☞ Know conditions for $AXB - CXD = E$ and $AX + X^\star D = E$: in terms of **spectral properties** of **matrix pencils** constructed from the coefficient matrices.

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☞ Know conditions for $AXB - CXD = E$ and $AX + X^{\star}D = E$: in terms of **spectral properties** of **matrix pencils** constructed from the coefficient matrices.

Q: Analogous characterization for $AXB + CX^{\star}D = E$ **??**

The vec approach

$\mathrm{vec}\,(AXB + CX^{\star}D) = \mathrm{vec}\,(E)$ leads to

- $\boxed{\star = \top}$: $\left[B^{\top} \otimes A + \Pi(C \otimes D^{\top})\right] \mathrm{vec}\,(X) = \mathrm{vec}\,(E)$

- $\boxed{\star = *}$: $(B^{\top} \otimes A)\mathrm{vec}\,(X) + \Pi(C \otimes D^{\top})\mathrm{vec}\,(\overline{X}) = \mathrm{vec}\,(E)$

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  Not linear over $\mathbb{C} \rightsquigarrow \mathrm{vec}\,(X) = [\mathrm{vec}\,(\mathrm{Re}\,X);\, \mathrm{vec}\,(\mathrm{Im}\,X)]$

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☞ $AXB + CX^{\star}D = E$ can be written as a linear system $MY = b$:

$$Y = \begin{cases} \mathrm{vec}(X), & \text{if } \star = \top \\ [\mathrm{vec}(\mathrm{Re}\, X); \mathrm{vec}(\mathrm{Im}\, X)], & \text{if } \star = * \end{cases}$$

The vec approach (cont.)

$$M \in \begin{cases} \mathbb{C}^{n^2 \times n^2}, & \text{if } \star = \top, \\ \mathbb{R}^{(2n^2)\times(2n^2)}, & \text{if } \star = * \end{cases}$$

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$$\boxed{\begin{array}{c} AXB + CX^\star D = E \text{ has a unique solution} \\ \Updownarrow \\ AXB + CX^\star D = 0 \text{ has a unique solution} \end{array}}$$

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$AXB + CX^\star D = E$ has a unique solution
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☞ We only need to look at the homogeneous equation!

Two basic preparatory results

Lemma 1

If $AXB + CX^\star D = 0$ has a unique solution, then

(a) At least one of A, C is **invertible**.

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Proof. (a) If A, C both singular, then $Au = 0 = Cv$, with $u, v \neq 0 \Rightarrow X = uv^\star$ is a nonzero solution.

(b) If B, D both singular, then $u^\star D = v^\star B = 0$ with $u, v \neq 0 \Rightarrow X = uv^\star$ is a nonzero solution $\square$

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☞ If both A, C or both B, D are singular, then $AXB + CX^\star D = 0$ has a **rank-1 solution**

☞ We will see that also one of A, D, and one of B, C must be **invertible**!

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If A, B invertible: $X + A^{-1}CX^\star DB^{-1} = 0 \rightsquigarrow \star$-Stein

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If C, D invertible: $C^{-1}AXBD^{-1} + X^\star = 0 \rightsquigarrow$ ⋆-Stein

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$AXB + X^{\star} = 0$ has a unique solution $\Leftrightarrow$ $AB^{\star}Y + Y^{\star} = 0$ has a unique solution

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Proof. $(\Leftarrow)$: $AXB + X^\star = 0$ $(X \neq 0) \Rightarrow (AB^\star)(X^\star A^\star) + AX = 0$, so
$Y = (AX)^\star \neq 0$ is solution of $AB^\star Y + Y^\star = 0$.
$(\Rightarrow)$: $AB^\star Y + Y^\star = 0$ $(Y \neq 0) \Rightarrow X = B^\star Y \neq 0$ is a solution of $AXB + X^\star = 0.\Box$

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Corollary

$AXB + CX^{\star}D = 0$ has a unique solution if and only if

(a) A is **invertible** and $D^{\star}A^{-1}CY + Y^{\star}B = 0$ has a unique solution, or

(b) C is **invertible** and $B^{\star}C^{-1}AY + Y^{\star}D = 0$ has a unique solution.

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Corollary $\star$-Sylvester!!!

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Characterization for $\star$-Sylvester (again)

Theorem (Uniqueness of solution for $\star$-Sylvester) [Byers-Kressner'06, Kressner-Schröder-Watkins'09]

$AX + X^\star D = E$ has unique solution if and only if $A - \lambda D^\star$ is **regular** and:

- $\boxed{\star = *}$: If $\lambda \in \Lambda(A - \lambda D^*)$, then $(1/\overline{\lambda}) \notin \Lambda(A - \lambda D^*)$.
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Two different proofs:

- [BK'06] ($\star = \top$): Relies on some continuity arguments of operators. [KSW'09] ($\star = *$)
- [D-Dopico-Guillery-Montealegre-Reyes'11]: Using The **Kronecker canonical form** of $A + \lambda B^\star$.

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Theorem (Uniqueness of solution for ⋆-Sylvester) [Byers-Kressner'06, Kressner-Schröder-Watkins'09]

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$S \subseteq \mathbb{C} \cup \{\infty\}$ is

- reciprocal free if $\lambda \neq \mu^{-1}$ for all $\lambda, \mu \in S$
- $*$-reciprocal free if $\lambda \neq (\overline{\mu})^{-1}$ for all $\lambda, \mu \in S$

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- $\boxed{\star = *}$: $\Lambda(A - \lambda D^*)$ is $*$-reciprocal free.
- $\boxed{\star = \top}$: $\Lambda(A - \lambda D^\top) \setminus \{1\}$ is reciprocal free, and $m_1(A - \lambda D^\top) \leq 1$.

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Characterization of uniqueness of solution

Theorem (Uniqueness for generalized $\star$-Sylvester)

$AXB + CX^\star D = E$ has a unique solution if and only if the pencil

$$P(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$$

is **regular** and:

- $\boxed{\star = *}$: $\Lambda(P)$ is $*$-reciprocal free.
- $\boxed{\star = \top}$: $\Lambda(P) \setminus \{\pm 1\}$ is reciprocal free and $m_1(P) = m_{-1}(P) \leq 1$.

Remark: $m_\lambda(P) = m_{-\lambda}(P)$

Characterization of uniqueness of solution

Theorem (Uniqueness for generalized $\star$-Sylvester)

$AXB + CX^\star D = E$ has a unique solution if and only if the pencil

$$P(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$$

is **regular** and:

- $\boxed{\star = *}$: $\Lambda(P)$ is $*$-reciprocal free.
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Proof of the main result

$AXB + CX^\star D = E$ has unique sol. $\Leftrightarrow P(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$ **regular** and $\boxed{\star = *}$ $\Lambda(P)$ $*$-rec, free; $\boxed{\star = \top}$ $\Lambda(P) \setminus \{\pm 1\}$ rec. free, $m_{\pm 1}(P) \leq 1$

Proof:

- A invertible: $\det P(\lambda) = \pm \det(A) \det(B^\star - \lambda^2 D^\star A^{-1} C)$

$$\begin{bmatrix} 0 & I \\ I & -\lambda D^\star A^{-1} \end{bmatrix} \begin{bmatrix} \lambda D^* & B^\star \\ A & \lambda C \end{bmatrix} = \begin{bmatrix} A & \lambda C \\ 0 & B^\star - \lambda^2 D^\star A^{-1} C \end{bmatrix}.$$

- C invertible: $\det P(\lambda) = \pm \det(C) \det(B^\star C^{-1} A - \lambda^2 D^\star)$

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$$\begin{bmatrix} \lambda I & -\lambda B^\star C^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} = \begin{bmatrix} \lambda^2 D^\star - B^\star C^{-1} A & 0 \\ A & \lambda C \end{bmatrix}.$$

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Recall:

$AXB + CX^\star D = 0$ has a unique solution iff

(a) A is **invertible** and $D^\star A^{-1} CY + Y^\star B = 0$ has a unique solution, or

(b) C is **invertible** and $B^\star C^{-1} AY + Y^\star D = 0$ has a unique solution.

$AX + X^\star D = E$ has unique solution iff $A - \lambda D^\star$ is **regular** and:

- $\boxed{\star = *}$: $\Lambda(A - \lambda D^*)$ is $*$-reciprocal free.
- $\boxed{\star = \top}$: $\Lambda(A - \lambda D^\top) \setminus \{1\}$ is reciprocal free, and $m_1(A - \lambda D^\top) \leq 1$.

The periodic Schur decomposition

Theorem [Bojanczyk-Golub-Van Dooren'92]

There are U_1, U_2, V_1, V_2 unitary such that

$$\begin{array}{ll} U_1 A V_1 = T_A, & U_1 C V_2 = T_C, \\ U_2 B^\star V_1 = T_B^\star, & U_2 D^\star V_2 = T_D^\star, \end{array}$$

with $T_A, T_B^\star, T_C, T_D^\star$ **upper triangular**.

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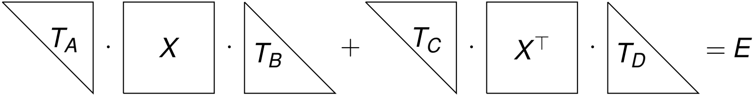
Connection with the pencil $P(\lambda)$:

$$\begin{bmatrix} U_2 & \\ & U_1 \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \begin{bmatrix} V_1 & \\ & V_2 \end{bmatrix} = \begin{bmatrix} \lambda T_D^\star & T_B^\star \\ T_A & \lambda T_C \end{bmatrix}$$

An $O(n^3)$ algorithm

(Based on the algorithm in [D-Dopico'11] for $AX + X^\top D = E$, outlined in [Chiang-Chu-Lin'12])



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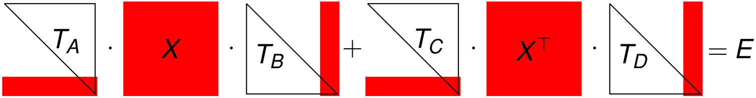

$$\begin{array}{|cccc|}
\hline
X_{11} & \cdots & X_{1,k-1} & X_{1k} \\
\vdots & \ddots & \vdots & \vdots \\
X_{k-1,1} & \cdots & X_{k-1,k-1} & X_{k-1,k} \\
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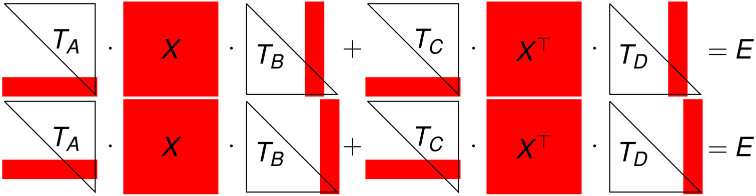

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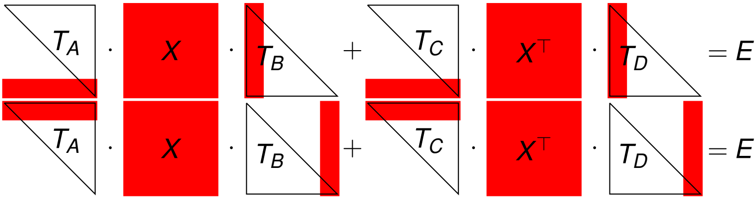

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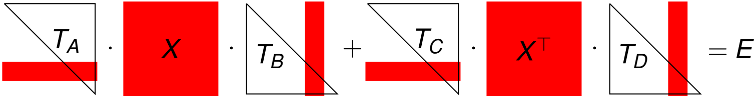

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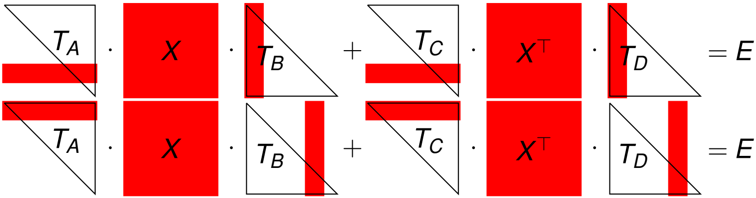

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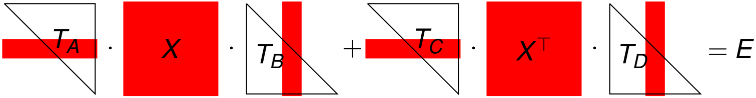

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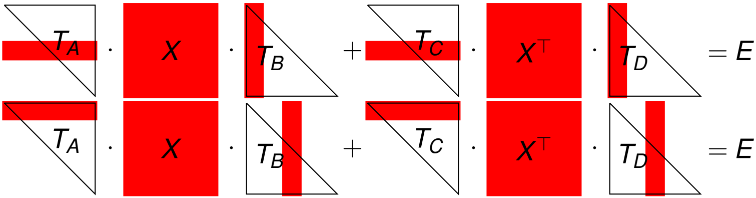

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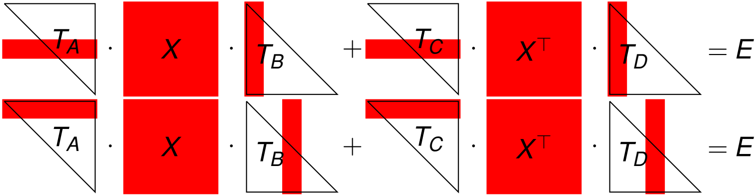

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Systems of generalized $\star$-Sylvester equations

Goal 1:
Obtain **necessary and sufficient conditions** for uniqueness of solution of systems of equations of the form $AXB + CX^{\star}D = E$ (with both $X = Y$ or $X \neq Y$) and $\star = 1, \top, *$.

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☞ More on this at the forthcoming ILAS2016 Conference in Leuven

F. De Terán, B. Iannazzo, *Uniqueness of solution of a generalized ⋆-Sylvester matrix equation*, LAA 493 (2016)

R. Byers, D. Kressner, *Structured condition numbers for invariant subspaces*, SIMAX 28 (2) (2006)

C.-Y. Chiang, K.-W. E. Chu, W.-W. Lin, On the ⋆-Sylvester equation $AX \pm X^{\star}B = C$, AMC 218 (2012)

F. De Terán, F. M. Dopico, *Consistency and efficient solution of the Sylvester equation for ⋆-congruence*, ELA 22 (2011)

F. De Terán, F. M. Dopico, N. Guillery, D. Montealegre, N. Z. Reyes, *The solution of the equation* $AX + X^{\star}B = 0$, LAA 438 (2011)

D. Kressner, C. Schröder, D. S. Watkins, *Implicit QR algorithms for palindromic and even eigenvalue problems*, NA 51(2) (2009)

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THANKS FOR YOUR ATTENTION !!!!!