Constructing strong ℓ-ifications

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Outline

1. Motivation. Basic definitions.
2. New construction of strong ℓ-ifications
3. Minimal index recovery
4. The case where ℓ divides d

Outline

Notation

$\mathbb{F}$ a field.

$\overline{\mathbb{F}}$: algebraic closure of $\mathbb{F}$.

$\mathbb{F}[\lambda]^{m\times n}$: ring of $m \times n$ matrices whose entries are polynomials in λ with coefficients over $\mathbb{F}$ (matrix polynomials).

$P(\lambda) = \lambda^d P_d + \lambda^{d-1} + \cdots + \lambda P_1 + P_0 \in \mathbb{F}[\lambda]^{m\times n}$: a given $m \times n$ matrix polynomial of degree d ($P_d \neq 0$).

Reversal polynomial of $P(\lambda)$: $\mathrm{rev}\, P := P_d + \lambda P_{d-1} + \cdots + \lambda^{d-1} P_1 + \lambda^d P_0$

Why ℓ-ifications?

(Companion) Linearizations have been quite useful in the **P**olynomial **E**igenvalue **P**roblem (**PEP**) but...

- They increase very much the size of the problem: $n \times n \longrightarrow (dn) \times (dn)$ (for all companion linearizations of square polynomials).
- Imposible to preserve certain structures using companion linearizations (for instance: T**-palindromic** for **even degree** polynomials).

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Strong ℓ-ifications

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$L(\lambda)$ a matrix polynomial of degree ℓ is an ℓ-ification of $P(\lambda)$ if

$$U(\lambda)\begin{bmatrix} I_s & \\ & L(\lambda) \end{bmatrix} V(\lambda) = \begin{bmatrix} I_t & \\ & P(\lambda) \end{bmatrix},$$

for some $s, t \geq 0$ and $U(\lambda), V(\lambda)$ **unimodular** matrix polynomials (constant nonzero determinant).

If, in addition, $\operatorname{rev} L$ is an ℓ-ification of $\operatorname{rev} P$, then $L(\lambda)$ is a strong ℓ-ification.

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Main features of strong ℓ-ifications

$$U(\lambda)\begin{bmatrix} I_s & \\ & L(\lambda) \end{bmatrix}V(\lambda) = \begin{bmatrix} I_t & \\ & P(\lambda) \end{bmatrix} \qquad (\ell\text{-ification})$$

- ℓ-ifications preserve: finite partial multiplicities + number of left / right minimal indices
- **Strong** ℓ-ifications also preserve the infinite partial multiplicities.
- However, the minimal indices are not necessarily preserved (and this is usually the case).
- One of s, t can be **always** chosen to be **zero**.
- The size of $P(\lambda)$ can be **larger** than the size of $L(\lambda)$ (**only** if $P(\lambda)$ is **singular**).
- $U(\lambda), V(\lambda)$ are essentially row and column elementary transformations.

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Example:

$$P(\lambda) = \begin{bmatrix} \lambda^2 & 1 & 0 \\ 0 & 0 & \lambda^2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad L(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & 0 \end{bmatrix}.$$

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Companion ℓ-ifications

$\mathcal{P}(d, m \times n, \mathbb{F})$ = space of all $m \times n$ matrix polynomials of fixed degree d.

Definition (Companion ℓ-ification)

A companion ℓ-ification for matrix polynomials $P(\lambda)$ in $\mathcal{P}(d, m \times n, \mathbb{F})$ is of the form $\mathcal{C}_P(\lambda) = \sum_{i=0}^{\ell} \lambda^i X_i$, satisfying:

- $\mathcal{C}_P(\lambda)$ is a **strong ℓ-ification for** P for *every* $P \in \mathcal{P}(d, m \times n, \mathbb{F})$.
- Each entry of X_i is either a **constant**, or a **constant multiple** of just **one** of the entries of $P(\lambda)$.

Example **[D., Dopico, Mackey, 2014]:** If $d = \ell k$,

$$C_1^\ell(\lambda) := \begin{bmatrix} B_k(\lambda) & B_{k-1}(\lambda) & B_{k-2}(\lambda) & \cdots & B_1(\lambda) \\ -I_n & \lambda^\ell I_n & 0 & \cdots & 0 \\ & -I_n & \lambda^\ell I_n & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & -I_n & \lambda^\ell I_n \end{bmatrix} \quad \text{and} \quad C_2^\ell(\lambda) := C_1^\ell(\lambda)^{\mathcal{B}}$$

$$\text{with :} \quad \begin{array}{l} B_1(\lambda) := \lambda^\ell P_\ell + \lambda^{\ell-1} P_{\ell-1} + \cdots + \lambda P_1 + P_0 \,, \\ B_j(\lambda) := \lambda^\ell P_{\ell j} + \lambda^{\ell-1} P_{\ell j-1} + \cdots + \lambda P_{\ell(j-1)+1} \,, \qquad \text{for } j = 2, \ldots, k \,. \end{array}$$

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Minimal bases

$N(\lambda) \in \mathbb{F}[\lambda]^{m\times n} \rightsquigarrow N_h$: *highest row degree coefficient* matrix.

Definition: $N(\lambda)$ is row reduced if N_h is of full row rank.
(Similar definition of column reduced with the highest column degree coefficient matrix).

Definition

The $m \times n$ matrix polynomial $N(\lambda)$, with $m \leq n$ is a minimal basis if:

(a) $N(\lambda)$ has **full row rank** for all $\lambda \in \overline{\mathbb{F}}$, and

(b) it is **row reduced**.

Remark: Similar definition with $m \geq n$, full column rank, and column reduced.

Example:

$$N(\lambda) = \begin{bmatrix} \lambda^3 & 1 & \lambda \\ \lambda & 3\lambda^2 + 2 & \lambda + 1 \end{bmatrix} \rightsquigarrow N_h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}.$$

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Row/column degrees

An important feature of a minimal basis are its row/column degrees.

For instance, for minimal bases of the right (resp., left) nullspace of $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$, $\mathcal{N}_r(P)$ (resp. $\mathcal{N}_\ell(P)$):

$$\begin{aligned} \mathcal{N}_r(P) &:= \left\{x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \;:\; P(\lambda)x(\lambda) \equiv 0\right\}, \\ \mathcal{N}_\ell(P) &:= \left\{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \;:\; y(\lambda)^T P(\lambda) \equiv 0^T\right\}, \end{aligned}$$

they are the right (resp. left) minimal indices of $P(\lambda)$.

Dual minimal bases and row degrees

Definition

$N_1(\lambda) \in \mathbb{F}[\lambda]^{m_1 \times n}$, $N_2(\lambda) \in \mathbb{F}[\lambda]^{m_2 \times n}$ are dual minimal bases if $N_1(\lambda)$ and $N_2(\lambda)$ are both minimal bases and:

$$m_1 + m_2 = n, \quad \text{and} \qquad N_1(\lambda) N_2(\lambda)^T = 0.$$

Theorem (D., Dopico, Mackey, Van Dooren, 2015)

Let $(\eta_1, \ldots, \eta_{m_1})$ and $(\varepsilon_1, \ldots, \varepsilon_{m_2})$, with $\varepsilon_i, \eta_j \geq 0$ and:

$$\sum_{j=1}^{m_1} \eta_j = \sum_{i=1}^{m_2} \varepsilon_i \, .$$

Then there always exist $N_1(\lambda) \in \mathbb{F}[\lambda]^{m_1 \times n}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{m_2 \times n}$, with $n = m_1 + m_2$, **dual minimal bases** whose row degrees are, respectively, $(\eta_1, \ldots, \eta_{m_1})$ and $(\varepsilon_1, \ldots, \varepsilon_{m_2})$.

☞ They can be built up using **zigzag matrices**.

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Basic quantities

☞ We focus on the case $k\ell = nd$ (i.e., ℓ **divides** nd).
(Similar construction for the case where ℓ divides md).

☞ Note that $\ell < d \Rightarrow k > n$

Set:

$$\widehat{d} := d - \ell, \qquad k := \widehat{n} + n \qquad (\widehat{d}, \widehat{n} > 0)$$

Then:

$$(\widehat{n} + n)\ell = nd \Leftrightarrow \widehat{n}\ell = n\widehat{d}$$

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Outline of construction

Step 1: Construct a pair of **dual minimal bases** $\widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}$ and $\widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ such that:

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Step 2: Find a solution, $\widetilde{L}(\lambda) \in \mathbb{F}[\lambda]^{m\times(\widehat{n}+n)}$, to

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Theorem

If $\widehat{L}(\lambda), \widetilde{L}(\lambda)$ are as above, then

$$L(\lambda) = \begin{bmatrix} \widehat{L}(\lambda) \\ \widetilde{L}(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(\widehat{n}+m)\times(\widehat{n}+n)}$$

is a strong ℓ-ification of $P(\lambda)$.

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$$\Rightarrow \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} \widehat{L} \\ \widetilde{L} \end{bmatrix} \begin{bmatrix} \widetilde{N}^T & | & \widehat{N}^T \end{bmatrix} = \begin{bmatrix} I & \\ & P \end{bmatrix}$$

Outline of construction

Step 1: Construct a pair of **dual minimal bases** $\widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}$ and $\widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ such that:

(i) All row degrees of $\widehat{L}(\lambda)$ are equal to ℓ.

(ii) All row degrees of $\widehat{N}(\lambda)$ are equal to $\widehat{d}$ $(= d - \ell)$.

Step 2: Find a solution, $\widetilde{L}(\lambda) \in \mathbb{F}[\lambda]^{m\times(\widehat{n}+n)}$, to

$$\widetilde{L}(\lambda)\widehat{N}(\lambda)^T = P(\lambda),$$

with $\deg \widetilde{L}(\lambda) \leq \ell$.

IDEA:

$$\left.\begin{array}{l} \widehat{L}\widehat{N}^T = 0 \\ \widetilde{L}\widehat{N}^T = P \end{array}\right\} \Rightarrow \begin{bmatrix} \widehat{L} \\ \widetilde{L} \end{bmatrix} \overbrace{\begin{bmatrix} \widetilde{N}^T \mid \widehat{N}^T \end{bmatrix}}^{unimodular} = \begin{bmatrix} I & 0 \\ X & P \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} \widehat{L} \\ \widetilde{L} \end{bmatrix} \begin{bmatrix} \widetilde{N}^T \mid \widehat{N}^T \end{bmatrix} = \begin{bmatrix} I & \\ & P \end{bmatrix}$$

(i)–(ii) guarantee that the ℓ-ification is **strong**.

Is it always possible to perform **Step 1** and **Step 2**?

Step 1: $\widehat{n}\ell = n\widehat{d} \;\Rightarrow\; \widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}, \widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ exist (by the inverse row degree theorem for dual minimal bases).

☞ One way to construct them is using **zigzag matrices** (recall Froilán's talk!).

Step 2: Set:

$$\widetilde{L}(\lambda) = \lambda^{\ell}\widetilde{L}_{\ell} + \lambda^{\ell-1}\widetilde{L}_{\ell-1} + \cdots + \lambda\widetilde{L}_1 + \widetilde{L}_0,$$
$$\widehat{N}(\lambda) = \lambda^{\widehat{d}}\widehat{N}_{\widehat{d}} + \lambda^{\widehat{d}-1}\widehat{N}_{\widehat{d}-1} + \cdots + \lambda\widehat{N}_1 + \widehat{N}_0,$$

and write the convolution equation:

It is always possible to perform **Step 1** and **Step 2**:

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and write the convolution equation:

$$\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} & \widetilde{L}_{\ell} \end{bmatrix} \begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & & \ddots & & \ddots & \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix} = \begin{bmatrix} P_0 & \dots & P_{d-1} & P_d \end{bmatrix}.$$

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$(\widehat{n}+n)(\ell+1) \times n(d+1)$
($\widehat{n}$ more rows than columns)

It is always possible to perform **Step 1** and **Step 2**:

Step 1: $\widehat{n}\ell = n\widehat{d} \;\Rightarrow\; \widehat{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}, \widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ exist (by the inverse row degree theorem for dual minimal bases).

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and write the convolution equation:

$$\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} & \widetilde{L}_{\ell} \end{bmatrix} \begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & \\ & & \ddots & & \ddots \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix} = \begin{bmatrix} P_0 & \dots & P_{d-1} & P_d \end{bmatrix}.$$

☞ It has infinitely many solutions.

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and write the convolution equation:

$$\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} & \widetilde{L}_{\ell} \end{bmatrix} \begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & & \ddots & & \ddots & \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix} = \begin{bmatrix} P_0 & \dots & P_{d-1} & P_d \end{bmatrix}.$$

☞ First solve: $\widetilde{L}_{\ell}\widehat{N}_{\widehat{d}}^T = P_d$ ($\widehat{N}_{\widehat{d}}^T$ has full column rank).

It is always possible to perform **Step 1** and **Step 2**:

Step 1: $\widehat{n}\ell = n\widehat{d} \Rightarrow \widetilde{L}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{n}\times(\widehat{n}+n)}, \widehat{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(\widehat{n}+n)}$ exist (by the inverse row degree theorem for dual minimal bases).

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and write the convolution equation:

(1)

$$\begin{bmatrix} \widetilde{L}_0 & \dots & \widetilde{L}_{\ell-1} \end{bmatrix} \begin{bmatrix} \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & & \\ & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T & & \\ & & \ddots & & \ddots & \\ & & & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}}^T \end{bmatrix} = \begin{bmatrix} P_0 & P_1 & \dots & P_{d-1} \end{bmatrix} - \widetilde{L}_{\ell} \begin{bmatrix} 0 \dots 0 & \widehat{N}_0^T & \dots & \widehat{N}_{\widehat{d}-1}^T \end{bmatrix}$$

☞ First solve: $\widetilde{L}_{\ell}\widehat{N}_{\widehat{d}}^T = P_d$ ($\widehat{N}_{\widehat{d}}^T$ has full column rank).

☞ Then solve (1).

Example

$P(\lambda)$ of size $m \times 2$ and degree $d = 3$, and $\ell = 2$.

☞ Following the **zigzag** construction for dual minimal bases $\widehat{L}(\lambda), \widehat{N}(\lambda)$ in **Step 1**, and with an appropriate choice of $\widetilde{L}_2$ in **Step 2**, we get the strong quadratification:

$$L(\lambda) = \begin{bmatrix} \widehat{L}(\lambda) \\ \widetilde{L}(\lambda) \end{bmatrix} = \lambda^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & P_3 e_1 & P_3 e_2 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -1 & 0 \\ P_1 e_1 - P_0 e_2 & P_2 e_1 & P_2 e_2 - P_3 e_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ P_0 e_1 & P_0 e_2 & P_1 e_2 - P_2 e_1 \end{bmatrix}.$$

Size

The size of the strong ℓ-ifications we construct is:

$$(\widehat{n}+m)\times(\widehat{n}+n) \qquad \text{(if } \ell | nd\text{)}$$

with

$$\widehat{n}=\frac{n(d-\ell)}{\ell},$$

or

$$(\widehat{m}+m)\times(\widehat{m}+n) \qquad \text{(if } \ell | md\text{)}$$

with

$$\widehat{m}=\frac{m(d-\ell)}{\ell}.$$

(Compare with the size of companion linearizations:

$$((d-1)s+m)\times((d-1)s+n),$$

where $s=\min\{m,n\}$).

Outline

Minimal indices of $L(\lambda)$ and $P(\lambda)$

Theorem

When $\ell | nd$, the construction in **Steps 1** and **2** always provides a strong ℓ-ification of $m \times n$ matrix polynomials of degree d. Moreover:

(i) If $\varepsilon_1, \ldots, \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then the right minimal indices of $L(\lambda)$ are $\varepsilon_1 + (d - \ell), \ldots, \varepsilon_p + (d - \ell)$.

(ii) If $\eta_1, \ldots, \eta_q$ are the left minimal indices of $P(\lambda)$, then the left minimal indices of $L(\lambda)$ are $\eta_1, \ldots, \eta_q$.

Remark: Similar result when $\ell | md$, replacing the roles of left/right minimal indices.

Outline

Set $d = k\ell$. We can take:

$$\widehat{L}(\lambda) = \left(\begin{bmatrix} \lambda^\ell & -1 & & \\ & \ddots & \ddots & \\ & & \lambda^\ell & -1 \end{bmatrix}_{(k-1)\times k} \right) \otimes I_n, \quad \text{and} \quad \widehat{N}(\lambda)^T = \begin{bmatrix} 1 \\ \lambda^\ell \\ \lambda^{2\ell} \\ \vdots \\ \lambda^{(k-1)\ell} \end{bmatrix} \otimes I_n.$$

and

$$\widetilde{L}_\ell = \begin{bmatrix} 0 & \dots & 0 & P_d \end{bmatrix} \in \mathbb{F}^{m \times nk},$$

to get:

$$L(\lambda) = \begin{bmatrix} \lambda^\ell I_n & -I_n & & \\ & \ddots & \ddots & \\ & & \lambda^\ell I_n & -I_n \\ D_0(\lambda) & \dots & D_{k-2}(\lambda) & D_{k-1}(\lambda) \end{bmatrix},$$

where

$$D_j(\lambda) = P_{j\ell} + \lambda P_{j\ell+1} + \cdots + \lambda^{\ell-1} P_{(j+1)\ell-1} \qquad (j = 0, \dots, k-2),$$
$$D_{k-1}(\lambda) = P_{(k-1)\ell} + \lambda P_{(k-1)\ell+1} + \cdots + \lambda^{\ell-1} P_{k\ell-1} + \lambda^\ell P_{k\ell}.$$

Compare:

$$L(\lambda) = \begin{bmatrix} \lambda^\ell I_n & -I_n & & & \\ & \ddots & \ddots & & \\ & & \lambda^\ell I_n & -I_n & \\ D_0(\lambda) & \dots & D_{k-2}(\lambda) & D_{k-1}(\lambda) & \end{bmatrix},$$

$$\begin{aligned} D_j(\lambda) &= P_{j\ell} + \lambda P_{j\ell+1} + \cdots + \lambda^{\ell-1} P_{(j+1)\ell-1} \qquad (j = 0, \dots, k-2), \\ D_{k-1}(\lambda) &= P_{(k-1)\ell} + \lambda P_{(k-1)\ell+1} + \cdots + \lambda^{\ell-1} P_{k\ell-1} + \lambda^\ell P_{k\ell} \end{aligned}$$

with

$$C_1^\ell(\lambda) = \begin{bmatrix} B_k(\lambda) & B_{k-1}(\lambda) & B_{k-2}(\lambda) & \cdots & B_1(\lambda) \\ -I_n & \lambda^\ell I_n & 0 & \cdots & 0 \\ & -I_n & \lambda^\ell I_n & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & -I_n & \lambda^\ell I_n \end{bmatrix}$$

$$\begin{aligned} B_1(\lambda) &:= \lambda^\ell P_\ell + \lambda^{\ell-1} P_{\ell-1} + \cdots + \lambda P_1 + P_0\,, \\ B_j(\lambda) &:= \lambda^\ell P_{\ell j} + \lambda^{\ell-1} P_{\ell j-1} + \cdots + \lambda P_{\ell(j-1)+1} \qquad (j = 2, \dots, k)\,. \end{aligned}$$

Conclusions

- We have provided a general construction of **strong ℓ-ifications**, $L(\lambda)$, of $m \times n$ matrix polynomials of degree d, $P(\lambda)$, valid for all $\ell | md$ or $\ell | nd$.
- If $\ell | nd$ (resp. $\ell | md$) then:
  - The left (resp., right) minimal indices of $L(\lambda)$ and $P(\lambda)$ coincide.
  - The right (resp. left) minimal indices of $L(\lambda)$ are the ones of $P(\lambda)$ increased by $(d - \ell)$ (each).
- When $\ell | d$ we get companion ℓ-ifications.

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