Flanders' theorem for many matrices under commutativity assumptions

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Joint work with **Ross A. Lippert**, **Yuji Nakatsukasa** & **Vanni Noferini**

Collaborators and remembrance

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D. E. Shaw Research-Simulation Tools

Vanni Noferini

The University of Manchester

Yuji Nakatsukasa

The University of Tokyo

Dedicated to the memory of:

Harley Flanders

Sept 13, 1925–July 26, 2013

Outline

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JCF(AB) vs JCF(BA)

Notation:

- JCF(M)=Jordan Canonical Form of M
- $\mathcal{S}_\lambda(M) = (n_1, n_2, \ldots, 0, 0, \ldots) =$ **Segré characteristic** of M at $\lambda \in \mathbb{C}$ (infinite sequence of ordered sizes $n_1 \geq n_2 \geq \ldots$ of Jordan blocks at λ in JCF(M))

Theorem (Flanders, 1951)

Given $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times m}$, set $M = AB, N = BA$.

(i) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(N)$ for all $\lambda \neq 0$.

(ii) $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq 1$.

Conversely, if $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ satisfy (i)–(ii), then $M = AB$ and $N = BA$, for some A, B.

In plain words: JCF(AB) and JCF(BA) can only differ in the J-blocks at 0, and the corresponding sizes differ, at most, by 1, and this happens **only** for matrices of the form AB and BA.

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Some history

Proved in:

H. Flanders
The elementary divisors of *AB* and *BA*.
Proc. Am. Math. Soc. 2 (1951) 871–874.

And later in:

W. V. Parker, B. E. Mitchell.
Elementary divisors of certain matrices.
Duke Math. J. 19 (1952) 483–485.

R. C. Thompson.
On the matrices *AB* and *BA*.
Linear Algebra Appl. 1 (1968) 43–58.

S. Bernau, A. Abian.
Jordan canonical forms of matrices *AB* and *BA*.
Rend. Istit. Mat. Univ. Trieste. 20 (1988) 101–108.

C. R. Johnson, E. S. Schreiner.
The relationship between *AB* and *BA*.
Amer. Math. Monthly 103 (1996) 578–581.

R. A. Lippert, G. Strang.
The Jordan form of *AB* and *BA*.
Electron. J. Linear Algebra 18 (2009) 281–288.

Flanders again: exhaustivity

Moreover:

Theorem (Flanders, 1951)

Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$, and $\boldsymbol{\mu}' = (\mu'_1, \mu'_2, \ldots)$ be two lists of integers with $\mu_1 \geq \mu_2 \geq \ldots \geq 0$, and $\mu'_1 \geq \mu'_2 \geq \ldots \geq 0$, with:

(i) $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty \leq 1$, and

(ii) $\|\boldsymbol{\mu}\|_1 = m$, $\|\boldsymbol{\mu}'\|_1 = n$.

Then, there are $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times m}$ with $\mathcal{S}_0(AB) = \boldsymbol{\mu}$ and $\mathcal{S}_0(BA) = \boldsymbol{\mu}'$.

The problem

What happens for **more than two** matrices?

JCF(ABC) and JCF(CBA) can be arbitrarily different !!

Notation: $$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{n\times n}.$$

Example

$A = \mathrm{diag}(1, 1/2, \ldots, 1/n)$, $B = -J_n(-1)^T$, $C = (AB)^{-1}J_n(0)$. Then:

- $ABC = J_n(0)$.
- The e-vals of CBA are: $0, \lambda_1, \ldots, \lambda_{n-1}$, with $\lambda_1 \cdots \lambda_{n-1} \neq 0$.

☞ We need to impose some **extra conditions** on A, B, C.

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Flanders pairs and bridges

Set $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$.

Definition

(M,N) is a Flanders pair if $M = AB$, $N = BA$, for some $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$.

There is a Flanders bridge between M and N if (M,N) is a Flanders pair.

Note: Not transitive !!!

Corollary (of Flanders' Theorem)

If $(M_1,M_2),(M_2,M_3),\ldots,(M_d,M_{d+1})$ are Flanders pairs, then:

(i) $\mathcal{S}_\lambda(M_1) = \mathcal{S}_\lambda(M_{d+1})$, for all $\lambda \neq 0$.

(ii) $\|\mathcal{S}_0(M_1) - \mathcal{S}_0(M_{d+1})\|_\infty \leq d$.

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The problems

Given $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$, we set:

$\mathcal{P}(A_1, \ldots, A_k) := \{A_{i_1} \cdots A_{i_k} : (i_1, \ldots, i_k) \text{ a permutation of } (1, \ldots, k)\}$

("Permuted products" of $A_1, \ldots, A_k$)

Three questions (after Flanders' Theorem):

- **Question 1**: Find necessary and sufficient conditions on $A_1, \ldots, A_k$ such that:
  - (i) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(N)$, for all $\lambda \neq 0$ and all $M, N \in \mathcal{P}(A_1, \ldots, A_k)$, and
  - (ii) $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq d$, for any $M, N \in \mathcal{P}(A_1, \ldots, A_k)$ and $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty = d$, for some $M, N \in \mathcal{P}(A_1, \ldots, A_k)$.
- **Question 2**: If M, N satisfy (i)–(ii), then $M, N \in \mathcal{P}(A_1, \ldots, A_k)$, for some $A_1, \ldots, A_k$ satisfying the conditions obtained in **Question 1**?
- **Question 3** (exhaustivity): Given two nonincreasing sequences of nonnegative integers $\boldsymbol{\mu}, \boldsymbol{\mu}'$ such that $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty = d$, find $A_1, \ldots, A_k$ satisfying the conditions obtained in **Question 1** and such that $\mathcal{S}_0(\Pi_1) = \boldsymbol{\mu},\ \mathcal{S}_0(\Pi_2) = \boldsymbol{\mu}'$, for some $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$.

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Permuted products of $A, B, C \in \mathbb{C}^{n\times n}$

$$\mathcal{P}(A,B,C) = \{ABC, ACB, BCA, BAC, CBA, CAB\}$$

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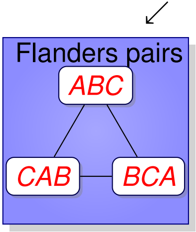

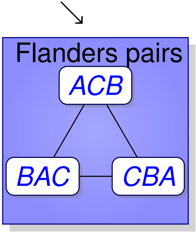

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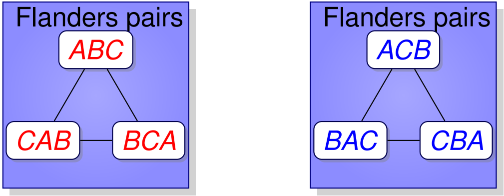
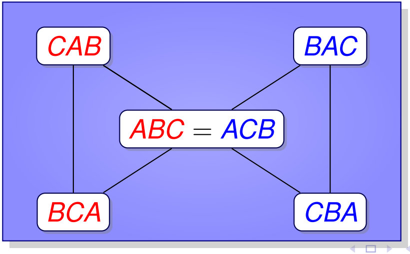


If $A(BC) = A(CB)$:



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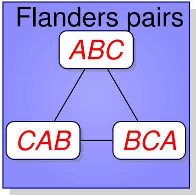
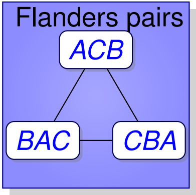
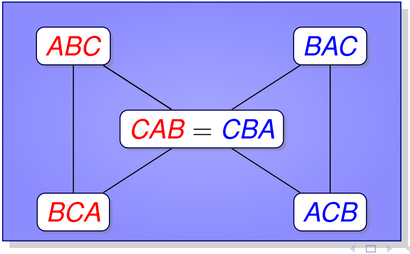




If $C(AB) = C(BA)$:



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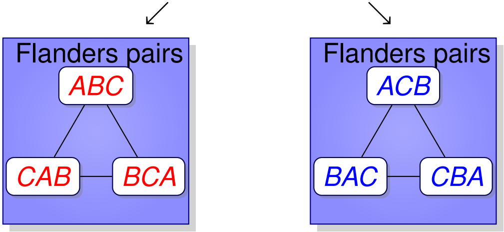
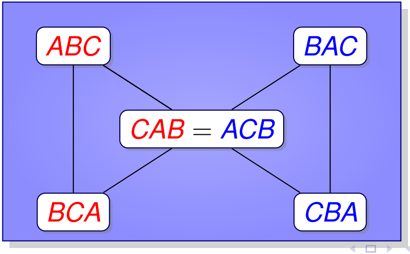


If $(CA)B = (AC)B$:



Commutativity relations

If at least **two of** A, B, C **commute** then, for any $\Pi_1, \Pi_2 \in \mathcal{P}(A, B, C)$:

(i) $\mathcal{S}_\lambda(\Pi_1) = \mathcal{S}_\lambda(\Pi_2)$, for all $\lambda \neq 0$.

(ii) $\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty \leq 2$.

☞ commutativity of (A, B) **or** (A, C), **or** (B, C) is the answer to **Question 1** for three matrices.

☞ Moreover, it is the answer to **Question 3**:

Theorem

Let μ, μ' be two nonincreasing sequences of nonnegative integers such that

(i) $\|\mu - \mu'\|_\infty \leq 2$, and

(ii) $\|\mu\|_1 = \|\mu'\|_1 = n$.

Then, there are three matrices $A, B, C \in \mathbb{C}^{n \times n}$, such that $AC = CA$ and

$$\mathcal{S}_0(ABC) = \mu, \qquad \text{and} \qquad \mathcal{S}_0(CBA) = \mu'.$$

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Answer to Question 2?

As for Question 2, we have:

Corollary

Let $M, N \in \mathbb{C}^{n\times n}$. Then the following are equivalent:

(a) There is $Q \in \mathbb{C}^{n\times n}$ such that (M, Q) and (Q, N) are Flanders pairs.

(b) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(N)$, for all $\lambda \neq 0$, and $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq 2$.

(c) There are $A, B, C \in \mathbb{C}^{n\times n}$ such that $AC = CA$, M is similar to ABC, and N is similar to CBA.

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Not necessarily: $M = ABC$ and $N = CBA$!!!

Outline

Basic definitions

Path of a graph: Sequence of adjacent edges containing no cycles. Its length is the number of edges.

Forest: A graph containing no cycles.

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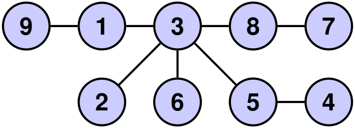

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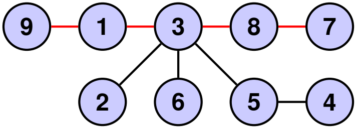

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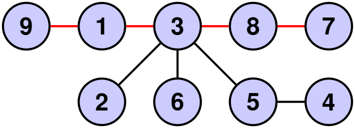

Definition

The graph of non-commutativity relations of $A_1, \ldots, A_k$ is the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, \ldots, k\}$, such that $\{i, j\} \in E$ if and only if $A_i A_j \neq A_j A_i$, for $1 \leq i, j \leq k$ with $i \neq j$.

Sequences of Flanders bridges

Definition

$M_1, M_{d+1} \in \mathbb{C}^{n\times n}$ are connected by a sequence of Flanders bridges if $(M_1, M_2), (M_2, M_3), \ldots, (M_d, M_{d+1})$ are Flanders pairs, for some $M_2, \ldots, M_d$.

$\mathcal{G}$: the graph of non-commutativity relations of $A_1, \ldots, A_k$.

Then, if products in $\mathcal{P}(A_1, \ldots, A_k)$ are considered as **formal products**:

Theorem

Any two products in $\mathcal{P}(A_1, \ldots, A_k)$ are related by a sequence of Flanders bridges $\Leftrightarrow$ $\mathcal{G}$ is a forest.

Hence: If $\mathcal{G}$ is a forest ($\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$):

- $\mathcal{S}_\lambda(\Pi_1) = \mathcal{S}_\lambda(\Pi_2)$, for all $\lambda \neq 0$.
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Sequences of Flanders bridges

Definition

$M_1, M_{d+1} \in \mathbb{C}^{n\times n}$ are connected by a sequence of Flanders bridges if $(M_1, M_2), (M_2, M_3), \ldots, (M_d, M_{d+1})$ are Flanders pairs, for some $M_2, \ldots, M_d$.

$\mathcal{G}$: the graph of non-commutativity relations of $A_1, \ldots, A_k$.

Then, if products in $\mathcal{P}(A_1, \ldots, A_k)$ are considered as **formal products**:

Theorem

Any two products in $\mathcal{P}(A_1, \ldots, A_k)$ are related by a sequence of Flanders bridges $\Leftrightarrow$ $\mathcal{G}$ is a forest.

Hence: If $\mathcal{G}$ is a forest ($\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$):

- $\mathcal{S}_\lambda(\Pi_1) = \mathcal{S}_\lambda(\Pi_2)$, for all $\lambda \neq 0$.
- $\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty \leq d$. ...$d$?

The main result

Theorem

$\mathcal{G}$ a forest. Set d= length of the longest path in $\mathcal{G}$.
Given $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$:

$$\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty \leq d. \tag{1}$$

This bound is **attainable**: Let $\mathcal{G}$ be any forest with k vertices, and let $d \leq k$ be the length of the longest path in $\mathcal{G}$. Then there are $A_1, \ldots, A_k \in \mathbb{C}^{n \times n}$ whose graph of non-commutativity relations is $\mathcal{G}$, and $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$ with

$$\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty = d.$$

Comment on the **Proof:**

- For (1): Uses tools from theory of permutations and graph theory.
- For the attainability: Constructive, just matrix manipulations.

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Theorem

$\mathcal{G}$ a forest. Set d= length of the longest path in $\mathcal{G}$.
Given $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \dots, A_k)$:

(1) $$\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty \leq d.$$

This bound is **attainable**: Let $\mathcal{G}$ be any forest with k vertices, and let $d \leq k$ be the length of the longest path in $\mathcal{G}$. Then there are $A_1, \dots, A_k \in \mathbb{C}^{n \times n}$ whose graph of non-commutativity relations is $\mathcal{G}$, and $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \dots, A_k)$ with

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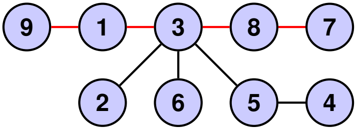
Example



Set:

$$\begin{array}{lll} A_1 = \mathrm{diag}(\widetilde{A}_1, I_8), & A_2 = \mathrm{diag}(I_7, D_2^{(2)}, I_4), & A_3 = \mathrm{diag}(\widetilde{A}_3, D_3^{(1)}, D_3^{(2)}, D_3^{(3)}, I_2), \\ A_4 = \mathrm{diag}(I_{11}, D_4^{(4)}), & A_5 = \mathrm{diag}(I_9, D_5^{(3)}, D_5^{(4)}), & A_6 = \mathrm{diag}(I_5, D_6^{(1)}, I_6), \\ A_7 = \mathrm{diag}(\widetilde{A}_7, D_2^{(2)}, I_4), & A_8 = \mathrm{diag}(\widetilde{A}_8, I_8), & A_9 = (\widetilde{A}_9, I_8), \end{array}$$

with:

$$\begin{array}{lll} \widetilde{A}_9 = \mathrm{diag}(I_3, J_2(0)) & \widetilde{A}_1 = \mathrm{diag}(I_2, J_2(0), 1), & \widetilde{A}_3 = \mathrm{diag}(1, J_2(0), I_2), \\ \widetilde{A}_8 = \mathrm{diag}(J_2(0), I_3), & \widetilde{A}_7 = \mathrm{diag}(0, I_4), & \widetilde{A}_i = I_5, \text{ for } i \neq 1, 3, 7, 8, 9, \end{array}$$

and $D_j^{(i)} \in \mathbb{C}^{2\times 2}$ nonsingular such that $D_3^{(1)} D_6^{(1)} \neq D_6^{(1)} D_3^{(1)}$, $D_3^{(2)} D_2^{(2)} \neq D_2^{(2)} D_3^{(2)}$, $D_3^{(3)} D_5^{(3)} \neq D_5^{(3)} D_3^{(3)}$, and $D_4^{(4)} D_5^{(4)} \neq D_5^{(4)} D_4^{(4)}$. Then:

$$\Pi_1 = (A_9 A_1 A_3 A_8 A_7) A_6 A_2 A_5 A_4 = \mathrm{diag}(J_5(0), J), \Pi_2 = (A_7 A_8 A_3 A_1 A_9) A_6 A_2 A_5 A_4 = \mathrm{diag}(0_5, J),$$

with $J = \mathrm{diag}\left(D_3^{(1)} D_6^{(1)}, D_3^{(2)} D_2^{(2)}, D_3^{(3)} D_5^{(3)}, D_5^{(4)} D_4^{(4)}\right)$, nonsingular.

Hence: $\mathcal{S}_0(\Pi_1) = (5)$ and $\mathcal{S}_0(\Pi_2) = (1, 1, 1, 1, 1)$, so $\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty = 4$.

Open Problems

1. Given $d \geq 4$ and two nonincreasing sequences $\boldsymbol{\mu}, \boldsymbol{\mu}'$ of nonnegative integers such that $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty \leq d-1$, is it always possible to find d matrices, $A_1, \ldots, A_d$, such that $\mathcal{G}$ is a path, and $\mathcal{S}_0(A_1 \cdots A_d) = \boldsymbol{\mu}$, $\mathcal{S}_0(A_d \cdots A_1) = \boldsymbol{\mu}'$?
2. If $M, Q \in \mathbb{C}^{n\times n}$ are such that $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(Q)$, for all $\lambda \neq 0$, and $\|\mathcal{S}_0(M) - \mathcal{S}_0(Q)\|_\infty \leq 2$, are there three matrices $A, B, C \in \mathbb{C}^{n\times n}$ with $AC = CA$, such that $M = ABC$ and $Q = CBA$?

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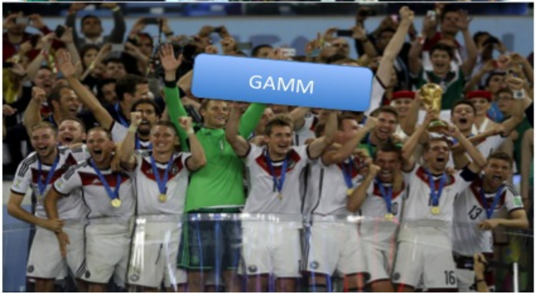

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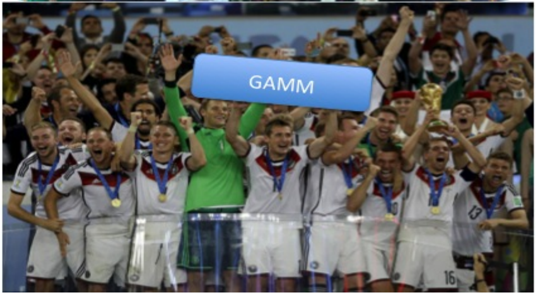

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