Solution of Sylvester-like equations and systems: consistency, uniqueness, and some applications

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Sylvester-like equations

$AX + XD = E$ (Sylvester)

$AX + X^\star D = E$ ($\star$-Sylvester) $\star = \top, *$

(Special attention to $AX + X^\star A = 0$)

$AXB + CXD = E$ (generalized Sylvester)

$AXB + CX^\star D = E$ (generalized $\star$-Sylvester)

Systems of all previous equations:

$$A_i X_j B_i + C_i X_k^{\blacktriangle} D_i = E_i, \qquad i, j, k \qquad \blacktriangle = 1, \top, *$$

I am NOT going to talk about...

- Large scale systems.
- Explicit (closed-form) formulas for the solution.
- Structured coefficients/solutions (symmetric, hermitian, centro-symmetric, centro-hermitian,...).
- Numerical methods...Well, just a little bit.

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Then...what am I going to talk about?

- Necessary and sufficient conditions for consistency.
- Necessary and sufficient conditions for uniqueness of solution.
- Connection with applied issues.
  - Orbit theory.
  - Block (anti-)diagonalization of block (anti-)triangular matrices.
- Dimension and expression (not closed-form) of the solution space.

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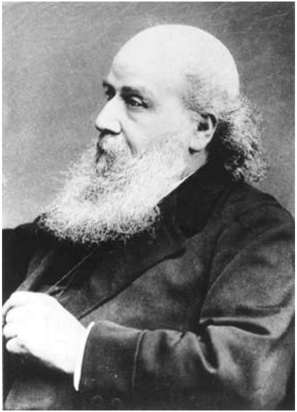


James Joseph Sylvester
(London, 1814–1897)

ANALYSE MATHÉMATIQUE. — *Sur l'équation en matrices $px = xq$*; par M. SYLVESTER.

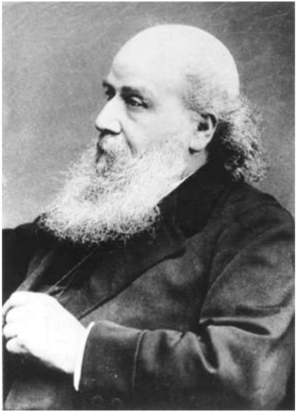
« Soient p et q deux matrices de l'ordre ω.

» Pour résoudre l'équation $px = xq$, on obtiendra ω^2 équations homogènes linéaires entre les ω^2 éléments de l'inconnue x et les éléments de p et de q, de sorte que, afin que l'équation donnée soit résoluble, les éléments de p et de q doivent être liés ensemble par une et une seule équation.

» Mais, si l'*équation identique* en p est écrite sous la forme

$$p^\omega + Bp^{\omega-1} + Cp^{\omega-2} + \ldots + L = 0,$$

Comptes Rendus Acad. Sci. 99 (1884)



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(68)

on aura apparemment, en vertu de l'équation $p = xqx^{-1}$

$$xq^{\omega}x^{-1} + Bxq^{\omega-1}x^{-1} + Cxq^{\omega-2}x^{-1} + \ldots + L = 0$$

ou bien

$$q^{\omega} + Bq^{\omega-1} + Cq^{\omega-2} + \ldots + L = 0;$$

donc les ω racines de q seront identiques avec celles de p et, au lieu d'une seule équation, on aura en apparence (*au moins*) ω équations entre les éléments de p et de q.

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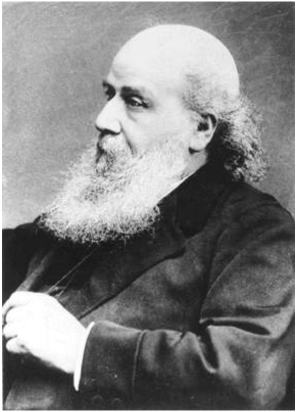

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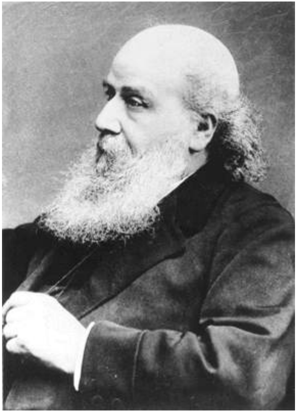

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si une des racines latentes de p est égale à une de q, l'équation $px = xq$ est résoluble et de plus, sans que cette condition soit satisfaite, l'équation est irrésoluble. Soient donc $\lambda_1, \lambda_2, \ldots, \lambda_\omega$ les racines latentes de β et $\mu_1, \mu_2, \ldots, \mu_\omega$ de q et supposons que $\lambda_1 = \mu$, alors

$$(p - \lambda_1)x = x(q - \mu_1),$$

et l'on peut satisfaire à cette équation en écrivant

$$x = (p - \lambda_2)(p - \lambda_3)\ldots(p - \lambda_\omega)(q - \mu_2)(q - \mu_3)\ldots(q - \mu_\omega).$$

Comptes Rendus Acad. Sci. 99 (1884)

Some recent activity about Sylvester equations (ILAS)

- ILAS 2014 (Korea): MS on Solution of Sylvester-like equations and canonical forms (co-organized by Stefan Johansson and F. DT.)

- Related talks in this meeting:
  - M. Karow, Mon 11:30–12:00
  - J. E. Román, Mon 12:00-12:30
  - D. Kressner, Tue 10:30–11:00
  - E. Jarlebring, Tue 16:30–17:00
  - D. Palitta, Tue 14:30–15:00
  - F. Uhlig, Tue 15:00–15:30
  - K. Meerbergen, Thu 10:30–11:00, Room AV 04.17
  - E. Ringh, Thu 14:30–15:00, Room AV 00.17

We will pay attention to...

- **Size** of the matrices:
  - In most cases, the only requirement is the product to be well-defined, but...
  - Not always (regarding **uniqueness**)!
- **Base field**: Desideratum: $\mathbb{F}$ arbitrary field, but...
  - $\mathbb{F} = \mathbb{C}$ mostly (for $\star = *$ and **uniqueness** issue).
  - char $(\mathbb{F}) \neq 2$ in some instances (regarding **consistency**).

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The vec approach

$\mathrm{vec}(AXB + CX^{\blacktriangle}D) = \mathrm{vec}(E)$ leads to

- $\boxed{\blacktriangle = 1}$: $[B^\top \otimes A + (C \otimes D^\top)]\,\mathrm{vec}(X) = \mathrm{vec}(E)$
- $\boxed{\blacktriangle = \top}$: $[B^\top \otimes A + \Pi(C \otimes D^\top)]\,\mathrm{vec}(X) = \mathrm{vec}(E)$

- $\boxed{\blacktriangle = *}$: $(B^\top \otimes A)\mathrm{vec}(X) + \Pi(C \otimes D^\top)\mathrm{vec}(\overline{X}) = \mathrm{vec}(E)$

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  Not linear over $\mathbb{C} \rightsquigarrow \mathrm{vec}\,(X) = [\mathrm{vec}\,(\mathrm{Re}\,X); \mathrm{vec}\,(\mathrm{Im}\,X)]$

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☞ $AXB + CX^{\blacktriangle}D = E$ can be written as a **linear system** $MY = b$:

$$Y = \begin{cases} \mathrm{vec}\,(X), & \text{if } \blacktriangle = \top, 1 \\ [\mathrm{vec}\,(\mathrm{Re}\,X); \mathrm{vec}\,(\mathrm{Im}\,X)], & \text{if } \blacktriangle = * \end{cases}$$

The vec approach (cont.)

$$\boxed{AXB + CX^{\blacktriangle}D = E \Leftrightarrow MY = b} \text{ (all } n \times n\text{, for simplicity)}$$

$$M \in \begin{cases} \mathbb{F}^{n^2 \times n^2}, & \text{if } \blacktriangle = 1, \top, \\ \mathbb{R}^{(2n^2)\times(2n^2)}, & \text{if } \blacktriangle = * \end{cases}$$

☹ Too large!

☹ Not easy to handle with

Combined with:

- Appropriate permutation of rows.
- Periodic Schur decomposition ($\mathbb{F} = \mathbb{R}, \mathbb{C}$).

☺ It will be **useful**!!

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Linearity and uniqueness of solution

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- If M is square:

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$$\boxed{\begin{array}{c} A_i X_{j_i} B_i + C_i X_{k_i}^{\blacktriangle} D_i = E_i \text{ has a unique solution} \\ \Updownarrow \\ A_i X_{j_i} B_i + C_i X_{k_i}^{\blacktriangle} D_i = 0 \text{ has a unique solution} \end{array}}$$

Linearity and uniqueness of solution

$$A_i X_{j_i} B_i + C_i X_{k_i}^{\blacktriangle} D_i = E_i \Leftrightarrow MY = b$$

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☞ We only need to look at the **homogeneous** equation!

Outline

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$A\in\mathbb{F}^{m\times m}, D\in\mathbb{F}^{n\times n}, E\in\mathbb{F}^{m\times n}$, $\mathbb{F}$ an **arbitrary field**.

Existence:

Theorem [Roth, 1952], [Flanders-Wimmer, 1977]

$AX+XD=E$ is **consistent** iff

$$\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} = P \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} P^{-1} \qquad \text{(Roth's criterion)}$$

for some invertible P.

Uniqueness ($\mathbb{F}=\mathbb{C}$):

Theorem

$AX-XD=0$ has a **unique solution** iff $\Lambda(A)\cap\Lambda(D)=\emptyset$

$X\in\mathbb{F}^{m\times n}$

☞ **Size**: Most general setting.

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Existence: Will be provided later.

Uniqueness: $A, C \in \mathbb{C}^{m\times m}$, $B, D \in \mathbb{C}^{n\times n}$.

Theorem [Chu, 1987]

$AXB + CXD = E$ has a **unique solution** iff the pencils $A - \lambda C$ and $D - \lambda B$ are regular and have disjoint spectra.

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☞ **Size**: Not the most general setting. It could be

$$A,C\in\mathbb{C}^{p\times m},\ B,D\in\mathbb{C}^{n\times q}$$

Outline

Existence of solution

$\mathbb{F}$ a field with $\operatorname{char}\mathbb{F} \neq 2$, $A \in \mathbb{F}^{n\times m}, B \in \mathbb{F}^{m\times n}, C \in \mathbb{F}^{m\times m}$

Theorem [Wimmer 1994], [DT-Dopico 2011]

$AX + X^\star B = C$ is **consistent** iff

$$P^\star \begin{bmatrix} C & A \\ B & 0 \end{bmatrix} P = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix},$$

for some nonsingular P.

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Uniqueness of solution

Definition: $\mathscr{S} \in \mathbb{C} \cup \{\infty\}$ is:

(a) reciprocal free if $\lambda\mu \neq 1$, for any $\lambda, \mu \in \mathscr{S}$.

(b) $*$-reciprocal free if $\lambda\overline{\mu} \neq 1$, for any $\lambda, \mu \in \mathscr{S}$.

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$A, D \in \mathbb{C}^{n\times n}$

Theorem [Byers-Kressner 2006], [Kressner-Schröder-Watkins, 2009]

$AX + X^\star D = E$ has a **unique solution** iff $A + \lambda D^\star$ is regular, and

- $\star = *$: $\Lambda(A + \lambda D^*)$ is $*$-reciprocal free.
- $\star = \top$: $\Lambda(A + \lambda D^\top) \setminus \{1\}$ is reciprocal free and $m_1(A + \lambda D^\top) \leq 1$.

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$A, D^\star\in\mathbb{C}^{n\times m}$

Theorem [Byers-Kressner 2006], [Kressner-Schröder-Watkins, 2009]

$AX+X^\star D=E$ has a **unique solution**, for any E, iff $A+\lambda D^\star$ is regular, and

- $\star=*$: $\Lambda(A+\lambda D^*)$ is $*$-reciprocal free.
- $\star=\top$: $\Lambda(A+\lambda D^\top)\setminus\{1\}$ is reciprocal free and $m_1(A+\lambda D^\top)\leq 1$.

☞ **Size**: Most general setting.

Outline

Existence: Will be provided later.

Uniqueness: $A,B,C,D\in\mathbb{C}^{n\times n}$.

Theorem [DT-Iannazzo, 2016]

$AXB+CX^\star D=E$ has a **unique solution** iff the pencil

$$\mathscr{P}(\lambda)=\begin{bmatrix}\lambda D^\star & B^\star\\ A & \lambda C\end{bmatrix}$$

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$X\in\mathbb{C}^{n\times n}$

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☞ **Size**: Not the most general setting. It could be

$$X\in\mathbb{C}^{m\times n}, A\in\mathbb{C}^{p\times m}, B\in\mathbb{C}^{n\times q}, C\in\mathbb{C}^{p\times n}, D\in\mathbb{C}^{m\times q}$$

Outline

Sylvester $AX - XD = 0$: Dimension of solution

$\mathscr{S} = \{X : \; AX - XD = 0\}$ (solution space)

$J_M \equiv$ Jordan canonical form of M,

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}_{k \times k}$$

$$J_A = \widetilde{J}_A \oplus J_{p_{i,\lambda}}(\lambda), \qquad J_D = \widetilde{J}_D \oplus J_{q_{j,\lambda}}(\lambda)$$

$$\dim \mathscr{S} = \sum_{\lambda} \sum_{i,j} \min\{p_{i,\lambda}, q_{j,\lambda}\}$$

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Sylvester $AX - XD = 0$, expression of solution

$P^{-1}AP = J_A, \qquad Q^{-1}DQ = J_D$

☹ Depends on P, Q !!!

(See [Gantmacher, 1959])

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$$AX - XD = 0$$

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$P^{-1}AP = J_A, \qquad Q^{-1}DQ = J_D$

$$(P^{-1}AP)(P^{-1}XQ) - (P^{-1}XQ)(Q^{-1}DQ) = 0$$

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$$J_A = \bigoplus_{i=1}^{p} J_{\ell_i}(\lambda_i), \qquad J_D = \bigoplus_{j=1}^{q} J_{k_j}(\mu_j)$$

Then $Y = \left[Y_{ij}\right]$, $1 \leq i \leq p$, $1 \leq j \leq q$, with

$$Y_{ij} = \begin{cases} 0_{\ell_i \times k_j} & \text{if } \lambda_i \neq \mu_j \\ \left[\begin{array}{c|cccc} & a_1 & a_2 & \dots & a_{\ell_i} \\ & & a_1 & \ddots & \vdots \\ 0_{\ell_i \times (k_j - \ell_i)} & & & \ddots & a_2 \\ & & & & a_1 \end{array}\right], \left[\begin{array}{cccc} \multicolumn{4}{c}{0_{(\ell_i - k_j) \times k_j}} \\ \hline a_1 & a_2 & \dots & a_{k_j} \\ & a_1 & \ddots & \vdots \\ & & \ddots & a_2 \\ & & & a_1 \end{array}\right] & \text{if } \lambda_i = \mu_j \end{cases}$$

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$\star$-Sylvester: $AX + X^\star D = 0$

$P(A+\lambda D^\star)Q = K_A + \lambda K_D^\star$ (Chronicle canonical form of $A+\lambda D^\star$)

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$$(PAQ)(Q^{-1}XP^\star) + (PX^\star Q^{-\star})(Q^\star DP^\star) = 0$$

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If $K_A + \lambda K_D^\star = \bigoplus_{i=1}^{p} K_A^{(i)} + \lambda (K_D^\star)^{(i)}$

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If $K_A + \lambda K_D^\star = \bigoplus_{i=1}^{p} K_A^{(i)} + \lambda (K_D^\star)^{(i)}$ 4 different types of blocks

Set $Y = \left[Y_{ij} \right]$, $1 \leq i,j \leq p$. The equation decouples into:

$$\begin{array}{ll} i=j: & K_A^{(i)} Y_{ii} + Y_{ii}^\star (K_D^\star)^{(i)} = 0 \\ i \neq j: & \left\{ \begin{array}{l} K_A^{(i)} Y_{ij} + Y_{ji}^\star (K_D^\star)^{(j)} = 0 \\ K_A^{(j)} Y_{ji} + Y_{ij}^\star (K_D^\star)^{(i)} = 0 \end{array} \right. \end{array}$$

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- We have solved each equation/system (**14** different ones!!).

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[DT-Dopico-Guillery-Montealegre-Reyes, 2011]

$\star$-Sylvester $AX + X^{\star}D = 0$: Dimension of the solution

Theorem [DT-Dopico-Guillery-Montealegre-Reyes, 2011]

The **dimension** of the solution space of $AX + X^{\top}B = 0$ is:

$$\dim \mathscr{S}(A,B) = \sum_{i=1}^{a} \varepsilon_i + \sum_{\mu_i=1} \lfloor k_i/2 \rfloor + \sum_{\mu_j=-1} \lceil k_j/2 \rceil +$$

$$\sum_{\substack{i,j=1\\ i<j}}^{a} (\varepsilon_i + \varepsilon_j) + \sum_{\substack{i<j\\ \mu_i\mu_j=1}} \min\{k_i, k_j\}$$

$$+ \sum_{i,j} (\eta_j - \varepsilon_i + 1) +$$

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Sizes of the right singular blocks in $KCF(A + \lambda B^\top)$

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Sizes of the left singular blocks in $KCF(A + \lambda B^\top)$

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Sizes of Jordan blocks in $KCF(A + \lambda B^\top)$

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Sizes of infinite Jordan blocks in $KCF(A + \lambda B^{\top})$

The other equations

With similar techniques, we get **expressions** for the solution (and its dimension) of:

1. $AXB + CXD = 0$: Depends on $\mathrm{KCF}(A - \lambda C)$ and $\mathrm{KCF}(D - \lambda B)$
   [Hernández-Gassó, 1989]
2. $AX + X^\star A = 0$: Depends on the Canonical form for congruence of A
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☞ In all cases:

- The expression for the solution depends on the change matrices. ☹
- $\mathbb{F} = \mathbb{C}$ (for other fields, it would require to have appropriate versions of canonical forms)
- **Size**: Most general situation, except in 1: $A, C \in \mathbb{C}^{m\times m}, B, D, \in \mathbb{C}^{n\times n}$

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Outline

Block (anti)-diagonalization

$$\overbrace{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{-1}}^{P^{-1}} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \overbrace{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}}^{P} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

Block (anti)-diagonalization

$$\overbrace{\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}}^{P^{-1}} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \overbrace{\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}}^{P} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$\Leftrightarrow A_{11}X - XA_{22} = A_{12} \rightsquigarrow \text{Sylvester equation}$$

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$$\overbrace{\begin{bmatrix} X_{11}^\star & X_{21}^\star \\ X_{12}^\star & X_{22}^\star \end{bmatrix}}^{P^\star} \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \overbrace{\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}}^{P} = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

Block (anti)-diagonalization

$$\overbrace{\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}}^{P^{-1}} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \overbrace{\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}}^{P} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$\Leftrightarrow A_{11}X - XA_{22} = A_{12} \rightsquigarrow$ Sylvester equation

Analogously:

$$\overbrace{\begin{bmatrix} I & 0 \\ (-X)^\star & I \end{bmatrix}}^{P^\star} \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \overbrace{\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}}^{P} = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$\Leftrightarrow A_{21}X + X^\star A_{12} = A_{22} \rightsquigarrow \star$-Sylvester equation

Orbit theory

Set:

$$\mathscr{O}_c(A) = \left\{ PAP^\top \ : \ P \text{ nonsingular} \right\} \quad \text{Congruence orbit of } A$$
$$\mathscr{O}_s(A) = \left\{ PAP^{-1} \ : \ P \text{ nonsingular} \right\} \quad \text{Similarity orbit of } A$$

Then:

$$T_{\mathscr{O}_c(A)}(A) = \{ XA + AX^\top \ : \ X \in \mathbb{C}^{n\times n} \} \quad \text{Tangent space of } \mathscr{O}_c(A) \text{ at } A$$
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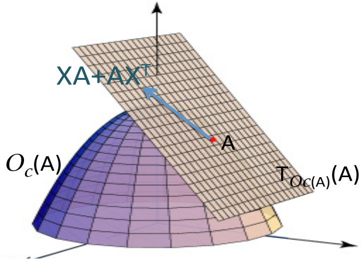

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(a) $\operatorname{codim} \mathcal{O}_c(A) = \operatorname{codim} T_{\mathcal{O}_c(A)}(A) = \dim(X : \; XA + AX^\top = 0)$

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A necessary condition...

Let

$$\begin{array}{rcll} A_iX_k - X_jD_i & = & E_i & i=1,\dots,n_1 \\ F_{i'}X_{k'} + X_{j'}^{\star}K_{i'} & = & L_{i'} & i'=1,\dots,n_2 \end{array}$$

for $j,k,j',k' \in \{1,\dots,m\}$.

If the system is **consistent**, set

$$P_\ell = \begin{bmatrix} I & -X_\ell \\ 0 & I \end{bmatrix} \quad \ell = 1,\dots,m$$

Then:

$$P_j^{-1}\begin{bmatrix} A_i & E_i \\ 0 & D_i \end{bmatrix} P_k = \begin{bmatrix} A_i & 0 \\ 0 & D_i \end{bmatrix}$$

$$P_{j'}^{\star}\begin{bmatrix} 0 & G_{i'} \\ K_{i'} & L_{i'} \end{bmatrix} P_{k'} = \begin{bmatrix} 0 & F_{i'} \\ K_{i'} & 0 \end{bmatrix}$$

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Is the converse true??

...and sufficient as well

$A_i, D_i, E_i, F_{i'}, K_{i'}, L_{i'}$ matrices over $\mathbb{F}$, with $\mathrm{char}\,\mathbb{F} \neq 2$

Theorem [Dmytryshyn-Kågström, 2015]

The system

$$\begin{array}{rcll} A_i X_k - X_j D_i & = & E_i & i = 1, \dots, n_1 \\ F_{i'} X_{k'} + X_{j'}^\star K_{i'} & = & L_{i'} & i' = 1, \dots, n_2 \end{array}$$

for $j, k \in \{1, \dots, m\}$ is **consistent** iff

$$P_j^{-1} \begin{bmatrix} A_i & E_i \\ 0 & D_i \end{bmatrix} P_k = \begin{bmatrix} A_i & 0 \\ 0 & D_i \end{bmatrix}$$
$$P_{j'}^\star \begin{bmatrix} 0 & G_{i'} \\ K_{i'} & L_{i'} \end{bmatrix} P_{k'} = \begin{bmatrix} 0 & F_{i'} \\ K_{i'} & 0 \end{bmatrix}$$

for some invertible $P_1, \dots, P_m$.

- [DT, arXiv, 2014] for just one unknown.
- Strongly based on [Wimmer, 1994].

Relevant features of this result

$$\begin{array}{rcl} A_i X_k - X_j D_i & = & E_i \\ F_{i'} X_{k'} + X_{j'}^\star K_{i'} & = & L_{i'} \end{array} \quad \textbf{consistent} \Leftrightarrow \begin{array}{l} P_j^{-1} \begin{bmatrix} A_i & E_i \\ 0 & D_i \end{bmatrix} P_k = \begin{bmatrix} A_i & 0 \\ 0 & D_i \end{bmatrix} \\ P_{j'}^\star \begin{bmatrix} 0 & G_{i'} \\ K_{i'} & L_{i'} \end{bmatrix} P_{k'} = \begin{bmatrix} 0 & F_{i'} \\ K_{i'} & 0 \end{bmatrix} \end{array}$$

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- **Extends** nicely and directly the result for **one** single equation (both Sylvester and $\star$-Sylvester).
- The only if part is true for arbitrary $\mathbb{F}$.
- **Size**: **Most general setting** (only restriction: the products must be well-defined).
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SIAM Student Paper Prize, 2015 (A. Dmytryshyn)

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In this setting, we can restrict ourselves to “periodic systems":

$$\begin{array}{l} A_k X_k B_k - C_k X_{k+1} D_k = E_k, \quad k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1^{\blacktriangle} D_r = E_r, \end{array}$$

with $\blacktriangle = 1, \star$.

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The case $\blacktriangle = *$: reduction to $\blacktriangle = 1$

Lemma

The system

$$(1)\qquad \begin{array}{c} A_kX_kB_k - C_kX_{k+1}D_k = 0, \quad k = 1,\dots,r-1, \\ A_rX_rB_r - C_rX_1^*D_r = 0, \end{array}$$

has a unique solution iff the system

$$(2)\qquad \begin{array}{cc} A_kX_kB_k - C_kX_{k+1}D_k = 0, & k = 1,\dots,r-1, \\ A_rX_rB_r - C_rX_{r+1}D_r = 0, & \\ B_k^*X_{r+k}A_k^* - D_k^*X_{r+k+1}C_k^* = 0, & k = 1,\dots,r-1, \\ B_r^*X_{2r}A_r^* - D_r^*X_1C_r^* = 0 & \end{array}$$

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Proof: $\Leftarrow$: If (1) has a nonzero solution $(X_1, \ldots, X_r)$, then $(X_1, \ldots, X_r, X_1^*, \ldots, X_r^*)$ is a nonzero solution of (2).

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Proof: $\Rightarrow$: If $(X_1, \dots, X_r, X_{r+1}, \dots, X_{2r})$ is a nonzero solution of (2), then $(X_1 + X_{r+1}^*, \dots, X_r + X_{2r}^*)$ is a solution of (1). If $(X_1 + X_{r+1}^*, \dots, X_r + X_{2r}^*) = 0$, then $X_{r+i} = -X_i^*$, $i = 1, \dots, r$, and then $\sqrt{-1}(X_1, \dots, X_r)$ is a nonzero solution of (1).$\square$

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has a unique solution.

The result is **FALSE** replacing $*$ by $\top$:

$$x + x^\top = 0 \Leftrightarrow x = 0 \qquad \text{but} \qquad \begin{array}{c} x + y = 0 \\ y + x = 0 \end{array} \Leftrightarrow y = -x$$

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☞ The case $\blacktriangle = *$ can be reduced to $\blacktriangle = 1$.

Characterization of the uniqueness of solution ($\blacktriangle = 1$)

Theorem [Byers-Rhee, 1995]

The system

$$\begin{aligned} A_k X_k B_k - C_k X_{k+1} D_k = 0, \quad k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_1 D_r = 0 \end{aligned}$$

has a **unique solution** iff the pencils

$$\begin{bmatrix} \lambda A_r & 0 & \dots & 0 & C_r \\ C_{r-1} & \lambda A_{r-1} & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & C_2 & \lambda A_2 & 0 \\ 0 & \dots & 0 & C_1 & \lambda A_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda D_r & 0 & \dots & 0 & B_r \\ B_{r-1} & \lambda D_{r-1} & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & B_1 & \lambda D_2 & 0 \\ 0 & \dots & 0 & B_1 & \lambda D_1 \end{bmatrix}$$

are regular and have disjoint spectra.

Characterization of the uniqueness of solution ($\blacktriangle = \star$)

Theorem [DT-Iannazzo-Poloni-Robol, 2016]

The system:

$$\begin{array}{c} A_k X_k B_k - C_k X_{k+1} D_k = 0, \quad k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1^\star D_r = 0 \end{array}$$

has a **unique solution** iff the pencil

$$\mathscr{P}(\lambda) = \begin{bmatrix} \lambda A_1 & -C_r^\star & & & & & \\ & \ddots & \ddots & & & & \\ & & \lambda A_r & -C_r^\star & & & \\ & & & \lambda B_1^\star & -D_1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & -D_{r-1} \\ -D_r^\star & & & & & & \lambda B_r^\star \end{bmatrix}$$

is regular and

- $\star = \top$: $\Lambda(\mathscr{P}) \backslash \{e^{2\pi k i/r} : k = 0 : r-1\}$ is reciprocal free and $m_{e^{2\pi k i/r}}(\mathscr{P}) \leq 1$.
- $\star = *$: $\Lambda(\mathscr{P})$ is $*$-reciprocal free.

An $O(rn^3)$ algorithm

- For systems $\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_1 D_r = 0 \end{cases}$
- Based on [D-Dopico'11] for $AX + X^\star D = E$, outlined in [Chiang-Chu-Lin'12] for a single equation $\rightsquigarrow$ (Bartels-Stewart).
- Uses the periodic Schur decomposition [Bojanczyk-Golub-Van Dooren, 1992] $\rightsquigarrow O(rn^3)$

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- Based on [D-Dopico'11] for $AX + X^\star D = E$, outlined in [Chiang-Chu-Lin'12] for a single equation $\rightsquigarrow$ (Bartels-Stewart).
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Outline

Existence and uniqueness (characterization)

Equation	Consistency	Uniqueness
$AX + XD = E$	Roth, 1952	Gantmacher, 1959
$AX + X^\star D = E$	Wimmer, 1994 ($\mathbb{F} = \mathbb{C}$) DT-Dopico, 2011 char$\mathbb{F} \neq 2$	Byers-Kressner, 2006 ($\star = \top$) Kressner-Schröder-Watkins, 2009 ($\star = *$) ($\mathbb{F} = \mathbb{C}$)
$AX + CX^\star = E$	Dmytryshyn-Kågström, 2016 char$\mathbb{F} \neq 2$	DT, 2013 ($\mathbb{F} = \mathbb{C}$)
$AXB + CXD = E$	Dmytryshyn-Kågström, 2016	Chu, 1987 $\mathbb{F} = \mathbb{R}, \mathbb{C}$ square coeffs.
$AXB + CX^\star D = E$	Dmytryshyn-Kågström, 2016 char$\mathbb{F} \neq 2$	DT-Iannazzo, 2016 square coeffs (same size)
$A_i X_j B_i + C_i X_k^\blacktriangle D_i = E_i$ (systems, $\blacktriangle = 1, \star$)	Dmytryshyn-Kågström, 2016 char$\mathbb{F} \neq 2$	DT-Iannazzo-Poloni-Robol 201? $\mathbb{F} = \mathbb{C}$ square coeffs. (same size)

Some open problems

- Necessary and sufficient conditions for the uniqueness of solution of $AXB + CXD = 0$ when A, B, C, D are **rectangular**.
  ☞ **In progress** (with B. Iannazzo, F. Poloni, L. Robol).
- Necessary and sufficient conditions for the uniqueness of solution of $AXB + CX^{\star}D = 0$ when A, B, C, D are **rectangular** (or square with different size).
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- Find the dimension of the solution space of $AXB + CX^{\star}D = 0$.
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Conclusions

- Solution of Sylvester-like equations is an interesting problem from both a theoretical and applied point of view.
- Elementary linear algebra techniques lead to beautiful and non-trivial results.
- **Elementary characterization** of the consistency and uniqueness of solution in terms of the coefficient matrices (or spectral properties of pencils/matrices constructed easily from them).
- Dimension and expression of the solution is known in some cases.
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THANKS FOR YOUR ATTENTION !!!!!