Uniqueness of solution of systems of generalized Sylvester and $\star$-Sylvester equations

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Outline

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Generalized Sylvester equations

$AXB - CXD = E$ (generalized Sylvester)

$AXB - CX^{\star}D = E$ (generalized $\star$-Sylvester) $\star = \top, *$

Particular cases:

$AX - XD = E$ (Sylvester)

$AX - X^{\star}D = E$ ($\star$-Sylvester)

☞ We are interested in:

Systems of all previous equations (coupled):

$$A_iX_jB_i - C_iX_k^{\blacktriangle}D_i = E_i, \qquad i,j,k \qquad \blacktriangle = 1, \star$$

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All $\mathbb{C}^{n\times n}$ matrices

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Main goals

G1

When does

$$A_i X_j B_i - C_i X_k^{\blacktriangle} D_i = E_i, \qquad i,j,k \qquad \blacktriangle = 1, \star$$

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G2

Provide an $O(n^3)$ algorithm to compute the (unique) solution.

The vec approach

$\mathrm{vec}\,(AXB - CX^{\blacktriangle}D) = \mathrm{vec}\,(E)$ leads to

- $\boxed{\blacktriangle = 1}$: $\left[B^\top \otimes A - (C \otimes D^\top)\right] \mathrm{vec}\,(X) = \mathrm{vec}\,(E)$
- $\boxed{\blacktriangle = \top}$: $\left[B^\top \otimes A - \Pi(C \otimes D^\top)\right] \mathrm{vec}\,(X) = \mathrm{vec}\,(E)$
- $\boxed{\blacktriangle = *}$: $(B^\top \otimes A)\mathrm{vec}\,(X) - \Pi(C \otimes D^\top)\mathrm{vec}\,(\overline{X}) = \mathrm{vec}\,(E)$

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☞ $AXB - CX^{\blacktriangle}D = E$ can be written as a **linear system** $MY = b$:

$$Y = \begin{cases} \mathrm{vec}\,(X), & \text{if } \blacktriangle = \top, 1 \\ [\mathrm{vec}\,(\mathrm{Re}\,X); \mathrm{vec}\,(\mathrm{Im}\,X)], & \text{if } \blacktriangle = * \end{cases}$$

The vec approach (cont.)

$$\boxed{AXB - CX^{\blacktriangle}D = E \Leftrightarrow MY = b}$$

$$M \in \begin{cases} \mathbb{F}^{n^2 \times n^2}, & \text{if } \blacktriangle = 1, \top, \\ \mathbb{R}^{(2n^2)\times(2n^2)}, & \text{if } \blacktriangle = * \end{cases}$$

☹ Too large!

☹ Not easy to handle with

☞ Combined with:

- Appropriate permutation of rows/columns
- Periodic Schur decomposition ($\mathbb{F} = \mathbb{R}, \mathbb{C}$).

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Linearity and uniqueness of solution

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☞ We only need to look at the **homogeneous** equation!

Related work

- Systems of generalized Sylvester equations:
  - Uniqueness (periodic systems): [Byers-Rhee'95]
  - Consistency, uniqueness (structured coefficients/solutions/equations, matrices over other sets, ...): [Wang-Sun-Li'02], [Lee-Vu'12], [He-etal'16], ...
- Systems of coupled generalized Sylvester and $\star$-Sylvester equations:
  - Iterative methods (structured coefficients and solution): [Dehghan-Hajarian'11], [Song-Chen-Zhao'11], [Wu-etal'11], [Wu-etal'11], [Beik-etal'13], [Song-etal'14], ...
  - Consistency: [Dmytryshyn-Kågström'16]
  - Uniqueness: **???**

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Most general setting !!!

Motivation: the case $r = 1$

Theorem [Chu'87]

$AXB - CXD = 0$ has only the trivial solution iff $A - \lambda C$ and $D - \lambda B$ are regular and have disjoint spectra.

Theorem [DT-Iannazo'16]

$AXB - CX^{\star}D = 0$ has only the trivial solution iff

$$\mathscr{Q}(\lambda) = \begin{bmatrix} \lambda D^{\star} & B^{\star} \\ -A & \lambda C \end{bmatrix}$$

is regular and

- $\boxed{\star = *}$: $\lambda_i \overline{\lambda_j} \neq 1$ (λ_i, λ_j e-vals of $\mathscr{Q}$).
- $\boxed{\star = \top}$: $\lambda_i \lambda_j \neq 1$ ($\lambda_i, \lambda_j \neq \pm 1$ e-vals of $\mathscr{Q}$) and $\lambda = \pm 1$ have multiplicity ≤ 1.

Outline

Most general setting

$$\boxed{A_i X_j B_i - C_i X_k^{\blacktriangle} D_i = E_i, \qquad i, j, k}$$

- r (matrix) equations and s (matrix) unknowns.
- The unknowns X_j, X_k can be **equal** or **different**.
- $\blacktriangle = 1, \star$
- M square $\Rightarrow r = s$

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Irreducible systems

$\mathbb{S}$: a system of (matrix) equations. Then

$$\begin{array}{ll} \mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \cup \cdots \cup \mathbb{S}_\ell & \\ \mathbb{S}_1, \ldots, \mathbb{S}_\ell & \text{irreducible} \end{array}$$

- $\mathbb{S}$ has a unique solution iff $\mathbb{S}_i$ has a unique solution, for all $i = 1, \ldots, \ell$.
- If $\mathbb{S}$ has a unique solution, then $\mathbb{S}_i$ has the same number of equations and unknowns.

☞ We can focus on **irreducible systems**.

All unknowns appear exactly twice

If some X_j appears just once in $\mathbb{S}$ (with unique solution), say in $A_j X_j B_j + C_j X_k^\blacktriangle D_j = 0$, then

- A_j, B_j are **invertible**.
- $\mathbb{S}$ is equivalent to: $\begin{cases} X_j = -A_j^{-1} C_j X_k^\blacktriangle D_j B_j^{-1} \\ \mathbb{S}_{r-1} \end{cases}$
- $\mathbb{S}_{r-1}$ irreducible with $r-1$ equations in the $r-1$ unknowns $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_r$

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☞ We can **remove** the equations corresponding to unknowns appearing just once.

☞ In the new system, **all unknowns appear exactly twice.**

Reduction to a periodic system with at most one $\star$

Given the **irreducible** system

$$A_i X_j B_i + C_i X_k^{\blacktriangle} D_i = E_i, \qquad i,j,k, \qquad \blacktriangle = 1,\star$$

with each unknown appearing exactly **twice**.

1. There is an equivalent system (**periodic**)

$$\begin{array}{rcl} \widetilde{A}_i X_i \widetilde{B}_i + \widetilde{C}_i X_{i+1}^{\blacktriangle} \widetilde{D}_i & = & \widetilde{E}_i, \quad i = 1,\ldots,r, \\ X_{r+1} & = & X_1 . \end{array}$$

(Relabelling the variables and applying $\star$, if necessary)

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(Applying $\star$, and changing variables $X_i \mapsto X_i^\star$, if necessary)

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$$\boxed{\begin{array}{rcl} A_i X_i B_i + C_i X_{i+1} D_i & = & E_i, \quad i = 1,\dots,r-1, \\ A_r X_r B_r + C_r X_1^{\blacktriangle} D_r & = & E_r . \end{array}}$$

Outline

The periodic Schur decomposition

Theorem [Bojanzyck-Golub-VanDooren'92]

Given $M_k, N_k \in \mathbb{C}^{n\times n}$, $k = 1, \ldots, r$. There are Q_k, Z_k **unitary**, for $k = 1, \ldots, r$, such that

$$Q_k^* M_k Z_k = T_k, \qquad Q_k^* N_k Z_{k+1} = R_k,$$

(Periodic Schur decomposition)

where T_k, R_k are upper triangular and $Z_{r+1} = Z_1$.

Eigenvalues of formal products

Given the *formal product*

$$\Pi = N_r^{-1} M_r N_{r-1}^{-1} M_{r-1} \cdots N_1^{-1} M_1$$

through the periodic Schur decomposition of M_i, N_i,

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we define its eigenvalues

$$\lambda_i = \frac{\prod_{k=1}^r (T_k)_{ii}}{\prod_{k=1}^r (R_k)_{ii}}, \quad i = 1, 2, \ldots, n.$$

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$(Z_1^{-1} \Pi Z_1 = R_r^{-1} T_r R_{r-1}^{-1} T_{r-1} \cdots R_1^{-1} T_1)$

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Definition: Π is singular if: $\prod_{k=1}^r (T_k)_{ii} = \prod_{k=1}^r (R_k)_{ii} = 0$, for some $i \in \{1, 2, \ldots, n\}$ (and regular otherwise).

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► Considered by several authors: [Bojanzyck-Golub-VanDooren'92], [Benner-Mehrmann-Xu'02], [Granat-Kågström'06a–b], [Granat-Kågström-Kressner'07a–b], ...

Main result (first formulation). The case $\blacktriangle = 1$

Theorem

The system

$$\left\{\begin{array}{ll} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1 D_r = 0 \end{array}\right.$$

has only the trivial solution iff

$$C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1 \quad \text{and} \quad D_r B_r^{-1} D_{r-1}^{-1} B_{r-1}^{-1} \cdots D_1 B_1^{-1}$$

are regular and have no common e-vals.

Main result (first formulation). The case $\blacktriangle = \star$

Theorem

The system

$$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_1^{\star} D_r = 0 \end{cases}$$

has only the trivial solution iff

$$\Pi = D_r^{-\star} B_r^{\star} D_{r-1}^{-\star} B_{r-1}^{\star} \cdots D_1^{-\star} B_1^{\star} C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1$$

is regular and

- $\boxed{\star = *}$: $\lambda_i \overline{\lambda}_j \neq 1$ (λ_i, λ_j e-vals of Π).
- $\boxed{\star = \top}$: $\lambda_i \lambda_j \neq 1$ ($\lambda_i, \lambda_j \neq -1$ e-vals of Π), and $\lambda = -1$ has multiplicity ≤ 1.

Outline

The case $\blacktriangle = \star$

Theorem

The system $\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_1^\star D_r = 0. \end{cases}$ has only the trivial solution iff the matrix pencil

$$\mathcal{Q}(\lambda) := \begin{bmatrix} \lambda A_1 & C_1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \lambda A_r & C_r & & & \\ & & & \lambda B_1^\star & D_1^\star & & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & D_{r-1}^\star \\ -D_r^\star & & & & & & \lambda B_r^\star \end{bmatrix}$$

is regular and

$\boxed{\star = *}$: $\lambda_i \overline{\lambda_j} \neq 1$ (λ_i, λ_j e-vals of $\mathcal{Q}$).

$\boxed{\star = \top}$: $\lambda_i \lambda_j \neq 1$ ($\lambda_i \neq \lambda_j$ e-vals of $\mathcal{Q}$) and $\lambda^{2r} \neq -1$ for any λ e-val of $\mathcal{Q}$.

The case ▲ $=1$

Theorem [Byers-Rhee'95]

The system

$$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1,\ldots,r-1, \\ A_r X_r B_r - C_r X_1 D_r = 0 \end{cases}$$

has only the trivial solution iff the matrix pencils

$$\begin{bmatrix} \lambda A_1 & C_1 & & \\ & \lambda A_2 & \ddots & \\ & & \ddots & C_{r-1} \\ C_r & & & \lambda A_r \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda D_1 & B_1 & & \\ & \lambda D_2 & \ddots & \\ & & \ddots & B_{r-1} \\ B_r & & & \lambda D_r \end{bmatrix}$$

are regular and have no common e-vals.

Outline

Two basic ideas ($\blacktriangle = 1, \top$)

☞ Main procedure:

1. Get an equivalent system with A_i, C_i upper triangular and B_i, D_i lower triangular (using the periodic Schur).
2. Rearrange the equations / unknowns of the big linear system to get a **block-diagonal** matrix.

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1. Get an equivalent system with A_i, C_i upper triangular and B_i, D_i lower triangular (using the periodic Schur).
2. Rearrange the equations / unknowns of the big linear system to get a **block-diagonal** matrix.

   Choose an appropriate ordering!!

An equivalent system with triangular coeffs. ($\blacktriangle = 1$)

$$Q_k^* A_k Z_k = \widehat{A}_k, \quad Q_k^* C_k Z_{k+1} = \widehat{C}_k,$$

$\widehat{A}_k, \widehat{C}_k$ upper triangular $\rightsquigarrow$ **periodic Schur form** of $C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1$

$$\widehat{Q}_k^* B_k^* \widehat{Z}_k = \widehat{B}_k^*, \quad \widehat{Q}_k^* D_k^* \widehat{Z}_{k+1} = \widehat{D}_k^*,$$

$\widehat{B}_k^*, \widehat{D}_k^*$ upper triangular $\rightsquigarrow$ **periodic Schur form** of $D_r^{-*} B_r^* D_{r-1}^{-*} B_{r-1}^* \cdots D_1^{-*} B_1^*$

Then

$$A_k X_k B_k - C_k X_{k+1} D_k = E_k \qquad (k = 1, \ldots, r)$$

is equivalent to:

$$\begin{aligned} \widehat{A}_k \widehat{X}_k \widehat{B}_k - \widehat{C}_k \widehat{X}_{k+1} \widehat{D}_k &= Q_k^* A_k Z_k \widehat{X}_k \widehat{Z}_k^* B_k \widehat{Q}_k - Q_k^* C_k Z_{k+1} \widehat{X}_{k+1} \widehat{Z}_{k+1}^* D_k \widehat{Q}_k \\ &= Q_k^* (A_k X_k B_k - C_k X_{k+1} D_k) \widehat{Q}_k \\ &= Q_k^* E_k \widehat{Q}_k = \widehat{E}_k. \end{aligned}$$

An equivalent system with triangular coeffs. ($\blacktriangle = \top$)

$$Q_k^* A_k Z_k = \widehat{A}_k, \qquad Q_k^* C_k Z_{k+1} = \widehat{C}_k, \qquad Z_{2r+1} = Z_1,$$
$$Q_{r+k}^* B_k^\top Z_{r+k} = \widehat{B}_k^\top, \qquad Q_{r+k}^* D_k^\top Z_{r+k+1} = \widehat{D}_k^\top, \qquad k = 1, 2, \ldots, r,$$

($\widehat{A}_k, \widehat{C}_k, \widehat{B}_k^*, \widehat{D}_k^*$ upper triangular) $\rightsquigarrow$ **periodic Schur form** of

$$D_r^{-\top} B_r^\top D_{r-1}^{-\top} B_{r-1}^\top \cdots D_1^{-\top} B_1^\top C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1.$$

Then

$$\begin{array}{l} A_k X_k B_k - C_k X_{k+1} D_k = E_k \quad (k = 1, \ldots, r-1) \\ A_r X_r B_r - C_r X_1 D_r = E_r \end{array}$$

is equivalent to:

$$\begin{aligned}
\widehat{A}_k \widehat{X}_k \widehat{B}_k - \widehat{C}_k \widehat{X}_{k+1} \widehat{D}_k &= Q_k^* A_k Z_k \widehat{X}_k Z_{r+k}^\top B_k \bar{Q}_{r+k} - Q_k^* C_k Z_{k+1} \widehat{X}_{k+1} Z_{r+k+1}^\top D_k \bar{Q}_{r+k} \\
&= Q_k^* (A_k X_k B_k - C_k X_{k+1} D_k) \bar{Q}_{r+k} \\
&= Q_k^* E_k \bar{Q}_{r+k} = \widehat{E}_k, \\
\widehat{A}_r \widehat{X}_r \widehat{B}_r - \widehat{C}_r \widehat{X}_1 \widehat{D}_r &= Q_r^* A_r Z_r \widehat{X}_r Z_{2r}^\top B_r \bar{Q}_{2r} - Q_r^* C_r Z_{r+1} \widehat{X}_1^\top Z_1^\top D_r \bar{Q}_{2r} \\
&= Q_r^* (A_r X_r B_r - C_r X_1^\top D_r) \bar{Q}_{2r} \\
&= Q_r^* E_r \bar{Q}_{2r} = \widehat{E}_r.
\end{aligned}$$

Choosing an appropriate ordering (I)

The *left-angle* property
(for an order on $\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}$):

$$\begin{bmatrix} & & & & & & \\ & & & \times & \circ & \circ & \circ \\ & & & \circ & \circ & \circ & \circ \\ & & & \circ & \circ & \circ & \circ \\ \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{bmatrix}_{n\times n}$$

- $\times \leq \circ$
- (i,j) and (j,i) ($\times$) are consecutive.

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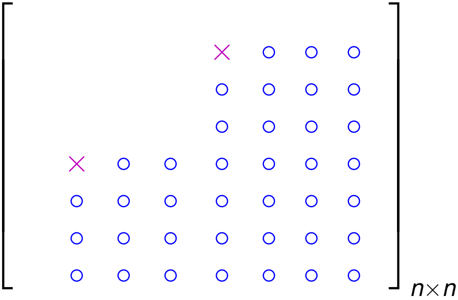

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- $\times \leq \circ$
- (i,j) and (j,i) ($\times$) are consecutive.

Choosing an appropriate ordering (II)

$$(i,j,k): \begin{cases} (i,j) \text{ entry of } X_k & \rightsquigarrow \mathscr{U} \text{ (unknowns)} \\ e_i^\top (A_k X_k B_k - C_k X_{k+1} D_k) e_j = (E_k)_{ij} & \rightsquigarrow \mathscr{E} \text{ (equations)} \end{cases}$$

If $\leq$ is an order on both $\mathscr{U}$ and $\mathscr{E}$ satisfying:

$$(i,j,k) \leq (i',j',k') \text{ whenever } (i,j) \leq_2 (i',j'),$$

with $\leq_2$ satisfying the left-angle property, then:

M is block-diagonal with $r \times r$ and $(2r) \times (2r)$ diagonal blocks.

- $r \times r$ blocks: Correspond to $(X_{ii})_1, \ldots, (X_{ii})_r$.
- $(2r) \times (2r)$ blocks: Correspond to $(X_{ij})_1, \ldots, (X_{ij})_r$, $i \neq j$.

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Choosing an appropriate ordering (II)

$$(i,j,k):\begin{cases}(i,j)\text{ entry of }X_k & \rightsquigarrow \mathscr{U}\text{ (unknowns)}\\ e_i^\top(A_kX_kB_k-C_kX_{k+1}D_k)e_j=(E_k)_{ij} & \rightsquigarrow \mathscr{E}\text{ (equations)}\end{cases}$$

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Some particular orderings

For each $1 \leq k \leq r$:

$$\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \boxtimes \end{bmatrix}$$

$\leq_{CR}$: Column-row ordering $\quad$ $\leq_S$: Squaring ordering $\quad$ $\leq_A$: Anti-diagonal ordering

Some particular orderings

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$$\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & {\color{red}\boxtimes} \\ \times & \times & \times & \times & {\color{red}\boxtimes} & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & {\color{red}\boxtimes} \\ \times & \times & \times & \times & {\color{red}\boxtimes} & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & {\color{red}\boxtimes} \\ \times & \times & \times & \times & {\color{red}\boxtimes} & \boxtimes \end{bmatrix}$$

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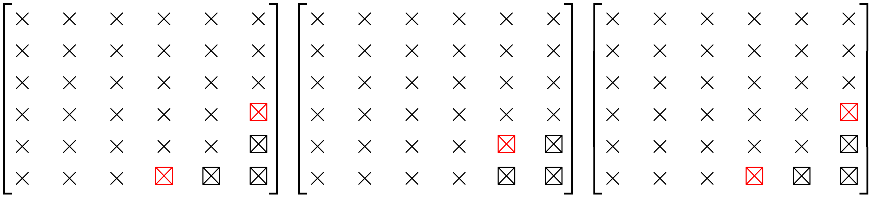

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$$
\begin{bmatrix}
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\end{bmatrix}
$$

$\leq_{CR}$: Column-row ordering $\quad$ $\leq_S$: Squaring ordering $\quad$ $\leq_A$: Anti-diagonal ordering

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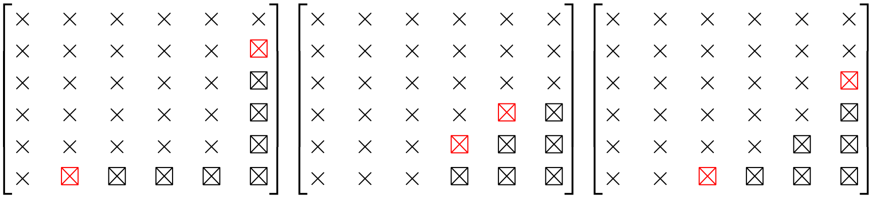

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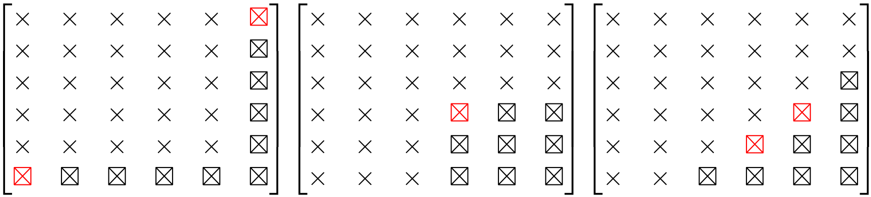

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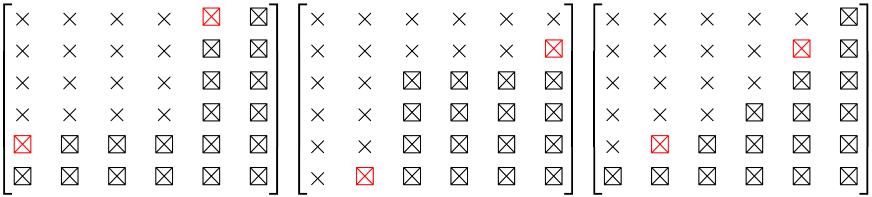

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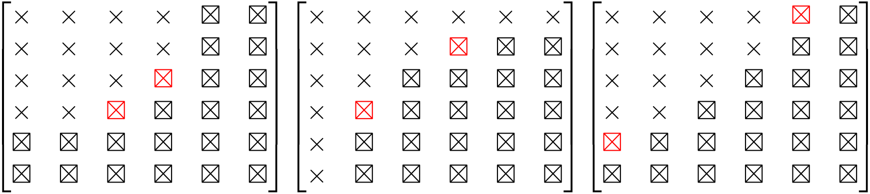
$$

$\leq_{CR}$: Column-row ordering $\leq_S$: Squaring ordering $\leq_A$: Anti-diagonal ordering

Some particular orderings

For each $1 \leq k \leq r$:



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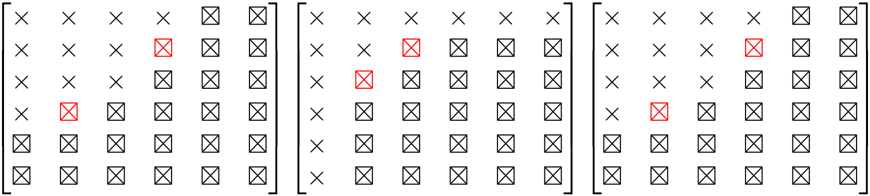

$\leq_{CR}$: Column-row ordering $\leq_S$: Squaring ordering $\leq_A$: Anti-diagonal ordering

Some particular orderings

For each $1 \leq k \leq r$:

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$$

$\leq_{CR}$: Column-row ordering $\leq_S$: Squaring ordering $\leq_A$: Anti-diagonal ordering

Some particular orderings

For each $1 \leq k \leq r$:

$$\begin{bmatrix} \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times & \times & {\color{red}\boxtimes} \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes \\ \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix}$$

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Some particular orderings

For each $1 \leq k \leq r$:

$$\begin{bmatrix} \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes \\ \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times & {\color{red}\boxtimes} & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes \\ \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix}$$

$\leq_{CR}$: Column-row ordering $\leq_S$: Squaring ordering $\leq_A$: Anti-diagonal ordering

Some particular orderings

For each $1 \leq k \leq r$:

$$\begin{bmatrix} \times & \times & \textcolor{red}{\boxtimes} & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \textcolor{red}{\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \times & \textcolor{red}{\boxtimes} & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \textcolor{red}{\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \textcolor{red}{\boxtimes} & \boxtimes & \boxtimes & \boxtimes \\ \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \textcolor{red}{\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix}$$

$\leq_{CR}$: Column-row ordering $\leq_S$: Squaring ordering $\leq_A$: Anti-diagonal ordering

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$$\begin{bmatrix} \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes \\ \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & \times & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix}$$

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For each $1 \leq k \leq r$:

$$\begin{bmatrix} \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix} \begin{bmatrix} \times & {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ {\color{red}\boxtimes} & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{bmatrix}$$

$\leq_{CR}$: Column-row ordering $\quad$ $\leq_S$: Squaring ordering $\quad$ $\leq_A$: Anti-diagonal ordering

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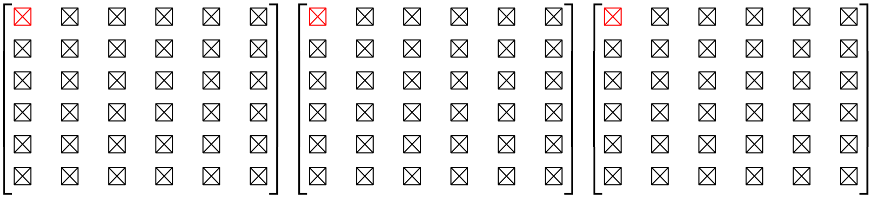
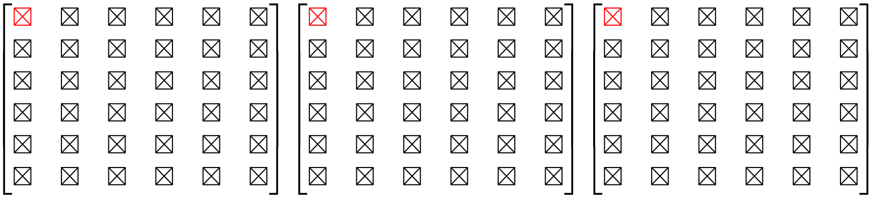
For each $1 \leq k \leq r$:



$\leq_{CR}$: Column-row ordering $\leq_{S}$: Squaring ordering $\leq_{A}$: Anti-diagonal ordering

Some particular orderings

For each $1 \leq k \leq r$:



$\leq_{CR}$: Column-row ordering $\leq_S$: Squaring ordering $\leq_A$: Anti-diagonal ordering

- We choose an ordering $\leq$ on (i,j,k) such that $(i,j,k) \leq (i,j,k')$ for $k \leq k'$

Diagonal blocks: $\blacktriangle = 1$

With any of these orderings, the diagonal blocks are:

$$M_{ii} := \begin{bmatrix} (A_1)_{ii}(B_1)_{ii} & -(C_1)_{ii}(D_1)_{ii} & & \\ & \ddots & \ddots & \\ & & (A_{r-1})_{ii}(B_{r-1})_{ii} & -(C_{r-1})_{ii}(D_{r-1})_{ii} \\ -(C_r)_{ii}(D_r)_{ii} & & & (A_r)_{ii}(B_r)_{ii} \end{bmatrix}$$

and

$$M_{ij} := \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & (A_{r-1})_{ii}(B_{r-1})_{jj} & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ -(C_r)_{ii}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}$$

Diagonal blocks: ▲ = 1

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$$\det M_{ii} = \prod_{k=1}^{r}(A_k)_{ii}(B_k)_{ii} - \prod_{k=1}^{r}(C_k)_{ii}(D_k)_{ii}$$

and

$$M_{ij} := \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & (A_{r-1})_{ii}(B_{r-1})_{jj} & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ -(C_r)_{ii}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}$$

$$\det M_{ij} = \prod_{k=1}^{r}(A_k)_{ii}(B_k)_{jj} - \prod_{k=1}^{r}(C_k)_{ii}(D_k)_{jj}$$

Diagonal blocks: $\blacktriangle = \top$

With these orderings, the diagonal blocks are:

$$M_{ij} := \begin{bmatrix} \mathscr{B}_{ij} & -(C_r)_{ii}(D_r)_{jj} e_r e_1^\top \\ -(C_1)_{jj}(D_1)_{ii} e_r e_1^\top & \mathscr{B}_{ji} \end{bmatrix},$$

where

$$\mathscr{B}_{ij} = \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & \ddots & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}.$$

(M_{ii} as for $\blacktriangle = 1$)

Diagonal blocks: $\blacktriangle = \top$

With these orderings, the diagonal blocks are:

$$M_{ij} := \begin{bmatrix} \mathscr{B}_{ij} & -(C_r)_{ii}(D_r)_{jj} e_r e_1^\top \\ -(C_1)_{jj}(D_1)_{ii} e_r e_1^\top & \mathscr{B}_{ji} \end{bmatrix},$$

where

$$\mathscr{B}_{ij} = \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & \ddots & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}.$$

$\det M_{ij} = \prod_{k=1}^{r} (A_k)_{ii}(B_k)_{ii}(A_k)_{jj}(B_k)_{jj} - \prod_{k=1}^{r} (C_k)_{ii}(D_k)_{ii}(C_k)_{jj}(D_k)_{jj}$

(M_{ij} as for $\blacktriangle = 1$)

The pencil approach: idea of the proof

1. $\det \mathscr{Q}(\lambda) = \prod_{i=1}^{n} \left(\lambda^{2r} \prod_{k=1}^{r} (A_k)_{ii} (B_k^\star)_{ii} + \prod_{k=1}^{r} (C_k)_{ii} (D_k^\star)_{ii}\right)$

2. $\boxed{\Lambda(\mathscr{Q}) = \sqrt[2r]{\mathscr{S}}}$, where

$$\mathscr{S} := \left\{ -\prod_{k=1}^{r} \frac{(C_k)_{ii}(D_k^\star)_{ii}}{(A_k)_{ii}(B_k^\star)_{ii}} \; , \quad i = 1, \ldots, n \right\}.$$

The case $\blacktriangle = *$

Lemma

The system

$$(1) \quad \begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1^* D_r = 0. \end{cases}$$

has a unique solution if and only if the system

$$(2) \quad \begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_{r+1} D_r = 0, & \\ B_k^* X_{r+k} A_k^* - D_k^* X_{r+k+1} C_k^* = 0, & k = 1, \ldots, r-1, \\ B_r^* X_{2r} A_r^* - D_r^* X_1 C_r^* = 0 & \end{cases}$$

has a unique solution.

The case $\blacktriangle = *$

Lemma

The system

(1) $$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_1^* D_r = 0. \end{cases}$$

has a unique solution if and only if the system

(2) $$\begin{cases} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_{r+1} D_r = 0, & \\ B_k^* X_{r+k} A_k^* - D_k^* X_{r+k+1} C_k^* = 0, & k = 1, \dots, r-1, \\ B_r^* X_{2r} A_r^* - D_r^* X_1 C_r^* = 0 & \end{cases}$$

has a unique solution.

Proof $(X_1, \dots, X_r) \neq 0$ solution of (1) $\Rightarrow (X_1, \dots, X_r, X_1^*, \dots, X_r^*) \neq 0$ solution of (2). $(X_1, \dots, X_r, X_{r+1}, \dots, X_{2r})$ nonzero solution of (2) $\Rightarrow (X_1 + X_{r+1}^*, \dots, X_r + X_{2r}^*)$ solution of (1). If $(X_1 + X_{r+1}^*, \dots, X_r + X_{2r}^*) = 0$, then $X_{r+i} = -X_i^*$, for $i = 1, \dots, r$, and $\imath(X_1, \dots, X_r)$ is a nonzero solution of (1). $\square$

The case $\blacktriangle = *$

Lemma

The system

(1) $$\left\{ \begin{array}{ll} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1^* D_r = 0. & \end{array} \right.$$

has a unique solution if and only if the system

(2) $$\left\{ \begin{array}{ll} A_k X_k B_k - C_k X_{k+1} D_k = 0, & k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_{r+1} D_r = 0, & \\ B_k^* X_{r+k} A_k^* - D_k^* X_{r+k+1} C_k^* = 0, & k = 1, \ldots, r-1, \\ B_r^* X_{2r} A_r^* - D_r^* X_1 C_r^* = 0 & \end{array} \right.$$

has a unique solution.

Not true for $\top$ instead of $*$!!!

Counterexample: $x_1 + x_1^\top = 2x_1 = 0$ vs $\left\{ \begin{array}{l} z_1 + z_2 = 0 \\ z_1 + z_2 = 0 \end{array} \right.$

The case $\blacktriangle = *$ (ctd.)

The results for $\blacktriangle = *$ follow from the ones for $\blacktriangle = 1$:

$$\boxed{\begin{array}{l} A_k X_k B_k - C_k X_{k+1} D_k = 0, \\ A_r X_r B_r - C_r X_1^* D_r = 0 \end{array}} \text{ unique sol.} \Leftrightarrow \boxed{\begin{array}{c} A_k X_k B_k - C_k X_{k+1} D_k = 0, \\ A_r X_r B_r - C_r X_{r+1} D_r = 0, \\ B_k^* X_{r+k} A_k^* - D_k^* X_{r+k+1} C_k^* = 0, \\ B_r^* X_{2r} A_r^* - D_r^* X_1 C_r^* = 0 \end{array}} \text{ unique sol.}$$

Applying the result for $\blacktriangle = 1$, this is equivalent to:

$$\Pi_1 = D_r^{-*} B_r^* D_{r-1}^{-*} B_{r-1}^* \cdots D_1^{-*} B_1^* C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1$$

and

$$\Pi_2 = C_r^* A_r^{-*} C_{r-1}^* A_{r-1}^{-*} \cdots C_1^* A_1^{-*} D_r B_r^{-1} D_{r-1} B_{r-1}^{-1} \cdots D_1 B_1^{-1}$$

are regular and have no common eigenvalues.

☞ e-vals of Π_1: $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \Rightarrow$ e-vals of Π_2: $\{(\overline{\lambda}_1)^{-1}, (\overline{\lambda}_2)^{-1}, \ldots, (\overline{\lambda}_n)^{-1}\}$, so they are disjoint if and only if $\lambda_i \overline{\lambda}_j \neq 1$.

The case $\blacktriangle = *$ (ctd.)

The results for $\blacktriangle = *$ follow from the ones for $\blacktriangle = 1$:

$$\boxed{\begin{array}{c} A_kX_kB_k - C_kX_{k+1}D_k = 0, \\ A_rX_rB_r - C_rX_1^*D_r = 0 \end{array}} \text{ unique sol.} \Leftrightarrow \boxed{\begin{array}{c} A_kX_kB_k - C_kX_{k+1}D_k = 0, \\ A_rX_rB_r - C_rX_{r+1}D_r = 0, \\ B_k^*X_{r+k}A_k^* - D_k^*X_{r+k+1}C_k^* = 0, \\ B_r^*X_{2r}A_r^* - D_r^*X_1C_r^* = 0 \end{array}} \text{ unique sol.}$$

Applying the result for $\blacktriangle = 1$, this is equivalent to:

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The case $\blacktriangle = *$ (ctd.)

The results for $\blacktriangle = *$ follow from the ones for $\blacktriangle = 1$:

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Outline

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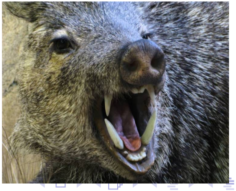

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THANKS FOR YOUR ATTENTION !!!!!