Matrix polynomials with completely prescribed eigenstructure☆

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Abstract

We present necessary and sufficient conditions for the existence of a matrix polynomial when its degree, its finite and infinite elementary divisors, and its left and right minimal indices are prescribed. These conditions hold for arbitrary infinite fields and are determined mainly by the "index sum theorem", which is a fundamental relationship between the rank, the degree, the sum of all partial multiplicities, and the sum of all minimal indices of any matrix polynomial. The proof developed for the existence of such polynomial is constructive and, therefore, solves a very general inverse problem for matrix polynomials with prescribed complete eigenstructure. This result allows us to fix the problem of the existence of ℓ-ifications of a given matrix polynomial, as well as to determine all their possible sizes and eigenstructures.

Keywords: matrix polynomials, index sum theorem, invariant polynomials, ℓ-ifications, minimal indices, inverse polynomial eigenvalue problems

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1. Introduction

Matrix polynomials are essential in the study of dynamical problems described by systems of differential or difference equations with constant coefficient matrices

$$P_d\, \Delta^d x(t) + \cdots + P_1\, \Delta x(t) + P_0\, x(t) = y(t), \tag{1}$$

where $P_i \in \mathbb{F}^{m\times n}$, $\mathbb{F}$ is an arbitrary field, $P_d \neq 0$, and Δ^j denotes the jth differential operator or the jth difference operator, depending on the context. More precisely, the system (1) is associated to the matrix polynomial of degree d

$$P(\lambda) = P_d\lambda^d + \cdots + P_1\lambda + P_0. \tag{2}$$

The importance of matrix polynomials in different applications is widely recognized and is discussed in classic references [13, 17, 28], as well as in more recent surveys [30]. These references consider infinite fields, but it is worth to emphasize that matrix polynomials over finite fields are also of interest in applications like convolutional codes [11].

A matrix polynomial $P(\lambda)$ is *regular* when $P(\lambda)$ is square and the scalar polynomial $\det P(\lambda)$ has at least one nonzero coefficient. Otherwise $P(\lambda)$ is said to be *singular*. When $P(\lambda)$ is regular, the solutions of the system of differential/difference equations (1) depend on the *eigenvalues* and *elementary divisors* of $P(\lambda)$. When $P(\lambda)$ is singular, the solutions of (1) are also determined by the *left* and *right null-spaces* of $P(\lambda)$, which describe, respectively, constraints on the differential or difference equations to have

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compatible solutions, and also degrees of freedom in the solution set (see [34] for the case of polynomials of degree 1). The key concepts of left and right *minimal indices* are related to the null-spaces of $P(\lambda)$ [11, 17] and, so, the elementary divisors, together with the left and right minimal indices are called the complete *eigenstructure* of $P(\lambda)$ [32].

The problem of computing the complete eigenstructure of matrix polynomials has been widely studied since the 1970's (see [32] and the references therein) and, in particular, it has received a lot of attention in the last decade in order to find algorithms which are efficient, are stable, are able to preserve structures that are important in applications, may give reliable information on the complete eigenstructure under perturbations (stratifications), and are able to deal with large scale matrix polynomial problems. This computational activity has motivated, in addition, the revision of many theoretical concepts on matrix polynomials to make them more amenable to computational purposes. To present a complete list of recent references on matrix polynomials is out of the scope of this work. So we just mention here the following small sample that may help the reader to look for many other recent references in this area: [1, 2, 3, 4, 8, 14, 16, 21, 24, 29, 31].

In the context of the problems addressed in this paper, we would like to emphasize just a few aspects of the recent research on matrix polynomials. First, that considerable activity has been devoted to the study of *linearizations* of matrix polynomials, due to the fact that the standard way of computing the eigenstructure of a matrix polynomial $P(\lambda)$ is through the use of linearizations [13, 32]. Linearizations of a matrix polynomial $P(\lambda)$ are matrix polynomials of degree 1 having the same elementary divisors and the same dimension of the right and left null-spaces as $P(\lambda)$ ([7, Lemma 2.3], [9, Theorem 4.1]). Recent research has revealed some drawbacks of linearizations as, for instance, that they cannot preserve the structure for some classes of structured polynomial eigenproblems which are important in applications (see [9, Section 7] and the references therein). This has motivated the introduction of the new notion of ℓ-ification in [9].

An *ℓ-ification* of a matrix polynomial $P(\lambda)$ of degree d is a matrix polynomial $Q(\lambda)$ of degree ℓ such that $P(\lambda)$ and $Q(\lambda)$ have the same elementary divisors and the same dimensions of the left and right null-spaces [9, Theorem 4.1]. An ℓ-ification is *strong* if it additionally has the same elementary divisors at ∞ as $P(\lambda)$. Hence, (strong) linearizations are just (strong) ℓ-ifications with $\ell = 1$. Unlike what happens with strong linearizations, it has been shown that for a fixed value $\ell > 1$ not every matrix polynomial of degree d has a strong ℓ-ification [9, Theorem 7.5]. This poses the problem of the existence of ℓ-ifications, which is clearly related to the inverse problem of characterizing when a list of elementary divisors can be realized by a polynomial of given degree ℓ and given dimensions of its left and right null-spaces.

Inverse polynomial eigenvalue problems of given degree have received attention in the literature since the 70's [23, Theorem 5.2], they have been also considered in classic references [13, Theorem 1.7], and very recently new and strong results have been obtained in [16, Theorem 5.2], [18], and [29, Section 5] (see also [22, Section 9.1]). Among all these references only [16] considers the existence and construction of matrix polynomials of given degree, with given elementary divisors, and *given minimal indices*, although only in the particular case of constructing polynomials with full rank, i.e., with only left or only right minimal indices but not with both. Quadratic inverse matrix polynomial eigenvalue problems have been considered also in [5] and, as a consequence of the study of quasi-canonical forms, in the ongoing works [10, 20].

In this paper, we consider the general inverse polynomial complete eigenstructure problem of given degree and we present necessary and sufficient conditions for the existence of a matrix polynomial of given degree, given finite and infinite elementary divisors, and given left and right minimal indices. These necessary and sufficient conditions are determined mainly by the "index sum theorem", a result discovered in early 90's for real polynomials [25, 27] and recently rediscovered, baptized, and extended to arbitrary fields in [9]. These necessary and sufficient conditions hold for arbitrary infinite fields and the proof of our main result is constructive, assuming that a procedure for constructing minimal bases is available (see [11, Section 4]). Therefore, matrix polynomials with the desired properties can indeed be constructed, although the procedure on which our proof relies is not efficient from a computational point of view. Finally, the solution of this general inverse matrix polynomial eigenstructure problem allows us to completely solve the problem of the existence of ℓ-ifications of a given polynomial $P(\lambda)$, as well as to determine all their possible sizes and eigenstructures.

The paper is organized as follows. In Section 2, we review basic notions. Section 3 states and proves the main results of the paper, which requires to develop some auxiliary lemmas. In Section 4, existence, sizes, and minimal indices of ℓ-ifications are studied. Finally, Section 5 presents the conclusions and some directions of future research.

2. Preliminaries

The most important results in this paper hold for any infinite field $\mathbb{F}$. However, many auxiliary lemmas and definitions are valid for arbitrary fields $\mathbb{F}$ (finite or infinite). Therefore, we adopt the following convention for stating results: if the field is not explicitly mentioned in a certain result, then such result is valid for an arbitrary field. Otherwise, we will indicate explicitly that the field is infinite.

Next, we introduce some basic notation. The *algebraic closure* of the field $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$. By $\mathbb{F}[\lambda]$ we denote the ring of polynomials in the variable λ with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)$ denotes the field of fractions of $\mathbb{F}[\lambda]$. Vectors with entries in $\mathbb{F}[\lambda]$ will be termed as *vector polynomials* and the *degree* of a vector polynomial is the highest degree of all its entries. I_n denotes the $n \times n$ identity matrix and $0_{m\times n}$ the null $m \times n$ matrix.

Throughout the paper, we assume that the leading coefficient P_d of a matrix polynomial $P(\lambda) = \sum_{i=0}^{d} P_i\lambda^i$ is nonzero and then we say that the *degree* of $P(\lambda)$ is d, denoted $\deg(P) = d$. Some references say in this situation that $P(\lambda)$ has "exact degree" d [16], but we will not follow this convention for brevity.

Unimodular matrix polynomials will be often used. They are defined as follows [12].

Definition 2.1. A square matrix polynomial $U(\lambda)$ is said to be *unimodular* if $\det U(\lambda)$ is a nonzero constant.

Note that the inverse of a unimodular matrix polynomial is also a unimodular matrix polynomial and that products of unimodular matrix polynomials are also unimodular. Therefore unimodular matrix polynomials form a transformation group and under this group a unique canonical form of an arbitrary matrix polynomial can be obtained [12].

Definition 2.2. The *Smith normal form* of an $m \times n$ matrix polynomial $P(\lambda)$ is the following diagonal matrix obtained under unimodular transformations $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \left[\begin{array}{cccc|c} p_1(\lambda) & 0 & \dots & 0 & \\ 0 & p_2(\lambda) & \ddots & \vdots & 0_{r\times(n-r)} \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \dots & 0 & p_r(\lambda) & \\ \hline & 0_{(m-r)\times r} & & & 0_{(m-r)\times(n-r)} \end{array}\right], \tag{3}$$

where $p_1(\lambda), \dots, p_r(\lambda)$ are monic scalar polynomials and $p_j(\lambda)$ is a divisor of $p_{j+1}(\lambda)$, for $j = 1, \dots, r-1$. The polynomials $p_j(\lambda)$ are unique and are called the *invariant polynomials* of $P(\lambda)$. An invariant polynomial $p_j(\lambda)$ is *trivial* if $p_j(\lambda) = 1$, otherwise $p_j(\lambda)$ is *non-trivial* (i.e., $\deg(p_j(\lambda)) \geq 1$).

A *finite eigenvalue* of $P(\lambda)$ is a number $\alpha \in \overline{\mathbb{F}}$ such that $p_j(\alpha) = 0$, for some $j = 1, \dots, r$. The *partial multiplicity sequence* of $P(\lambda)$ at the finite eigenvalue α is the sequence

$$0 \leq \delta_1(\alpha) \leq \delta_2(\alpha) \leq \cdots \leq \delta_r(\alpha), \tag{4}$$

such that $p_j(\lambda) = (\lambda - \alpha)^{\delta_j(\alpha)}\, q_j(\lambda)$ with $q_j(\alpha) \neq 0$, for $j = 1, \dots, r$. The *elementary divisors* of $P(\lambda)$ for the finite eigenvalue α are the collection of factors $(\lambda - \alpha)^{\delta_j(\alpha)}$ with $\delta_j(\alpha) > 0$, including repetitions.

The number r in (3) is the *rank* of $P(\lambda)$, which is denoted by $\operatorname{rank}(P)$.

The rank of $P(\lambda)$ is often called its "normal rank", but we will not use the adjective "normal" for brevity. An $m \times n$ matrix polynomial $P(\lambda)$ is said to have *full rank* if $\operatorname{rank}(P) = \min\{m, n\}$. Observe that some of the partial multiplicities at α appearing in (4) may be zero, but that for defining the elementary divisors for α we only consider the partial multiplicities that are positive.

Given a matrix polynomial $P(\lambda)$ over a field $\mathbb{F}$ which is not algebraically closed, its elementary divisors for an eigenvalue α may not be polynomials over $\mathbb{F}$ according to Definition 2.2. To avoid this fact, we define, following [12, Chapter VI], the set of *elementary divisors of* $P(\lambda)$ (note that we do not specify here any eigenvalue) as the set of positive powers of monic irreducible scalar polynomials different from 1 over $\mathbb{F}$ appearing in the decomposition of each invariant polynomial $p_j(\lambda)$ of $P(\lambda)$, $j = 1, \ldots, r$, into irreducible factors over $\mathbb{F}$. So, for instance, if $\mathbb{F} = \mathbb{R}$ then the elementary divisors of $P(\lambda)$ may be positive powers of real scalar polynomials of degree one or of real quadratic scalar polynomials with two complex conjugate roots.

Matrix polynomials may have eigenvalues at infinity, denoted by $\mu = \infty$. There are several different definitions for eigenvalues at infinity (see, for instance, [17, p. 450], where eigenvalues are termed as *zeros*). We use here the definition based on the so-called *reversal* matrix polynomial [21].

Definition 2.3. Let $P(\lambda) = \sum_{j=0}^{d} P_j \lambda^j$ be a matrix polynomial of degree d, the *reversal* matrix polynomial $\mathrm{rev}P(\mu)$ of $P(\lambda)$ is

$$\mathrm{rev}P(\mu) := \mu^d P\left(\frac{1}{\mu}\right) = P_d + P_{d-1}\,\mu + \cdots + P_0\,\mu^d \,. \tag{5}$$

We emphasize that in this paper the reversal is always taken with respect to the degree of the original polynomial. Note that other options are considered in [9, Definition 2.12].

Definition 2.4. We say that $\mu = \infty$ is an eigenvalue of the matrix polynomial $P(\lambda)$ if 0 is an eigenvalue of $\mathrm{rev}P(\mu)$, and the partial multiplicity sequence of $P(\lambda)$ at ∞ is the same as that of the eigenvalue 0 in $\mathrm{rev}P(\mu)$. The elementary divisors μ^{γ_j}, $\gamma_j > 0$, for the eigenvalue $\mu = 0$ of $\mathrm{rev}P(\mu)$ are the *elementary divisors for* ∞ of $P(\lambda)$.

Remark 2.5. It is well known that $\mathrm{rank}(P) = \mathrm{rank}(\mathrm{rev}P)$ (see, for instance, [22, Prop. 3.29]). Therefore, the length of the partial multiplicity sequence of $P(\lambda)$ at ∞ is equal to $\mathrm{rank}(P)$, as for any other eigenvalue of $P(\lambda)$. So, it follows from Definition 2.4 that $P(\lambda)$ is a matrix polynomial having no eigenvalues at ∞ if and only if its highest degree coefficient matrix P_d has rank equal to $\mathrm{rank}(P)$.

The simple Lemma 2.6 will be used to establish certain conditions for the existence of matrix polynomials of given degree d.

Lemma 2.6. *Let $P(\lambda)$ be a matrix polynomial with rank r. Then $P(\lambda)$ has strictly less than r elementary divisors for the eigenvalue ∞.*

Proof. Let $P(\lambda) = \sum_{j=0}^{d} P_j \lambda^j$ with $P_d \neq 0$. Then $\mathrm{rev}P(0) = P_d \neq 0$, and this implies that $\mathrm{rev}P(\lambda)$ has less than r elementary divisors for the eigenvalue 0, since otherwise $\mathrm{rev}P(0) = 0$. □

To avoid confusion with finite eigenvalues, the variable μ is used instead of λ for the infinite eigenvalue and its elementary divisors, and γ_j is used instead of δ_j for the partial multiplicities at ∞.

An $m \times n$ matrix polynomial $P(\lambda)$ whose rank r is smaller than m and/or n has non-trivial *left* and/or *right null-spaces* over the field $\mathbb{F}(\lambda)$:

$$\begin{aligned}
\mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\}, \\
\mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda)x(\lambda) \equiv 0 \right\}.
\end{aligned}$$

It is well known that every subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$ has bases consisting entirely of vector polynomials. In order to define the singular structure of $P(\lambda)$, we need to introduce *minimal bases* of $\mathcal{V}$.

Definition 2.7. Let $\mathcal{V}$ be a subspace of $\mathbb{F}(\lambda)^n$. A *minimal basis* of $\mathcal{V}$ is a basis of $\mathcal{V}$ consisting of vector polynomials whose sum of degrees is minimal among all bases of $\mathcal{V}$ consisting of vector polynomials.

It can be seen [11, 17] that the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same. These degrees are then called the *minimal indices* of $\mathcal{V}$. This leads to the definition of the minimal indices of a matrix polynomial.

Definition 2.8. Let $P(\lambda)$ be an $m \times n$ singular matrix polynomial with rank r over a field $\mathbb{F}$, and let the sets $\{y_1(\lambda)^T, \ldots, y_{m-r}(\lambda)^T\}$ and $\{x_1(\lambda), \ldots, x_{n-r}(\lambda)\}$ be minimal bases of $\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$, respectively, ordered so that $0 \leq \deg(y_1) \leq \cdots \leq \deg(y_{m-r})$ and $0 \leq \deg(x_1) \leq \cdots \leq \deg(x_{n-r})$. Let $\eta_i = \deg(y_i)$ for $i = 1, \ldots, m-r$ and $\varepsilon_j = \deg(x_j)$ for $j = 1, \ldots, n-r$. Then the scalars $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_{m-r}$ and $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_{n-r}$ are, respectively, the *left* and *right minimal indices of* $P(\lambda)$.

In order to give a practical characterization of minimal bases, we introduce Definition 2.9. In the following, when referring to the *column* (resp., *row*) *degrees* $d_1, \ldots, d_n$ (resp., $d'_1, \ldots, d'_m$) of an $m \times n$ matrix polynomial $P(\lambda)$, we mean that d_j (resp., d'_j) is the degree of the jth column (resp., row) of $P(\lambda)$.

Definition 2.9. Let $N(\lambda)$ be an $n \times r$ matrix polynomial with column degrees $d_1, \ldots, d_r$. The *highest-column-degree coefficient matrix* of $N(\lambda)$, denoted by N_{hc}, is the $n \times r$ constant matrix whose jth column is the coefficient of λ^{d_j} in the jth column of $N(\lambda)$. $N(\lambda)$ is said to be *column reduced* if N_{hc} has full column rank.

Similarly, let $M(\lambda)$ be an $r \times n$ matrix polynomial with row degrees $d_1, \ldots, d_r$. The *highest-row-degree coefficient matrix* of $M(\lambda)$, denoted by M_{hr}, is the $r \times n$ constant matrix whose jth row is the coefficient of λ^{d_j} in the jth row of $M(\lambda)$. $M(\lambda)$ is said to be *row reduced* if M_{hr} has full row rank.

Remark 2.10. Note that the rank of an $n \times r$ column reduced matrix polynomial $N(\lambda)$ is r [17, Chapter 6]. So, column reduced matrix polynomials have full column rank. A similar observation holds for row reduced matrix polynomials and full row rank.

Theorem 2.11 provides a characterization of those matrix polynomials whose columns or rows are minimal bases of the subspaces they span. Theorem 2.11 is a variation of [11, Main Theorem (2), p. 495], which we think is easier to use in practice when $\mathbb{F}$ is not algebraically closed. In [17, Theorem 6.5-10], we can find the corresponding result stated only for the complex field, but the proof remains valid for any algebraically closed field.

Theorem 2.11. *The columns (resp., rows) of a matrix polynomial $N(\lambda)$ over a field $\mathbb{F}$ are a minimal basis of the subspace they span if and only if $N(\lambda_0)$ has full column (resp., row) rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $N(\lambda)$ is column (resp., row) reduced.*

Proof. We only prove the result for columns, since for rows the proof is similar. Recall that $\mathbb{F} \subseteq \overline{\mathbb{F}}$, so $\overline{\mathbb{F}}$ is a field extension of $\mathbb{F}$. Let $n \times r$ be the size of $N(\lambda)$. Let $\mathrm{col}_{\mathbb{F}}(N)$ be the subspace of $\mathbb{F}(\lambda)^n$ spanned by the columns of $N(\lambda)$ over the field $\mathbb{F}(\lambda)$ and let $\mathrm{col}_{\overline{\mathbb{F}}}(N)$ be the subspace of $\overline{\mathbb{F}}(\lambda)^n$ spanned by the columns of $N(\lambda)$ over the field $\overline{\mathbb{F}}(\lambda)$.

We prove first that the columns of $N(\lambda)$ are a minimal basis of $\mathrm{col}_{\mathbb{F}}(N)$ if and only if the columns of $N(\lambda)$ are a minimal basis of $\mathrm{col}_{\overline{\mathbb{F}}}(N)$. It is immediate to see that the columns of $N(\lambda)$ are a basis of $\mathrm{col}_{\mathbb{F}}(N)$ if and only if the columns of $N(\lambda)$ are a basis of $\mathrm{col}_{\overline{\mathbb{F}}}(N)$, since the columns of $N(\lambda)$ are linearly independent over $\mathbb{F}(\lambda)$ if and only if they are linearly independent over $\overline{\mathbb{F}}(\lambda)$, because linear independence is equivalent to the existence of a nonzero $r \times r$ minor. Now assume that the columns of $N(\lambda)$ are a minimal basis of $\mathrm{col}_{\mathbb{F}}(N)$, then the columns of $N(\lambda)$ are a basis of $\mathrm{col}_{\overline{\mathbb{F}}}(N)$ and this basis has to be also minimal, because otherwise there would be a polynomials basis of $\mathrm{col}_{\overline{\mathbb{F}}}(N)$ with the sum of the degrees of its vectors smaller than the sum of the degrees of the columns of $N(\lambda)$. But this is a contradiction, since the minimal indices of $\mathrm{col}_{\overline{\mathbb{F}}}(N)$ are equal to those of $\mathrm{col}_{\mathbb{F}}(N)$ by [19, Theorem 5.1][1]. An analogous argument proves that if the columns of $N(\lambda)$ are a minimal basis of $\mathrm{col}_{\overline{\mathbb{F}}}(N)$, then they are a minimal basis of $\mathrm{col}_{\mathbb{F}}(N)$.

Theorem 2.11 now follows from [11, Main Theorem (2), p. 495] or [17, Theorem 6.5-10] applied to the columns of $N(\lambda)$ viewed as a basis of $\mathrm{col}_{\overline{\mathbb{F}}}(N)$. □

In this paper, for brevity, we often say that a $p \times q$ matrix polynomial $N(\lambda)$ with $p \geq q$ (resp., $p \leq q$) is a *minimal basis* if the columns (resp., rows) of $N(\lambda)$ are a minimal basis of the subspace they span.

Lemma 2.12 gathers some simple properties that will be used in the sequel.

[1]Theorem 5.1 in [19] is stated for right null-spaces of matrix polynomials, but note that $\mathrm{col}_{\mathbb{F}}(N)$ and $\mathrm{col}_{\overline{\mathbb{F}}}(N)$ can be seen as right null-spaces of any matrix polynomial with coefficients in $\mathbb{F}$ whose rows are any polynomial basis of the left null-space (over $\mathbb{F}$) of $N(\lambda)$.

Lemma 2.12. *Let $N(\lambda)$ be a matrix polynomial over a field $\mathbb{F}$. Then:*

(a) *If the columns of $N(\lambda)$ form a minimal basis and R is a nonsingular constant matrix, then the columns of $RN(\lambda)$ form a minimal basis with the same column degrees as $N(\lambda)$.*

(b) *If $N(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$, then there exists a matrix polynomial $Z(\lambda)$ such that $\begin{bmatrix} N(\lambda) & Z(\lambda) \end{bmatrix}$ is unimodular.*

(c) *If $N(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $P(\lambda)$ is any other matrix polynomial such that the product $N(\lambda)P(\lambda)$ is defined, then the invariant polynomials of $P(\lambda)$ and $N(\lambda)P(\lambda)$ are identical.*

Results analogous to those in (a), (b), *and* (c) *hold for rows.*

Proof. Part (a) follows from Theorem 2.11 and the observation that RN_{hc} is the highest-column-degree coefficient matrix of $RN(\lambda)$.

Part (b). Let $n \times r$, $n \geq r$, be the size of $N(\lambda)$, and observe that the Smith form of $N(\lambda)$ is the constant matrix $D = [I_r \ 0]^T \in \mathbb{F}^{n\times r}$. Therefore, by (3), $N(\lambda) = U(\lambda)DV(\lambda)$, with $U(\lambda)$ and $V(\lambda)$ unimodular matrix polynomials, and $N(\lambda) = U(\lambda)\mathrm{diag}(V(\lambda), I_{n-r})D$. Finally, define $\begin{bmatrix} N(\lambda) & Z(\lambda) \end{bmatrix} := U(\lambda)\mathrm{diag}(V(\lambda), I_{n-r})$ and note that this matrix is unimodular.

Part (c). Assume that $N(\lambda) \in \mathbb{F}^{n\times r}$ and $P(\lambda) \in \mathbb{F}^{r\times t}$, and let $P(\lambda) = Q(\lambda)D(\lambda)Y(\lambda)$ be such that $D(\lambda)$ is the Smith form of $P(\lambda)$, and $Q(\lambda)$ and $Y(\lambda)$ are unimodular. Then, by part (b),

$$N(\lambda)P(\lambda) = \begin{bmatrix} N(\lambda) & Z(\lambda) \end{bmatrix} \begin{bmatrix} Q(\lambda) & \\ & I_{n-r} \end{bmatrix} \begin{bmatrix} D(\lambda) \\ 0_{(n-r)\times t} \end{bmatrix} Y(\lambda),$$

which proves the result, since $\begin{bmatrix} N(\lambda) & Z(\lambda) \end{bmatrix} \mathrm{diag}(Q(\lambda), I_{n-r})$ is unimodular. □

We close this section by introducing the eigenstructure of a matrix polynomial.

Definition 2.13. Given an $m \times n$ matrix polynomial $P(\lambda)$ with rank r, the *eigenstructure* of $P(\lambda)$ consists of the following lists of scalar polynomials and non-negative integers:

(i) the invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$, with degrees $\delta_1, \ldots, \delta_r$ (finite structure),

(ii) the partial multiplicity sequence at ∞, $\gamma_1, \ldots, \gamma_r$ (infinite structure),

(iii) the right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$ (right singular structure), and

(iv) the left minimal indices $\eta_1, \ldots, \eta_{m-r}$ (left singular structure).

We emphasize that some of the integers in Definition 2.13 can be zero and/or can be repeated. Even all integers in some of the lists (i)–(iv) can be zero. In some recent references (see [9]) the eigenstructure of $P(\lambda)$ is defined to consist only of the lists in parts (i) and (ii) of Definition 2.13, while the name "singular structure" is used for the right and left minimal indices. We adopt here the shorter and classical terminology used for instance in [32].

3. Matrix polynomials of given degree with prescribed eigenstructure

We first recall the Index Sum Theorem. This theorem establishes a simple relationship between the eigenstructure, the degree, and the rank of any matrix polynomial with coefficients over an arbitrary field. Up to our knowledge, it was first stated in [25, 27] for matrix polynomials over the real field, and recently extended in [9] to matrix polynomials with coefficients over arbitrary fields.

Theorem 3.1. (Index Sum Theorem). *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree d and rank r having the following regular and singular eigenstructure:*

- *r invariant polynomials $p_j(\lambda)$ of degrees δ_j, for $j = 1, \ldots, r$,*
- *r infinite partial multiplicities $\gamma_1, \ldots, \gamma_r$,*

- *$n-r$ right minimal indices $\varepsilon_1, \dots, \varepsilon_{n-r}$, and*
- *$m-r$ left minimal indices $\eta_1, \dots, \eta_{m-r}$,*

where some of the degrees, partial multiplicities or indices can be zero, and/or one or both of the lists of minimal indices can be empty. Then

$$\sum_{j=1}^{r} \delta_j + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr. \tag{6}$$

The proof of Theorem 3.1 in [9] uses a particular linearization of $P(\lambda)$, together with the corresponding result for pencils. For matrix pencils (i.e., when $d = 1$), Theorem 3.1 follows from the Kronecker Canonical Form (KCF) [12] and using this canonical form, one can show also that there exist $m \times n$ matrix pencils with given rank r having all possible regular and singular eigenstructures satisfying the constraint (6). However, a similar result for matrix polynomials of degree $d > 1$ is not available in the literature and it is not obvious to prove it, since there is no anything similar to the KCF for polynomials of degree larger than one[2]. To establish this result is the main contribution of the present paper. This is presented in Theorem 3.2.

Theorem 3.2. *Let m, n, d, and $r \leq \min\{m, n\}$ be given positive integers. Let $p_1(\lambda), \dots, p_r(\lambda)$ be r arbitrary monic polynomials with coefficients in an infinite field $\mathbb{F}$ and with respective degrees $\delta_1, \dots, \delta_r$, and such that $p_j(\lambda)$ divides $p_{j+1}(\lambda)$ for $j = 1, \dots, r-1$. Let $0 \leq \gamma_1 \leq \cdots \leq \gamma_r$, $0 \leq \varepsilon_1 \leq \cdots \leq \varepsilon_{n-r}$, and $0 \leq \eta_1 \leq \cdots \leq \eta_{m-r}$ be given lists of nonnegative integers. Then, there exists an $m \times n$ matrix polynomial $P(\lambda)$ with coefficients in $\mathbb{F}$, with rank r, with degree d, with invariant polynomials $p_1(\lambda), \dots, p_r(\lambda)$, with partial multiplicities at ∞ equal to $\gamma_1, \dots, \gamma_r$, and with right and left minimal indices respectively equal to $\varepsilon_1, \dots \varepsilon_{n-r}$ and $\eta_1, \dots, \eta_{m-r}$ if and only if (6) holds and $\gamma_1 = 0$.*

The necessity of (6) and $\gamma_1 = 0$ in Theorem 3.2 follows immediately from the Index Sum Theorem and Lemma 2.6. However, the proof of the sufficiency of these conditions is nontrivial and requires a number of auxiliary lemmas, which will be presented in Subsection 3.1. This proof is based on techniques that were used in [16, Theorem 5.2] to prove the particular case of Theorem 3.2 in which $P(\lambda)$ has full row or full column rank. We will see that to prove the general case demands considerable more effort.

In Subsection 3.2, we present an alternative statement of Theorem 3.2 which is more convenient for proving the results on ℓ-ifications included in Section 4.

3.1. Auxiliary lemmas and proof of Theorem 3.2

Lemma 3.3 will allow us to reduce the proof of Theorem 3.2 to the case when all partial multiplicities at ∞ are zero, i.e., when there are no infinite eigenvalues. Lemma 3.3 is a particular case of [22, Proposition 3.29, Theorems 5.3 and 7.5] or [26, Theorem 4.1]. The statement of Lemma 3.3 is not usual, but it is stated exactly as it will be used in the proof of Theorem 3.2.

Lemma 3.3. *Let $\mathbb{F}$ be an infinite field, let $\lambda_1, \dots, \lambda_s \in \overline{\mathbb{F}}$, with $\lambda_i \neq \lambda_j$ if $i \neq j$, and let $\omega \in \mathbb{F}$ be such that $\omega \neq \lambda_i$, for $i = 1, \dots, s$. Let $P_\omega(\lambda)$ be an $m \times n$ matrix polynomial with coefficients in $\mathbb{F}$, of degree d, and rank r. Suppose that $P_\omega(\lambda)$ has not infinite eigenvalues, that $1/(\lambda_1 - \omega), \dots, 1/(\lambda_s - \omega)$ are the nonzero finite eigenvalues of $P_\omega(\lambda)$, and that $\lambda_0 = 0$ is an eigenvalue of $P_\omega(\lambda)$ with less than r associated elementary divisors. Let us define the following $m \times n$ matrix polynomial over $\mathbb{F}$*

$$P(\lambda) := (\lambda - \omega)^d \, P_\omega \left(\frac{1}{\lambda - \omega} \right). \tag{7}$$

Then:

(a) *$P(\lambda)$ has degree d and rank r.*

[2] It is interesting to mention here the significative advances on Kronecker-like quasicanonical forms for quadratic matrix polynomials obtained in [20].

(b) *$\mu_0 = \infty$ is an eigenvalue of $P(\lambda)$ with partial multiplicity sequence equal to the partial multiplicity sequence of $\lambda_0 = 0$ in $P_\omega(\lambda)$.*

(c) *The finite eigenvalues of $P(\lambda)$ are $\lambda_1, \ldots, \lambda_s$ and, for each $j = 1, \ldots, s$, the partial multiplicity sequence of λ_j is equal to the partial multiplicity sequence of $1/(\lambda_j - \omega)$ in $P_\omega(\lambda)$.*

(d) *The minimal indices of $P(\lambda)$ and $P_\omega(\lambda)$ are identical.*

Proof. As a consequence of the results in [22, 26] mentioned above, we only need to prove that $P(\lambda)$ has the same degree as $P_\omega(\lambda)$, since Möbius transformations of matrix polynomials do not preserve the degree in general [22, p. 6]. In our case, if $P_\omega(\lambda) = Q_0 + \lambda Q_1 + \cdots + \lambda^d Q_d$, with $Q_d \neq 0$, then $Q_0 \neq 0$ because $P_\omega(\lambda)$ has less than r elementary divisors at 0. Since

$$P(\lambda) = (\lambda - \omega)^d Q_0 + (\lambda - \omega)^{d-1} Q_1 + \cdots + Q_d,$$

we obtain that $P(\lambda)$ has degree d. □

Remark 3.4. Lemma 3.3 is only valid for infinite fields and this is the reason why Theorem 3.2 is only valid for infinite fields as well. In the proof of Theorem 3.2, for proving the existence of a polynomial $P(\lambda)$ with infinite eigenvalues and prescribed complete eigenstructure, in particular with prescribed finite eigenvalues $\lambda_1, \ldots, \lambda_s$, we will prove instead the existence of a polynomial $P_\omega(\lambda)$ as the one in Lemma 3.3 and then the desired $P(\lambda)$ will be given by (7). The problem is that if $\mathbb{F}$ is finite, then $\lambda_1, \ldots, \lambda_s$ might be all the elements in the field and, in this case, we cannot choose ω as in Lemma 3.3.

Lemma 3.5 will be also used in the proof of Theorem 3.2, more precisely in the proof of the key Lemma 3.7. Lemma 3.5(a) follows from techniques in [17, Chapter 6] and Lemma 3.5(b) is in [11, p. 503]. We include a proof for completeness.

Lemma 3.5. *Let $M(\lambda)$ and $N(\lambda)$ be matrix polynomials of sizes $n \times r$ and $n \times (n-r)$, respectively, such that*

$$M(\lambda)^T N(\lambda) = 0, \tag{8}$$

and let M_{hc} and N_{hc} be the highest-column-degree coefficient matrices of $M(\lambda)$ and $N(\lambda)$, respectively. Then:

(a) *If $M(\lambda)$ and $N(\lambda)$ are both column reduced, then there exists a nonsingular $n \times n$ constant matrix R such that*

$$M_{hc}^T R^{-1} = \begin{bmatrix} 0_{r\times(n-r)} & I_r \end{bmatrix} \qquad \text{and} \qquad R N_{hc} = \begin{bmatrix} I_{n-r} \\ 0_{r\times(n-r)} \end{bmatrix}.$$

(b) *If $M(\lambda)$ and $N(\lambda)$ are both minimal bases with column degrees $\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_r$ and $\varepsilon_1, \ldots, \varepsilon_{n-r}$, respectively, then*

$$\sum_{i=1}^{r} \widehat{\varepsilon}_i = \sum_{j=1}^{n-r} \varepsilon_j.$$

Proof. (a) Let $\varepsilon_1, \ldots, \varepsilon_{n-r}$ be the degrees of the columns of $N(\lambda)$ and $\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_r$ be the degrees of the columns of $M(\lambda)$. Define $D_\varepsilon(\lambda) := \operatorname{diag}(\lambda^{\varepsilon_1}, \ldots, \lambda^{\varepsilon_{n-r}})$ and $\widehat{D}_\varepsilon(\lambda) := \operatorname{diag}(\lambda^{\widehat{\varepsilon}_1}, \ldots, \lambda^{\widehat{\varepsilon}_r})$. Then, we can write

$$M(\lambda) = M_{hc}\, \widehat{D}_\varepsilon(\lambda) + \widetilde{M}(\lambda) \qquad \text{and} \qquad N(\lambda) = N_{hc}\, D_\varepsilon(\lambda) + \widetilde{N}(\lambda). \tag{9}$$

Observe that

$$\deg(\operatorname{col}_j(\widetilde{M}(\lambda))) < \widehat{\varepsilon}_j, \text{ for } j = 1, \ldots, r, \quad \text{and} \quad \deg(\operatorname{col}_j(\widetilde{N}(\lambda))) < \varepsilon_j, \text{ for } j = 1, \ldots, n-r. \tag{10}$$

Since N_{hc} has full column rank, there exists a constant nonsingular $n \times n$ matrix R_1 such that

$$R_1\, N_{hc} = \begin{bmatrix} I_{n-r} \\ 0_{r\times(n-r)} \end{bmatrix}, \quad \text{so} \quad R_1\, N(\lambda) = \begin{bmatrix} I_{n-r} \\ 0_{r\times(n-r)} \end{bmatrix} D_\varepsilon(\lambda) + R_1 \widetilde{N}(\lambda).$$

Therefore, we have $M(\lambda)^T R_1^{-1} R_1 N(\lambda) = 0$, from (8), which, combined with (9), implies

$$\widehat{D}_\varepsilon(\lambda) M_{hc}^T R_1^{-1} \begin{bmatrix} I_{n-r} \\ 0_{r\times(n-r)} \end{bmatrix} D_\varepsilon(\lambda) + \widehat{D}_\varepsilon(\lambda) M_{hc}^T \widetilde{N}(\lambda) + \widetilde{M}(\lambda)^T N_{hc} D_\varepsilon(\lambda) + \widetilde{M}(\lambda)^T \widetilde{N}(\lambda) = 0\,. \tag{11}$$

The right and left-hand sides of (11) are $r\times(n-r)$ matrix polynomials and the (i,j) entry of (11) is

$$\left(M_{hc}^T R_1^{-1} \begin{bmatrix} I_{n-r} \\ 0_{r\times(n-r)} \end{bmatrix} \right)_{ij} \lambda^{\widehat{\varepsilon}_i + \varepsilon_j} + h_{ij}(\lambda) = 0,$$

where $\deg(h_{ij}(\lambda)) < \widehat{\varepsilon}_i + \varepsilon_j$ by (10), for $i = 1, \dots, r$ and $j = 1, \dots, n-r$. Therefore,

$$M_{hc}^T R_1^{-1} \begin{bmatrix} I_{n-r} \\ 0_{r\times(n-r)} \end{bmatrix} = 0, \qquad \text{which implies} \qquad M_{hc}^T R_1^{-1} = \begin{bmatrix} 0_{r\times(n-r)} & X \end{bmatrix}, \tag{12}$$

where X is an $r\times r$ nonsingular matrix, since M_{hc} has full column rank. Finally, define the $n\times n$ matrix $R_2 = \operatorname{diag}(I_{n-r}, X)$, note that

$$M_{hc}^T R_1^{-1} R_2^{-1} = \begin{bmatrix} 0_{r\times(n-r)} & I_r \end{bmatrix} \qquad \text{and} \qquad R_2 R_1 N_{hc} = \begin{bmatrix} I_{n-r} \\ 0_{r\times(n-r)} \end{bmatrix},$$

and take $R = R_2 R_1$.

(b) Let us partition $M(\lambda)^T = [M_1(\lambda)^T\,,\, M_2(\lambda)^T]$, with $M_2(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$, and $N(\lambda) = [N_1(\lambda)^T\,,\, N_2(\lambda)^T]^T$, with $N_1(\lambda) \in \mathbb{F}[\lambda]^{(n-r)\times(n-r)}$. Theorem 2.11 and Lemma 2.12(b) guarantee that $M(\lambda)^T$ can be completed to a unimodular matrix

$$\begin{bmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ M_1(\lambda)^T & M_2(\lambda)^T \end{bmatrix} \in \mathbb{F}[\lambda]^{n\times n}.$$

Note that

$$\begin{bmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ M_1(\lambda)^T & M_2(\lambda)^T \end{bmatrix} \begin{bmatrix} N_1(\lambda) \\ N_2(\lambda) \end{bmatrix} = \begin{bmatrix} Z(\lambda) \\ 0_{r\times(n-r)} \end{bmatrix}. \tag{13}$$

From Theorem 2.11, we get that $[I_{n-r}\,,\, 0_{(n-r)\times r}]^T$ is the Smith normal form of $N(\lambda)$, which implies that the matrix $Z(\lambda) \in \mathbb{F}[\lambda]^{(n-r)\times(n-r)}$ in (13) is unimodular. Let $R \in \mathbb{F}^{n\times n}$ be a nonsingular constant matrix as in part (a) and define

$$\begin{bmatrix} \widehat{U}_{11}(\lambda) & \widehat{U}_{12}(\lambda) \\ \widehat{M}_1(\lambda)^T & \widehat{M}_2(\lambda)^T \end{bmatrix} := \begin{bmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ M_1(\lambda)^T & M_2(\lambda)^T \end{bmatrix} R^{-1} \qquad \text{and} \qquad \begin{bmatrix} \widehat{N}_1(\lambda) \\ \widehat{N}_2(\lambda) \end{bmatrix} := R \begin{bmatrix} N_1(\lambda) \\ N_2(\lambda) \end{bmatrix}, \tag{14}$$

where, as a consequence of part (a), both $\widehat{M}_2(\lambda)$ and $\widehat{N}_1(\lambda)$ are square nonsingular matrix polynomials with highest-column-degree coefficient matrices equal to I_r and I_{n-r} respectively. Next consider the matrix polynomial

$$V(\lambda) := \begin{bmatrix} V_{11}(\lambda) & V_{12}(\lambda) \\ V_{21}(\lambda) & V_{22}(\lambda) \end{bmatrix} := \begin{bmatrix} \widehat{U}_{11}(\lambda) & \widehat{U}_{12}(\lambda) \\ \widehat{M}_1(\lambda)^T & \widehat{M}_2(\lambda)^T \end{bmatrix}^{-1}, \tag{15}$$

and combine this definition with (13) and (14) to get

$$\begin{bmatrix} \widehat{N}_1(\lambda) \\ \widehat{N}_2(\lambda) \end{bmatrix} = \begin{bmatrix} V_{11}(\lambda) & V_{12}(\lambda) \\ V_{21}(\lambda) & V_{22}(\lambda) \end{bmatrix} \begin{bmatrix} Z(\lambda) \\ 0 \end{bmatrix},$$

which implies $\widehat{N}_1(\lambda) = V_{11}(\lambda) Z(\lambda)$ and, since $Z(\lambda)$ is unimodular,

$$\deg(\det(\widehat{N}_1(\lambda))) = \deg(\det(V_{11}(\lambda))). \tag{16}$$

Next, since the matrices in (15) are unimodular, a standard property of minors of inverses [15, p. 21] implies that $\det(V_{11}(\lambda))/\det(\widehat{M}_2(\lambda)) = \det(V(\lambda))$ is a nonzero constant, which combined with (16) yields

$$\deg(\det(\widehat{M}_2(\lambda))) = \deg(\det(\widehat{N}_1(\lambda))).$$

Since the highest-column-degree coefficient matrices of $\widehat{M}_2(\lambda)$ and $\widehat{N}_1(\lambda)$ are I_r and I_{n-r}, respectively, we get, taking into account (14), that

$$\sum_{i=1}^{r} \widehat{\varepsilon}_i = \deg(\det(\widehat{M}_2(\lambda))) = \deg(\det(\widehat{N}_1(\lambda))) = \sum_{j=1}^{n-r} \varepsilon_j .$$

□

Lemma 3.6 is a technical result needed to prove Lemma 3.7.

Lemma 3.6. *Let k and d be nonnegative integers such that $k - d > 1$. Let $Q(\lambda)$ be a 2×2 matrix polynomial*

$$Q(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & e(\lambda) \end{bmatrix},$$

such that

$$\deg(a(\lambda)) = d, \quad \deg(b(\lambda)) \le d-1, \quad \deg(c(\lambda)) \le k-1, \quad \deg(e(\lambda)) = k,$$

and $a(\lambda)$ and $e(\lambda)$ are monic. Let $c(\lambda) = c_{k-1}\lambda^{k-1} + c_{k-2}\lambda^{k-2} + \cdots + c_0$ and

$$\begin{bmatrix} \widetilde{a}(\lambda) & \widetilde{b}(\lambda) \\ \widetilde{c}(\lambda) & \widetilde{e}(\lambda) \end{bmatrix} := \begin{bmatrix} 1 & 0 \\ (c_{k-1}+1)\,\lambda^{k-d-1} & -1 \end{bmatrix} Q(\lambda) \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.$$

Then,

$$\deg(\widetilde{a}(\lambda)) = d, \quad \deg(\widetilde{b}(\lambda)) = d+1, \quad \deg(\widetilde{c}(\lambda)) = k-1, \quad \deg(\widetilde{e}(\lambda)) \le k-1,$$

and $\widetilde{a}(\lambda)$, $\widetilde{b}(\lambda)$, and $\widetilde{c}(\lambda)$ are monic.

Proof. A direct computation yields

$$\begin{bmatrix} 1 & 0 \\ (c_{k-1}+1)\,\lambda^{k-d-1} & -1 \end{bmatrix} Q(\lambda) \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$
$$= \left[\begin{array}{c|c} a(\lambda) & \lambda a(\lambda) + b(\lambda) \\ \hline (c_{k-1}+1)\lambda^{k-d-1} a(\lambda) - c(\lambda) & (c_{k-1}+1)\lambda^{k-d-1}(\lambda a(\lambda) + b(\lambda)) - (\lambda c(\lambda) + e(\lambda)) \end{array} \right].$$

The result follows immediately from this expression. □

Lemma 3.7 shows that given any lists of invariant polynomials and right minimal indices there exists always a row reduced matrix polynomial $P(\lambda)$ with the degrees of its rows "as close as possible" and having precisely these invariant polynomials and right minimal indices. Observe that such $P(\lambda)$ has full row rank, i.e., it has not left minimal indices, since it is row reduced, and that if its row degrees were all equal, then $P(\lambda)$ would not have infinite eigenvalues. Therefore, by (6), the row degrees of $P(\lambda)$ can be all equal only if the sum of the degrees of the given invariant polynomials plus the given right minimal indices is a multiple of the number of invariant polynomials. Lemma 3.7 is the key piece in the proof of Theorem 3.2 and its proof is rather technical.

Lemma 3.7. *Let r, n be two positive integers with $r \le n$. Let $p_1(\lambda), \ldots, p_r(\lambda)$ be r monic scalar polynomials, with respective degrees $\delta_1 \le \cdots \le \delta_r$ and such that $p_j(\lambda)$ is a divisor of $p_{j+1}(\lambda)$ for $j = 1, \ldots, r-1$. Let $\varepsilon_1 \le \cdots \le \varepsilon_{n-r}$ be a list of $n-r$ nonnegative integers (which is empty if $n = r$). Define*

$$\delta := \sum_{j=1}^{r} \delta_j \qquad \text{and} \qquad \varepsilon := \sum_{j=1}^{n-r} \varepsilon_j ,$$

and write

$$\delta + \varepsilon = r q_\varepsilon + t_\varepsilon, \quad \text{where} \quad 0 \le t_\varepsilon < r.$$

Then there exists an $r \times n$ row reduced matrix polynomial $P(\lambda)$ with t_ε row degrees equal to $q_\varepsilon + 1$ and $r - t_\varepsilon$ row degrees equal to q_ε, and such that $p_1(\lambda), \ldots, p_r(\lambda)$ and $\varepsilon_1, \ldots, \varepsilon_{n-r}$ are, respectively, the invariant polynomials and the right minimal indices of $P(\lambda)$. The degree of $P(\lambda)$ is thus $q_\varepsilon + 1$ if $t_\varepsilon > 0$ and q_ε otherwise.

Proof. The proof follows closely the proof of Theorem 5.2 in [16]. We prove first in detail the most difficult case $r < n$ and sketch at the end the proof of the much easier case $r = n$. Let $\widetilde{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(n-r)}$ be any (right) minimal basis with column degrees $\varepsilon_1 \leq \cdots \leq \varepsilon_{n-r}$. An specific way to construct such $\widetilde{N}(\lambda)$ is to place on the main diagonal $\lambda^{\varepsilon_1}, \ldots, \lambda^{\varepsilon_{n-r}}$ and on the first subdiagonal $1, \ldots, 1$. Now, let $\widetilde{Q}(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ be any minimal basis of $\mathcal{N}_\ell(\widetilde{N})$ and let $\widehat{\varepsilon}_1 \leq \cdots \leq \widehat{\varepsilon}_r$ be its row degrees. So, $\widetilde{Q}(\lambda)\widetilde{N}(\lambda) = 0$ and according to Lemma 3.5(b),

$$\varepsilon = \sum_{i=1}^{r} \widehat{\varepsilon}_i = \sum_{j=1}^{n-r} \varepsilon_j \,. \tag{17}$$

Next, define $\widehat{Q}(\lambda) := \operatorname{diag}(p_1(\lambda), \ldots, p_r(\lambda))\, \widetilde{Q}(\lambda)$. Observe that $\widehat{Q}(\lambda)$ is row reduced, since $\widetilde{Q}(\lambda)$ is and $\widehat{Q}_{hr} = \widetilde{Q}_{hr}$, i.e., the highest-row-degree coefficient matrices are equal, $\widehat{Q}(\lambda)\widetilde{N}(\lambda) = 0$, the row degrees of $\widehat{Q}(\lambda)$ are $\widehat{\varepsilon}_1 + \delta_1 \leq \cdots \leq \widehat{\varepsilon}_r + \delta_r$, and $p_1(\lambda), \ldots, p_r(\lambda)$ are the invariant polynomials of $\widehat{Q}(\lambda)$, by Lemma 2.12(c) (for rows). Therefore, we can apply Lemma 3.5(a) to $\widehat{Q}(\lambda)\widetilde{N}(\lambda) = 0$ and define the following two polynomial matrices

$$Q(\lambda) = \widehat{Q}(\lambda) R^{-1} \in \mathbb{F}[\lambda]^{r\times n} \qquad \text{and} \qquad N(\lambda) = R\widetilde{N}(\lambda) \in \mathbb{F}[\lambda]^{n\times(n-r)}. \tag{18}$$

From the discussion above, it is easy to check that $Q(\lambda)$ and $N(\lambda)$ satisfy:

(i) $Q(\lambda)N(\lambda) = 0$,
(ii) $Q(\lambda)$ is row reduced and $N(\lambda)$ is a minimal basis of $\mathcal{N}_r(Q)$,
(iii) $Q_{hr} = \begin{bmatrix} 0_{r\times(n-r)} & I_r \end{bmatrix}$ and $N_{hc} = \begin{bmatrix} I_{n-r} & 0_{(n-r)\times r} \end{bmatrix}^T$,
(iv) $Q(\lambda)$ has invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$ and right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$,
(v) the sum of the row degrees of $Q(\lambda)$ is $\delta + \varepsilon$.

In addition, the matrix polynomial $Q(\lambda)$ has row degrees $\widehat{\varepsilon}_1 + \delta_1 \leq \cdots \leq \widehat{\varepsilon}_r + \delta_r$, which are denoted by

$$d_1 := \widehat{\varepsilon}_1 + \delta_1,\ d_2 := \widehat{\varepsilon}_2 + \delta_2, \ldots,\ d_{r-1} := \widehat{\varepsilon}_{r-1} + \delta_{r-1},\ d_r := \widehat{\varepsilon}_r + \delta_r, \tag{19}$$

for simplicity. So, $Q(\lambda)$ may have not the row degrees of the sought $P(\lambda)$ in the statement of Lemma 3.7, but $Q(\lambda)$ satisfies all the other properties of $P(\lambda)$. Observe, moreover, that the sum of the row degrees of $P(\lambda)$ is also $\delta + \varepsilon$.

If $d_r - d_1 \leq 1$, then the degrees of the r rows of $Q(\lambda)$ differ at most by 1 and their sum is $\delta + \varepsilon$, which implies that the row degrees of $Q(\lambda)$ are the ones in the statement and the proof is finished by taking $P(\lambda) = Q(\lambda)$. Therefore, let us assume $d_r - d_1 > 1$ in the rest of the proof. The strategy of the rest of the proof consists in applying very particular unimodular transformations $U(\lambda)$ and $V(\lambda)$ as follows,

$$Q(\lambda) \to U(\lambda)Q(\lambda)V(\lambda), \qquad N(\lambda) \to V(\lambda)^{-1}N(\lambda), \tag{20}$$

in such a way that $U(\lambda)Q(\lambda)V(\lambda)$ and $V(\lambda)^{-1}N(\lambda)$ satisfy the properties (i), (ii), (iii), (iv), and (v) above and, in addition, the row degrees of $U(\lambda)Q(\lambda)V(\lambda)$ are $d_1 + 1, d_2, \ldots, d_{r-1}, d_r - 1$. Clearly after a finite sequence of this type of transformations we will get a matrix polynomial $Q_{last}(\lambda)$ which will be $P(\lambda)$ in the statement.

Let us show how to construct the unimodular matrices $U(\lambda)$ and $V(\lambda)$ mentioned in the previous paragraph. For this purpose let us partition the matrix polynomials $Q(\lambda)$ and $N(\lambda)$ as follows

$$Q(\lambda) \;=\; \begin{array}{c} \\ 1 \\ r-2 \\ 1 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} n-r & 1 & r-2 & 1 \end{array} \\ \begin{bmatrix} Q_{11}(\lambda) & Q_{12}(\lambda) & Q_{13}(\lambda) & Q_{14}(\lambda) \\ Q_{21}(\lambda) & Q_{22}(\lambda) & Q_{23}(\lambda) & Q_{24}(\lambda) \\ Q_{31}(\lambda) & Q_{32}(\lambda) & Q_{33}(\lambda) & Q_{34}(\lambda) \end{bmatrix} \end{array} \qquad \text{and} \qquad N(\lambda) = \begin{array}{c} n-r \\ 1 \\ r-2 \\ 1 \end{array} \!\! \begin{bmatrix} N_1(\lambda) \\ N_2(\lambda) \\ N_3(\lambda) \\ N_4(\lambda) \end{bmatrix}. \tag{21}$$

Observe that, as a consequence of property (iii), i.e., of the structure of Q_{hr}, and (19), $\deg(Q_{12}(\lambda)) = d_1, \deg(Q_{14}(\lambda)) \leq d_1 - 1, \deg(Q_{32}(\lambda)) \leq d_r - 1$, and $\deg(Q_{34}(\lambda)) = d_r$. Therefore, by Lemma 3.6, there exists $\alpha \in \mathbb{F}$ such that

$$\begin{bmatrix} 1 & 0 \\ \alpha\, \lambda^{d_r - d_1 - 1} & -1 \end{bmatrix} \begin{bmatrix} Q_{12}(\lambda) & Q_{14}(\lambda) \\ Q_{32}(\lambda) & Q_{34}(\lambda) \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} J_{12}(\lambda) & J_{14}(\lambda) \\ J_{32}(\lambda) & J_{34}(\lambda) \end{bmatrix}, \tag{22}$$

with $\deg(J_{12}(\lambda)) = d_1, \deg(J_{14}(\lambda)) = d_1+1, \deg(J_{32}(\lambda)) = d_r - 1$, and $\deg(J_{34}(\lambda)) \leq d_r - 1$. Based on (22), we define the following unimodular matrices

$$\widetilde{U}(\lambda) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{r-2} & 0 \\ \alpha\lambda^{d_r-d_1-1} & 0 & -1 \end{bmatrix} \in \mathbb{F}[\lambda]^{r\times r} \quad \text{and} \quad \widetilde{V}(\lambda) := \begin{bmatrix} I_{n-r} & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & I_{r-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{n\times n}.$$

Direct computations taking into account (22) show that

$$\begin{aligned} J(\lambda) &:= \widetilde{U}(\lambda)Q(\lambda)\widetilde{V}(\lambda) \\ &= \begin{bmatrix} Q_{11}(\lambda) & J_{12}(\lambda) & Q_{13}(\lambda) & J_{14}(\lambda) \\ Q_{21}(\lambda) & Q_{22}(\lambda) & Q_{23}(\lambda) & Q_{24}(\lambda)+\lambda Q_{22}(\lambda) \\ \alpha\,\lambda^{d_r-d_1-1}Q_{11}(\lambda) - Q_{31}(\lambda) & J_{32}(\lambda) & \alpha\,\lambda^{d_r-d_1-1}Q_{13}(\lambda) - Q_{33}(\lambda) & J_{34}(\lambda) \end{bmatrix}, \end{aligned} \tag{23}$$

where, for $i = 1,3$ and $j = 2,4$, the $J_{ij}(\lambda)$ blocks are those in (22), and also

$$H(\lambda) := \widetilde{V}(\lambda)^{-1}N(\lambda) = \begin{bmatrix} I_{n-r} & 0 & 0 & 0 \\ 0 & 1 & 0 & -\lambda \\ 0 & 0 & I_{r-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1(\lambda) \\ N_2(\lambda) \\ N_3(\lambda) \\ N_4(\lambda) \end{bmatrix} = \begin{bmatrix} N_1(\lambda) \\ N_2(\lambda) - \lambda N_4(\lambda) \\ N_3(\lambda) \\ N_4(\lambda) \end{bmatrix}. \tag{24}$$

Taking into account the properties (i), (ii), (iii), (iv), and (v) of $Q(\lambda)$ and $N(\lambda)$, (19), (22), (23), and (24), we deduce easily the following properties for $J(\lambda)$ and $H(\lambda)$:

1. $J(\lambda)H(\lambda) = 0$;
2. $J(\lambda)$ has the same invariant polynomials as $Q(\lambda)$, that is, $p_1(\lambda), \dots, p_r(\lambda)$;
3. The row degrees of $J(\lambda)$ are $d_1+1, d_2, \dots, d_{r-1}, d_r - 1$ and its sum is equal to $\delta + \varepsilon$;
4. $J(\lambda)$ is row reduced, since
$$J_{hr} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & I_{r-2} & \mathtt{X} \\ \mathtt{X} & 1 & \mathtt{X} & \mathtt{X} \end{bmatrix},$$
where the entries in X are not specified;
5. The column degrees of $H(\lambda)$ are the same as those of $N(\lambda)$, that is $\varepsilon_1, \dots, \varepsilon_{n-r}$;
6. $H(\lambda)$ is column reduced since
$$H_{hc} = \begin{bmatrix} I_{n-r} \\ \mathtt{X} \\ 0 \\ 0 \end{bmatrix};$$
7. For all $\lambda_0 \in \overline{\mathbb{F}}$, $H(\lambda_0) = \widetilde{V}(\lambda_0)^{-1}N(\lambda_0)$ has full column rank, since $N(\lambda_0)$ has. Therefore, $H(\lambda)$ is a minimal basis of $\mathcal{N}_r(J)$ by Theorem 2.11.

Two additional operations are needed in order to get the unimodular transformations announced in (20). First, a permutation Π such that the rows of $\Pi J(\lambda) = \Pi\,\widetilde{U}(\lambda)Q(\lambda)\widetilde{V}(\lambda)$ are sorted with non-decreasing degrees. Second, to use Lemma 3.5(a) to prove that there exists a nonsingular constant matrix $\widetilde{R}$ such that $\Pi J(\lambda)\widetilde{R}^{-1} = \Pi\,\widetilde{U}(\lambda)Q(\lambda)\widetilde{V}(\lambda)\widetilde{R}^{-1}$ has highest-row-degree coefficient matrix equal to $\begin{bmatrix} 0_{r\times(n-r)} & I_r \end{bmatrix}$, and $\widetilde{R}H(\lambda) = \widetilde{R}\widetilde{V}(\lambda)^{-1}N(\lambda)$ has highest-column-degree coefficient matrix equal to $\begin{bmatrix} I_{n-r} & 0_{(n-r)\times r} \end{bmatrix}^T$. Therefore, the matrices $U(\lambda)$ and $V(\lambda)$ in (20) are

$$U(\lambda) = \Pi\,\widetilde{U}(\lambda) \qquad \text{and} \qquad V(\lambda) = \widetilde{V}(\lambda)\widetilde{R}^{-1},$$

and the proof of Lemma 3.7 is completed when $r < n$.

The proof of the case $r = n$ is much easier, since there are not right minimal indices, and we simply apply the unimodular transformations $U(\lambda)$ and $V(\lambda)$ considered above to $Q(\lambda) := \text{diag}(p_1(\lambda), \ldots, p_r(\lambda))$. □

If in Lemma 3.7 we take $p_j(\lambda) = 1$ for $j = 1, \ldots r$, and use Theorem 2.11, then we get directly Lemma 3.8.

Lemma 3.8. *Let r, n be two positive integers with $r \leq n$ and let $\varepsilon_1 \leq \cdots \leq \varepsilon_{n-r}$ be a list of $n - r$ nonnegative integers (which is empty if $n = r$). Define*

$$\varepsilon := \sum_{j=1}^{n-r} \varepsilon_j \,,$$

and write it as

$$\varepsilon = rq_\varepsilon + t_\varepsilon, \quad \text{where} \quad 0 \leq t_\varepsilon < r.$$

Then there exists an $r \times n$ row reduced matrix polynomial $P(\lambda)$ with t_ε row degrees equal to $q_\varepsilon + 1$ and $r - t_\varepsilon$ row degrees equal to q_ε, and such that all the invariant polynomials of $P(\lambda)$ are trivial and $\varepsilon_1 \leq \cdots \leq \varepsilon_{n-r}$ are its right minimal indices. The degree of $P(\lambda)$ is thus $q_\varepsilon + 1$ if $t_\varepsilon > 0$ and q_ε otherwise. In particular, the rows of $P(\lambda)$ form a minimal basis.

Lemma 3.8 implies just by transposition a similar result for the existence of column reduced matrix polynomials with prescribed left minimal indices. This is Lemma 3.9.

Lemma 3.9. *Let r, m be two positive integers with $r \leq m$ and let $\eta_1 \leq \cdots \leq \eta_{m-r}$ be a list of $m - r$ nonnegative integers (which is empty if $m = r$). Define*

$$\eta := \sum_{j=1}^{m-r} \eta_j \,,$$

and write it as

$$\eta = rq_\eta + t_\eta, \quad \text{where} \quad 0 \leq t_\eta < r.$$

Then there exists an $m \times r$ column reduced matrix polynomial $P(\lambda)$ with t_η column degrees equal to $q_\eta + 1$ and $r - t_\eta$ column degrees equal to q_η, and such that all the invariant polynomials of $P(\lambda)$ are trivial and $\eta_1 \leq \cdots \leq \eta_{m-r}$ are its left minimal indices. The degree of $P(\lambda)$ is thus $q_\eta + 1$ if $t_\eta > 0$ and q_η otherwise. In particular, the columns of $P(\lambda)$ form a minimal basis.

With Lemmas 3.9 and 3.7 at hand, we are in the position of proving the main result in this paper: Theorem 3.2.

Proof of Theorem 3.2. As we commented in the paragraph just below the statement of Theorem 3.2 the existence of $P(\lambda)$ implies immediately (6) and $\gamma_1 = 0$. So, it remains to prove that (6) and $\gamma_1 = 0$ imply that $P(\lambda)$ exists.

In the first place, let us show that if there are nonzero partial multiplicities at ∞, then we can reduce the proof to the case in which all partial multiplicities at ∞ are zero. For this purpose, let us express the invariant polynomials as

$$p_j(\lambda) = (\lambda - \lambda_1)^{\delta_j(\lambda_1)} \cdots (\lambda - \lambda_s)^{\delta_j(\lambda_s)}, \qquad j = 1, \ldots r, \tag{25}$$

where $\lambda_i \in \overline{\mathbb{F}}$, for $i = 1, \ldots s$, and $\lambda_i \neq \lambda_k$ if $i \neq k$. Let $\omega \in \mathbb{F}$ be such that $\omega \neq \lambda_i$, for $i = 1, \ldots s$, and define the polynomials

$$q_j(\lambda) = \lambda^{\gamma_j} \left(\lambda - \frac{1}{\lambda_1 - \omega}\right)^{\delta_j(\lambda_1)} \cdots \left(\lambda - \frac{1}{\lambda_s - \omega}\right)^{\delta_j(\lambda_s)}, \qquad j = 1, \ldots r, \tag{26}$$

which satisfy that $q_j(\lambda)$ divides $q_{j+1}(\lambda)$ for $j = 1, \ldots, r-1$. Note, in addition, that $q_j(\lambda)$ has coefficients in $\mathbb{F}$ (not in $\overline{\mathbb{F}}$). This latter property can be established as follows: if $0 \neq \beta \in \mathbb{F}$, then it easily follows that

$$q_j(\beta) = \beta^{k_j} \, \frac{p_j\left(\omega + \frac{1}{\beta}\right)}{p_j(\omega)} \in \mathbb{F},$$

where $k_j = \deg(q_j)$. Since $q_j(\lambda)$ is monic, one can take k_j different values for β and construct a linear system in $\mathbb{F}$ to determine the coefficients of $q_j(\lambda)$. Such linear system has obviously a unique solution in $\mathbb{F}^{k_j}$. Observe, that since $k_j = \deg(q_j) = \deg(p_j) + \gamma_j$, we have from (6) that

$$\sum_{j=1}^{r} k_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr.$$

Therefore, if we construct an $m \times n$ matrix polynomial $P_\omega(\lambda)$ with coefficients in $\mathbb{F}$, with rank r, with degree d, with invariant polynomials $q_1(\lambda), \ldots, q_r(\lambda)$, *without eigenvalues at* ∞, with right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$, and with left minimal indices $\eta_1, \ldots, \eta_{m-r}$, then, according to Lemma 3.3, the polynomial $P(\lambda)$ in (7) is the polynomial in Theorem 3.2 (with $\gamma_1, \ldots, \gamma_r$ partial multiplicities at ∞). Therefore, in the rest of the proof we assume that $\gamma_1 = \cdots = \gamma_r = 0$.

In this scenario, the hypothesis (6) becomes $\delta + \varepsilon + \eta = dr$, with $\delta = \sum_{j=1}^{r} \delta_j$, $\varepsilon = \sum_{j=1}^{n-r} \varepsilon_j$, and $\eta = \sum_{j=1}^{m-r} \eta_j$. Let $K(\lambda)$ be an $m \times r$ matrix polynomial with the properties of $P(\lambda)$ in Lemma 3.9 and *with columns sorted in non-decreasing order of degrees*, i.e., each of the first $r - t_\eta$ columns of $K(\lambda)$ has degree q_η and the remaining ones have degree $q_\eta + 1$. Let $M(\lambda)$ be an $r \times n$ matrix polynomial with the properties of $P(\lambda)$ in Lemma 3.7 and *with rows sorted in non-increasing order of degrees*, i.e., each of the first t_ε rows of $M(\lambda)$ has degree $q_\varepsilon + 1$ and the remaining ones have degree q_ε. Define the $m \times n$ matrix polynomial

$$P(\lambda) = K(\lambda) M(\lambda) \in \mathbb{F}[\lambda]^{m \times n}, \tag{27}$$

which satisfies the following properties:

1. $\operatorname{rank}(P) = r$, since both $K(\lambda)$ and $M(\lambda)$ have rank r;
2. The invariant polynomials of $P(\lambda)$ are identical to the invariant polynomials of $M(\lambda)$, that is, $p_1(\lambda), \ldots, p_r(\lambda)$, since $K(\lambda)$ is a minimal basis and Lemma 2.12(c) can be applied;
3. The right minimal indices of $P(\lambda)$ are $\varepsilon_1, \ldots, \varepsilon_{n-r}$, since $\mathcal{N}_r(P) = \mathcal{N}_r(M)$;
4. The left minimal indices of $P(\lambda)$ are $\eta_1, \ldots, \eta_{m-r}$, since $\mathcal{N}_\ell(P) = \mathcal{N}_\ell(K)$.

Therefore, it only remains to prove that $\deg(P) = d$ and that $P(\lambda)$ has not infinite eigenvalues.

For this purpose, note that $\eta = r q_\eta + t_\eta$, with $0 \leq t_\eta < r$, and $\delta + \varepsilon = r q_\varepsilon + t_\varepsilon$, with $0 \leq t_\varepsilon < r$, imply $dr = \delta + \varepsilon + \eta = r(q_\eta + q_\varepsilon) + (t_\eta + t_\varepsilon)$, with $0 \leq t_\eta + t_\varepsilon < 2r$. But $t_\eta + t_\varepsilon = r(d - q_\eta - q_\varepsilon)$ is a multiple of r and, therefore, $t_\eta + t_\varepsilon = 0$ or $t_\eta + t_\varepsilon = r$ and

$$d = \begin{cases} q_\eta + q_\varepsilon, & \text{if } t_\eta + t_\varepsilon = 0, \\ q_\eta + q_\varepsilon + 1, & \text{if } t_\eta + t_\varepsilon = r. \end{cases}$$

On the other hand, from (27)

$$P(\lambda) = \operatorname{col}_1(K) \operatorname{row}_1(M) + \cdots + \operatorname{col}_r(K) \operatorname{row}_r(M), \tag{28}$$

and observe that if $t_\eta + t_\varepsilon = 0$, then all summands in (28) have degree $d = q_\eta + q_\varepsilon$, while if $t_\eta + t_\varepsilon = r$, then all summands in (28) have degree $d = q_\eta + q_\varepsilon + 1$. In both cases the matrix coefficient of degree d in $P(\lambda)$ is

$$P_d = K_{hc} \, M_{hr},$$

where K_{hc} is the highest-column-degree coefficient matrix of $K(\lambda)$ and M_{hr} is the highest-row-degree coefficient matrix of $M(\lambda)$. Since $K(\lambda)$ is column reduced and $M(\lambda)$ is row reduced, we get that $\operatorname{rank}(P_d) = r$, which implies that $\deg(P) = d$ and that $P(\lambda)$ has no infinite eigenvalues. This completes the proof of Theorem 3.2. □

3.2. Theorem 3.2 expressed in terms of lists of elementary divisors and minimal indices

Theorem 3.11 below expresses Theorem 3.2 as a realizability result in the spirit of results included in [20] for quadratic matrix polynomials or, equivalently, as a general inverse eigenproblem of matrix polynomials. Theorem 3.11 has the advantage of not assuming in advance which are the rank and the size of the polynomial $P(\lambda)$ whose existence is established. This feature is very convenient for proving the results on ℓ-ifications included in Section 4.

Theorem 3.11 uses the concepts introduced in Definition 3.10.

Definition 3.10. *A list $\mathcal{L}$ of elementary divisors over a field $\mathbb{F}$* is the concatenation of two lists: a list $\mathcal{L}_{\text{fin}}$ of positive integer powers of monic irreducible polynomials of degree at least 1 with coefficients in $\mathbb{F}$ and a list $\mathcal{L}_\infty$ of elementary divisors $\mu^{\alpha_1}, \mu^{\alpha_2}, \ldots, \mu^{\alpha_{g_\infty}}$ at ∞. The length of the longest sublist of $\mathcal{L}_{\text{fin}}$ containing powers of the *same* irreducible polynomial is denoted by $g_{\text{fin}}(\mathcal{L})$ and the length of $\mathcal{L}_\infty$ is denoted by $g_\infty(\mathcal{L})$. The sum of the degrees of the elements in $\mathcal{L}_{\text{fin}}$ is denoted by $\delta_{\text{fin}}(\mathcal{L})$ and the sum of the degrees of the elements in $\mathcal{L}_\infty$ is denoted by $\delta_\infty(\mathcal{L})$.

Theorem 3.11. *Let $\mathcal{L}$ be a list of elementary divisors over an infinite field $\mathbb{F}$, let $\mathcal{M}_r = \{\varepsilon_1, \ldots, \varepsilon_p\}$ be a list of right minimal indices, let $\mathcal{M}_l = \{\eta_1, \ldots, \eta_q\}$ be a list of left minimal indices, and define*

$$S(\mathcal{L}) = \delta_{\text{fin}}(\mathcal{L}) + \delta_\infty(\mathcal{L}) + \sum_{i=1}^{q} \eta_i + \sum_{i=1}^{p} \varepsilon_i \,. \tag{29}$$

Then, there exists a matrix polynomial $P(\lambda)$ with coefficients in $\mathbb{F}$ of degree d and whose elementary divisors, right minimal indices, and left minimal indices are, respectively, those in the lists $\mathcal{L}$, $\mathcal{M}_r$, and $\mathcal{M}_l$, if and only if

$$\text{(i) } d \text{ is a divisor of } S(\mathcal{L}), \quad \text{(ii) } \frac{S(\mathcal{L})}{d} \geq g_{\text{fin}}(\mathcal{L}), \quad \text{and} \quad \text{(iii) } \frac{S(\mathcal{L})}{d} > g_\infty(\mathcal{L}). \tag{30}$$

In this case, the rank of $P(\lambda)$ is $S(\mathcal{L})/d$ and the size of $P(\lambda)$ is $(q + S(\mathcal{L})/d) \times (p + S(\mathcal{L})/d)$.

Proof. ($\Longrightarrow$) Let us assume that $P(\lambda)$ exists. Then Theorem 3.1 implies $d\,\text{rank}(P) = S(\mathcal{L})$, so, (30)-(i) follows and $\text{rank}(P) = S(\mathcal{L})/d$. The non-trivial invariant polynomials of $P(\lambda)$ are uniquely determined from the elementary divisors in the sublist $\mathcal{L}_{\text{fin}}$ of $\mathcal{L}$, as explained for instance in [12, Chap. VI, p. 142], and its number is precisely $g_{\text{fin}}(\mathcal{L})$. Now (30)-(ii) follows from $\text{rank}(P) \geq g_{\text{fin}}(\mathcal{L})$. Finally, (30)-(iii) follows from Lemma 2.6. The size of $P(\lambda)$ is obtained from the rank-nullity theorem.

($\Longleftarrow$) Let us assume that the three conditions in (30) hold and note that (30)-(iii) guarantees that $S(\mathcal{L}) > 0$. From (30)-(i) define the positive integer $r := S(\mathcal{L})/d$.

First, let us denote for simplicity $g_{\text{fin}} := g_{\text{fin}}(\mathcal{L})$ and construct with all the elements in $\mathcal{L}_{\text{fin}}$ a sequence of monic polynomials $q_j(\lambda)$, $j = 1, \ldots, g_{\text{fin}}$, of degree larger than zero and such that $q_j(\lambda)$ is a divisor of $q_{j+1}(\lambda)$ for $j = 1, \ldots, g_{\text{fin}} - 1$. This construction is easy and is explained in [12, Chap. VI, p. 142]. Taking into account that $r - g_{\text{fin}} \geq 0$ by (30)-(ii), we define the following list of r polynomials

$$(\, p_1(\lambda), \ldots, p_r(\lambda) \,) = (\, \underbrace{1, \ldots, 1}_{r - g_{\text{fin}}}, q_1(\lambda), \ldots, q_{g_{\text{fin}}}(\lambda) \,). \tag{31}$$

Observe that the sum of the degrees of $p_1(\lambda), \ldots, p_r(\lambda)$ is precisely $\delta_{\text{fin}}(\mathcal{L})$.

Second, denote for simplicity $g_\infty := g_\infty(\mathcal{L})$ and let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{g_\infty}$ be the exponents of the elements of $\mathcal{L}_\infty$, which are all positive. Taking into account that $r - g_\infty > 0$ by (30)-(iii), we define the following list of r nonnegative numbers

$$(\, \gamma_1, \ldots, \gamma_r \,) = (\, \underbrace{0, \ldots, 0}_{r - g_\infty}, \alpha_1, \alpha_2, \ldots, \alpha_{g_\infty} \,). \tag{32}$$

Observe that $\sum_{i=1}^{r} \gamma_i = \delta_\infty(\mathcal{L})$ and that $\gamma_1 = 0$.

Finally, note that if we define $r := S(\mathcal{L})/d$, $m := q + S(\mathcal{L})/d$, and $n := p + S(\mathcal{L})/d$, then (29) is precisely (6) for the polynomials defined in (31) and the partial multiplicity sequence at ∞ given in (32), therefore the existence of $P(\lambda)$ follows from Theorem 3.2. $\square$

4. Existence and possible sizes and eigenstructures of ℓ-ifications

The new concept of ℓ-ification of a matrix polynomial has been recently introduced in [9, Section 3], as a natural generalization of the classical definition of linearization [13]. One motivation for investigating ℓ-ifications is that there exist matrix polynomials with special structures arising in applications which do not have any linearization with the same structure [9, Section 7]. The results introduced in Section 3 allow us to establish easily in this section a number of new results on ℓ-ifications that would be difficult to prove by other means. Let us start by recalling the definition of ℓ-ification.

Definition 4.1. A matrix polynomial $R(\lambda)$ of degree $\ell > 0$ is said to be an ℓ-ification of a given matrix polynomial $P(\lambda)$ if for some $r, s \geq 0$ there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda)\begin{bmatrix} R(\lambda) & \\ & I_s \end{bmatrix} V(\lambda) = \begin{bmatrix} P(\lambda) & \\ & I_r \end{bmatrix}.$$

If, in addition, $\operatorname{rev} R(\lambda)$ is an ℓ-ification of $\operatorname{rev} P(\lambda)$, then $R(\lambda)$ is said to be a *strong* ℓ-ification of $P(\lambda)$.

In [9], ℓ-ifications are defined to have degree smaller than or equal to ℓ. However, since in most cases it is of interest to fix the exact degree of a matrix polynomial, we focus in this paper on ℓ-ifications with degree ℓ. It is natural to think that in applications the most interesting ℓ-ifications of $P(\lambda)$ should satisfy $\ell < \deg(P)$, but observe that the results presented in this section do not require such assumption and, so, they will be stated for any positive integer ℓ. This is one of the reasons of considering two identity matrices I_s and I_r in Definition 4.1, although only one of them is really needed [9, Corollary 4.4]. Observe also that, since we assume $\ell > 0$, zero degree polynomials are never considered ℓ-ifications in this work.

In this section we consider two matrix polynomials, $P(\lambda)$ and $R(\lambda)$, so it is convenient to adopt for brevity the following notation: for any matrix polynomial $Q(\lambda)$, $\delta_{\text{fin}}(Q)$ is the sum of the degrees of the invariant polynomials of $Q(\lambda)$, $\delta_\infty(Q)$ is the sum of the degrees of the elementary divisors at ∞ of $Q(\lambda)$, $g_{\text{fin}}(Q)$ is the number of nontrivial invariant polynomials of $Q(\lambda)$ (equivalently, the largest of the geometric multiplicities of the finite eigenvalues of $Q(\lambda)$), and $g_\infty(Q)$ is the number of elementary divisors of $Q(\lambda)$ at ∞ (equivalently, the geometric multiplicity of the infinite eigenvalue of $Q(\lambda)$). We will use very often this notation without explicitly referring to.

The results of this section generalize to ℓ-ifications of arbitrary degree what was proved in Theorems 4.10 and 4.12 of [9] only for linearizations. In addition, we solve for arbitrary infinite fields the conjecture posed in [9, Remark 7.6]. In the proofs of this section, we will use very often Theorem 4.1 in [9]. Therefore, we state below this result in a form specially adapted to our purposes.

Theorem 4.2. (Characterization of ℓ-ifications [9, Theorem 4.1])
Consider a matrix polynomial $P(\lambda)$ and another matrix polynomial $R(\lambda)$ of degree $\ell > 0$, and the following three conditions on $P(\lambda)$ and $R(\lambda)$:

(a) $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_r(R)$ *and* $\dim \mathcal{N}_\ell(P) = \dim \mathcal{N}_\ell(R)$
(*i.e.,* $P(\lambda)$ *and* $R(\lambda)$ *have the same* numbers *of right and left minimal indices*).
(b) $P(\lambda)$ *and* $R(\lambda)$ *have exactly the same* finite *elementary divisors.*
(c) $P(\lambda)$ *and* $R(\lambda)$ *have exactly the same* infinite *elementary divisors.*

Then:

(1) $R(\lambda)$ *is an ℓ-ification of $P(\lambda)$ if and only if conditions* (a) *and* (b) *hold.*
(2) $R(\lambda)$ *is a strong ℓ-ification of $P(\lambda)$ if and only if conditions* (a), (b), *and* (c) *hold.*

Observe that condition (b) in Theorem 4.2 is equivalent to "$P(\lambda)$ and $R(\lambda)$ have exactly the same non-trivial invariant polynomials".

4.1. Results for standard ℓ-ifications

Theorem 4.3 determines all possible sizes, all possible nonzero partial multiplicities at ∞, and all possible minimal indices of (possibly not strong) ℓ-ifications of a given matrix polynomial $P(\lambda)$. Recall that according to Theorem 4.2 the non-trivial invariant polynomials as well as the numbers of left and right minimal indices of any ℓ-ification of $P(\lambda)$ are the same as those of $P(\lambda)$. The set of positive integers is denoted by $\mathbb{Z}^+$ in Theorem 4.3.

Theorem 4.3. (Size range of ℓ-ifications) *Let $P(\lambda)$ be an $m \times n$ matrix polynomial over an infinite field $\mathbb{F}$ with rank equal to $r > 0$, let ℓ be a positive integer, and let*

$$\widetilde{r} = \min \left\{ y \in \mathbb{Z}^+ \,:\, y \geq \frac{\delta_{\text{fin}}(P)}{\ell} \quad \textit{and} \quad y \geq g_{\text{fin}}(P) \right\}.$$

Then:

(a) *There is an $s_1 \times s_2$ ℓ-ification of $P(\lambda)$ if and only if*

$$s_1 \geq (m-r) + \widetilde{r}, \quad s_2 \geq (n-r) + \widetilde{r}, \quad \textit{and} \quad s_1 - s_2 = m - n. \tag{33}$$

In particular, the minimum-size ℓ-ification of $P(\lambda)$ has sizes $s_1 = (m-r) + \widetilde{r}$ and $s_2 = (n-r) + \widetilde{r}$.

(b) *Let $0 \leq \widetilde{\eta}_1 \leq \cdots \leq \widetilde{\eta}_{m-r}$ be a list of left minimal indices, let $0 \leq \widetilde{\varepsilon}_1 \leq \cdots \leq \widetilde{\varepsilon}_{n-r}$ be a list of right minimal indices, let $0 < \widetilde{\gamma}_1 \leq \cdots \leq \widetilde{\gamma}_t$ be a (possibly empty) list of nonzero partial multiplicities at ∞, and define*

$$\widetilde{S} = \delta_{\text{fin}}(P) + \sum_{i=1}^{t} \widetilde{\gamma}_i + \sum_{i=1}^{m-r} \widetilde{\eta}_i + \sum_{i=1}^{n-r} \widetilde{\varepsilon}_i \,.$$

Then, there exists an ℓ-ification $R(\lambda)$ of $P(\lambda)$ having the previous lists of minimal indices and nonzero partial multiplicities at ∞ if and only if

$$\text{(i)} \ \ \ell \textit{ is a divisor of } \widetilde{S}, \quad \text{(ii)} \ \frac{\widetilde{S}}{\ell} \geq g_{\text{fin}}(P), \quad \textit{and} \quad \text{(iii)} \ \frac{\widetilde{S}}{\ell} > t. \tag{34}$$

The size of this $R(\lambda)$ is $s_1 \times s_2$, where

$$s_1 = (m-r) + \frac{\widetilde{S}}{\ell} \quad \textit{and} \quad s_2 = (n-r) + \frac{\widetilde{S}}{\ell}\,.$$

Proof. (a) ($\Rightarrow$): If $R(\lambda)$ has size $s_1 \times s_2$ and is an ℓ-ification of $P(\lambda)$, then $R(\lambda)$ has the same non-trivial invariant polynomials as $P(\lambda)$ and the same numbers, $m - r$ and $n - r$, of left and right minimal indices as $P(\lambda)$ by Theorem 4.2(1). Therefore, Theorem 3.1 applied to $R(\lambda)$ implies $\text{rank}(R) \geq \delta_{\text{fin}}(R)/\ell = \delta_{\text{fin}}(P)/\ell$, and Definition 2.2 implies $\text{rank}(R) \geq g_{\text{fin}}(R) = g_{\text{fin}}(P)$. Therefore, $\text{rank}(R) \geq \widetilde{r}$. Finally, from the rank-nullity theorem, we get

$$s_1 = (m-r) + \text{rank}(R) \geq (m-r) + \widetilde{r} \qquad \text{and} \qquad s_2 = (n-r) + \text{rank}(R) \geq (n-r) + \widetilde{r},$$

and $s_1 - s_2 = m - n$.

(a) ($\Leftarrow$): To see that there exists an ℓ-ification of $P(\lambda)$ for each size $s_1 \times s_2$ allowed by (33), note first that each of these sizes can be written as

$$s_1 = (m-r) + \widehat{r}, \qquad s_2 = (n-r) + \widehat{r},$$

for some $\widehat{r} \geq \widetilde{r}$. Therefore, $\widehat{r} \geq \delta_{\text{fin}}(P)/\ell$ and, so, there are nonnegative numbers $0 \leq \widetilde{\eta}_1 \leq \cdots \leq \widetilde{\eta}_{m-r}$ and $0 \leq \widetilde{\varepsilon}_1 \leq \cdots \leq \widetilde{\varepsilon}_{n-r}$ such that

$$\ell\,\widehat{r} = \delta_{\text{fin}}(P) + \sum_{i=1}^{m-r} \widetilde{\eta}_i + \sum_{i=1}^{n-r} \widetilde{\varepsilon}_i \,.$$

In addition, $\widehat{r} \geq g_{\text{fin}}(P)$. Combining all this information with Theorem 3.11, we get that there exists a matrix polynomial $R(\lambda)$ of degree ℓ, with no elementary divisors at ∞, with the same non-trivial invariant polynomials[3] as $P(\lambda)$, with left and right minimal indices equal, respectively, to $0 \leq \widetilde{\eta}_1 \leq \cdots \leq \widetilde{\eta}_{m-r}$ and $0 \leq \widetilde{\varepsilon}_1 \leq \cdots \leq \widetilde{\varepsilon}_{n-r}$, and with size $s_1 \times s_2$ such that

$$s_1 = (m-r) + \widehat{r} \qquad \text{and} \qquad s_2 = (n-r) + \widehat{r}.$$

[3]Recall that there is a bijection between the set of lists of finite elementary divisors and the set of lists of non-trivial invariant polynomials [12, Ch. VI, p. 142].

This $R(\lambda)$ is an ℓ-ification of $P(\lambda)$ by Theorem 4.2(1).

(b): It is an immediate consequence of Theorem 3.11 and Theorem 4.2(1). □

Remark 4.4. Observe that Theorem 4.3(a) guarantees, in particular, that every matrix polynomial has an ℓ-ification for any positive degree ℓ. Note, in addition, that ℓ-ifications can have arbitrarily large sizes. The three conditions in (33) are redundant, since the second one follows from the first and the third ones. We write all of them to make explicit both minimum sizes of ℓ-ifications.

Remark 4.5. In Theorem 4.3(b) the cardinalities of the lists of left and right minimal indices are $m-r$ and $n-r$, respectively, and so they are determined by the size and the rank of $P(\lambda)$. However, the cardinality t of the list of nonzero partial multiplicities at ∞ can be chosen, and in particular, this list can be chosen to be empty.

Remark 4.6. Let $0 \le \eta_1 \le \cdots \le \eta_{m-r}$ be the left minimal indices of $P(\lambda)$, let $0 \le \varepsilon_1 \le \cdots \le \varepsilon_{n-r}$ be the right minimal indices of $P(\lambda)$, and let $0 < \gamma_1 \le \cdots \le \gamma_{g_\infty(P)}$ be the nonzero partial multiplicities at ∞ of $P(\lambda)$, and define

$$S = \delta_{\text{fin}}(P) + \sum_{i=1}^{g_\infty(P)} \gamma_i + \sum_{i=1}^{m-r} \eta_i + \sum_{i=1}^{n-r} \varepsilon_i \,.$$

Theorem 4.3(b) implies that if ℓ is a divisor of S, $S/\ell \ge g_{\text{fin}}(P)$, and $S/\ell > g_\infty(P)$, then there exists an ℓ-ification $R(\lambda)$ of $P(\lambda)$ *with exactly the same complete eigenstructure as* $P(\lambda)$ (barring trivial invariant polynomials). In particular, this always happens if ℓ is a divisor of $d = \deg(P(\lambda))$, since in this case Theorem 3.1 implies $dr = S$ and, so, $S/\ell \ge r$, which can be combined with $r \ge g_{\text{fin}}(P)$ and $r > g_\infty(P)$ (see Lemma 2.6) to show that the three conditions in (34) hold. Observe that such $R(\lambda)$ is in fact a strong ℓ-ification of $P(\lambda)$ according to Theorem 4.2(2).

In particular, Theorem 4.3(b) implies that there always exist linearizations (just make $\ell = 1$ in the discussion above) which preserve *the complete* eigenstructure of $P(\lambda)$. A similar result was already obtained in [32] but using a different definition of eigenstructure at ∞. An advantage of their approach is that also a construction of such linearizations was provided there. It is an interesting open problem to derive such a construction for the definition of infinite structure used in this paper as well. We point out here that for the definition of eigenstructure at ∞ used in [32], there is also a corresponding index sum theorem described in [33].

4.2. Results for strong ℓ-ifications of regular polynomials

Our next result generalizes Theorem 7.5 in [9] in two senses. First, Theorem 4.7 is valid for strong ℓ-ifications of any degree, i.e., smaller than, equal to, or larger than, the degree of $P(\lambda)$, while Theorem 7.5 in [9] is valid only for strong ℓ-ifications with degree smaller than or equal to the degree of $P(\lambda)$. Second, Theorem 4.7 is valid for arbitrary infinite fields, while Theorem 7.5 in [9] is valid only for algebraically closed fields.

Theorem 4.7. (Size of strong ℓ-fications of regular polynomials) *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial over an infinite field $\mathbb{F}$ of degree d and let ℓ be a positive integer. Then, there exists a strong ℓ-ification $R(\lambda)$ of $P(\lambda)$ if and only if*

$$\text{(i) } \ell \text{ is a divisor of } dn, \quad \text{(ii) } \frac{dn}{\ell} \ge g_{\text{fin}}(P), \quad \text{and} \quad \text{(iii) } \frac{dn}{\ell} > g_\infty(P). \tag{35}$$

In addition, the size of any strong ℓ-ification of $P(\lambda)$ is $(dn)/\ell \times (dn)/\ell$. In particular, if $\ell > dn$, then there are no strong ℓ-ifications of $P(\lambda)$.

Proof. Recall throughout the proof that Theorem 3.1 implies that $dn = \delta_{\text{fin}}(P) + \delta_\infty(P)$, since $P(\lambda)$ is regular and has no minimal indices.

First, assume that the conditions in (35) hold. Therefore, Theorem 3.11 with $S(\mathcal{L}) = \delta_{\text{fin}}(P) + \delta_\infty(P)$ implies that there exists a regular matrix polynomial $R(\lambda)$ of degree ℓ, with the same non-trivial invariant polynomials as $P(\lambda)$, and with the same elementary divisors at ∞ as $P(\lambda)$. Such $R(\lambda)$ has size $(dn)/\ell \times (dn)/\ell$ and is an strong ℓ-ification of $P(\lambda)$ by Theorem 4.2(2).

Next, assume that there exists a strong ℓ-ification $R(\lambda)$ of $P(\lambda)$. According to Theorem 4.2(2), $R(\lambda)$ is regular and has the same non-trivial invariant polynomials as $P(\lambda)$, and the same elementary divisors at ∞ as $P(\lambda)$. Therefore, if $R(\lambda)$ has size $s \times s$, then Theorem 3.1 applied to $R(\lambda)$ implies $s\ell = \delta_{\text{fin}}(P) + \delta_\infty(P)$, which, together with Lemma 2.6, implies the conditions (35). □

The following two corollaries follow easily from Theorem 4.7. Corollary 4.8 is well known: both the result on existence and the result on size (see [6, Theorem 3.2] and [9, Theorem 4.12]). We include this corollary here just to show that these classical results can be retrieved from Theorem 4.7. Corollary 4.9 shows that Theorem 4.7 simplifies considerably if we assume that $\ell \leq \deg(P)$. Corollary 4.9 is related to Theorem 7.5 in [9].

Corollary 4.8. (Size of strong linearizations of regular polynomials) *For any $n \times n$ regular matrix polynomial $P(\lambda)$ over an infinite field $\mathbb{F}$ of degree $d \geq 1$, there always exists a strong linearization of $P(\lambda)$ and the size of any such strong linearization is $(dn) \times (dn)$.*

Proof. If $\ell = 1$, then (i) in (35) always holds, and the same happens for (ii) and (iii), since $n \geq g_{\text{fin}}(P)$ and $n > g_\infty(P)$. The size follows also from Theorem 4.7. □

Corollary 4.9. *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial of degree d over an infinite field $\mathbb{F}$ and let $1 \leq \ell \leq d$ be an integer. Then, there exists a strong ℓ-ification of $P(\lambda)$ if and only if ℓ divides dn. In addition, the size of any strong ℓ-ification of $P(\lambda)$ is $(dn)/\ell \times (dn)/\ell$.*

Proof. The result follows from Theorem 4.7 and the following observation: $n \geq g_{\text{fin}}(P)$ and $n > g_\infty(P)$ always hold, where the last strict inequality is because of Lemma 2.6. In addition, $d/\ell \geq 1$. Therefore, (ii) and (iii) in (35) automatically hold in this case. □

4.3. Results for strong ℓ-ifications of singular polynomials

Theorem 4.10 determines all possible sizes and all possible minimal indices of strong ℓ-ifications of a given singular matrix polynomial $P(\lambda)$. Recall that according to Theorem 4.2, the non-trivial invariant polynomials, the elementary divisors at ∞, as well as the numbers of left and right minimal indices of any strong ℓ-ification of $P(\lambda)$ are the same as those of $P(\lambda)$.

Theorem 4.10. (Size range of strong ℓ-fications of singular polynomials) *Let $P(\lambda)$ be an $m \times n$ singular matrix polynomial over an infinite field $\mathbb{F}$ with rank equal to $r > 0$, i.e., at least one of $m - r$ or $n - r$ is nonzero, let ℓ be a positive integer, and let*

$$\widetilde{r} = \min \left\{ y \in \mathbb{Z}^+ \, : \, y \geq \frac{\delta_{\text{fin}}(P) + \delta_\infty(P)}{\ell}, \quad y \geq g_{\text{fin}}(P), \quad \text{and} \quad y > g_\infty(P) \right\}.$$

Then:

(a) *There is an $s_1 \times s_2$ strong ℓ-fication of $P(\lambda)$ if and only if*

$$s_1 \geq (m - r) + \widetilde{r}, \quad s_2 \geq (n - r) + \widetilde{r}, \quad \text{and} \quad s_1 - s_2 = m - n. \tag{36}$$

In particular, the minimum-size strong ℓ-ification of $P(\lambda)$ has sizes $s_1 = (m - r) + \widetilde{r}$ and $s_2 = (n - r) + \widetilde{r}$.

(b) *Let $0 \leq \widetilde{\eta}_1 \leq \cdots \leq \widetilde{\eta}_{m-r}$ be a list of left minimal indices, let $0 \leq \widetilde{\varepsilon}_1 \leq \cdots \leq \widetilde{\varepsilon}_{n-r}$ be a list of right minimal indices, and define*

$$\widetilde{S} = \delta_{\text{fin}}(P) + \delta_\infty(P) + \sum_{i=1}^{m-r} \widetilde{\eta}_i + \sum_{i=1}^{n-r} \widetilde{\varepsilon}_i \, .$$

Then, there exists a strong ℓ-ification $R(\lambda)$ of $P(\lambda)$ having the previous lists of minimal indices if and only if

$$\text{(i) } \ell \textit{ is a divisor of } \widetilde{S}, \quad \text{(ii) } \frac{\widetilde{S}}{\ell} \geq g_{\text{fin}}(P), \quad \textit{and} \quad \text{(iii) } \frac{\widetilde{S}}{\ell} > g_{\infty}(P). \tag{37}$$

The size of this $R(\lambda)$ is $s_1 \times s_2$, where

$$s_1 = (m-r) + \frac{\widetilde{S}}{\ell} \quad \textit{and} \quad s_2 = (n-r) + \frac{\widetilde{S}}{\ell}\,.$$

Proof. (a) ($\Rightarrow$): If $R(\lambda)$ has size $s_1 \times s_2$ and is a strong ℓ-ification of $P(\lambda)$, then $R(\lambda)$ and $P(\lambda)$ have the same non-trivial invariant polynomials, the same elementary divisors at ∞, and the same numbers, $m-r$ and $n-r$, of left and right minimal indices by Theorem 4.2(2). Therefore, Theorem 3.1 applied to $R(\lambda)$ implies $\operatorname{rank}(R) \geq (\delta_{\text{fin}}(P) + \delta_{\infty}(P))/\ell$, Definition 2.2 implies $\operatorname{rank}(R) \geq g_{\text{fin}}(R) = g_{\text{fin}}(P)$, and Lemma 2.6 implies $\operatorname{rank}(R) > g_{\infty}(R) = g_{\infty}(P)$. Therefore, $\operatorname{rank}(R) \geq \widetilde{r}$. Finally, from the rank-nullity theorem, we get

$$s_1 = (m-r) + \operatorname{rank}(R) \geq (m-r) + \widetilde{r} \qquad \text{and} \qquad s_2 = (n-r) + \operatorname{rank}(R) \geq (n-r) + \widetilde{r},$$

and $s_1 - s_2 = m - n$.

(a) ($\Leftarrow$): To see that there exists a strong ℓ-ification of $P(\lambda)$ for each size $s_1 \times s_2$ allowed by (36), note that each of these sizes can be written as $s_1 = (m-r) + \widehat{r}, s_2 = (n-r) + \widehat{r}$, for some $\widehat{r} \geq \widetilde{r}$. Therefore, $\widehat{r} \geq (\delta_{\text{fin}}(P) + \delta_{\infty}(P))/\ell$ and, so, there are nonnegative numbers $0 \leq \widetilde{\eta}_1 \leq \cdots \leq \widetilde{\eta}_{m-r}$ and $0 \leq \widetilde{\varepsilon}_1 \leq \cdots \leq \widetilde{\varepsilon}_{n-r}$ such that

$$\ell\,\widehat{r} = \delta_{\text{fin}}(P) + \delta_{\infty}(P) + \sum_{i=1}^{m-r} \widetilde{\eta}_i + \sum_{i=1}^{n-r} \widetilde{\varepsilon}_i\,.$$

In addition, $\widehat{r} \geq g_{\text{fin}}(P)$ and $\widehat{r} > g_{\infty}(P)$. Combining all this information with Theorem 3.11, we get that there exists a matrix polynomial $R(\lambda)$ of degree ℓ, with the same non-trivial invariant polynomials as $P(\lambda)$, with the same elementary divisors at ∞ as $P(\lambda)$, with left and right minimal indices equal, respectively, to $0 \leq \widetilde{\eta}_1 \leq \cdots \leq \widetilde{\eta}_{m-r}$ and $0 \leq \widetilde{\varepsilon}_1 \leq \cdots \leq \widetilde{\varepsilon}_{n-r}$, and with size $s_1 \times s_2$ such that, $s_1 = (m-r) + \widehat{r}$ and $s_2 = (n-r) + \widehat{r}$. This $R(\lambda)$ is a strong ℓ-ification of $P(\lambda)$ by Theorem 4.2(2).

(b): It is an immediate consequence of Theorem 3.11 and Theorem 4.2(2). □

Remark 4.11. Observe that Theorem 4.10(a) guarantees, in particular, that every singular matrix polynomial has a strong ℓ-fication for any degree ℓ. Note, in addition, that strong ℓ-ifications of singular polynomials can have arbitrarily large sizes. These two results are in stark contrast with the situation for regular matrix polynomials, since Theorem 4.7 shows that, in this case, strong ℓ-ifications exist only for certain values of ℓ and that their sizes are fixed.

5. Conclusions

We have solved a very general inverse problem for matrix polynomials with *completely* prescribed eigenstructure. As far as we know, this is the first result of this type that considers the complete eigenstructure, i.e., both regular and singular, of a matrix polynomial of arbitrary degree. The solution of this inverse problem has been used to settle the existence problem of ℓ-fications of a given matrix polynomial, as well as to determine their possible sizes and eigenstructures.

The results presented in this work lead naturally to many related open problems. For instance, the solution of inverse problems for structured matrix polynomials with completely prescribed eigenstructure for classes of polynomials that are important in applications, such as symmetric, skew-symmetric, alternating, and palindromic matrix polynomials. Another open problem in this area is to develop efficient methods for constructing, when possible, matrix polynomials of a given degree, with given lists of finite and infinite elementary divisors, and with given lists of left and right minimal indices, since the construction used in the proof of Theorem 3.2 is not efficient in practice. This is closely related to the problem of constructing strong ℓ-ifications in the situations that are not covered by the ℓ-ifications presented in [9, Section 5]. Finally, the problem of extending to finite fields the results of this paper is still open as well.

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