GENERIC CHANGE OF THE PARTIAL MULTIPLICITIES OF REGULAR MATRIX PENCILS UNDER LOW-RANK PERTURBATIONS*

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Abstract. We describe the generic change of the partial multiplicities at a given eigenvalue λ_0 of a regular matrix pencil $A_0 + \lambda A_1$ under perturbations with low normal rank. More precisely, if the pencil $A_0 + \lambda A_1$ has exactly g nonzero partial multiplicities at λ_0, then for most perturbations $B_0 + \lambda B_1$ with normal rank $r < g$ the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$ has exactly $g - r$ nonzero partial multiplicities at λ_0, which coincide with those obtained after removing the largest r partial multiplicities of the original pencil $A_0 + \lambda A_1$ at λ_0. Though partial results on this problem had been previously obtained in the literature, its complete solution remained open.

Key words. regular matrix pencils, Weierstrass canonical form, low rank perturbations, matrix spectral perturbation theory, partial multiplicities

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1. Introduction.

Let $A_0, A_1 \in \mathbb{C}^{n\times n}$ and let $A_0+\lambda A_1$ be a regular matrix pencil having g nonzero partial multiplicities at the eigenvalue λ_0 (named the *Weierstrass structure* at λ_0 in [7]). Let $B_0 + \lambda B_1$ be another matrix pencil with *low rank*. We are interested in describing the *generic* nonzero partial multiplicities at λ_0 of the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$. For this, we first need to properly establish what we mean by "generic" and by "low rank". Both notions are related, and the genericity will depend on the notion of low rank we consider. These notions are in turn related to the way in which the perturbations $B_0+\lambda B_1$ are built up. In particular, the notion of genericity is closely related to the geometry of the set of perturbations. This has led us to introduce an appropriate notion of genericty in the set of perturbations which is consistent with the special geometric structure of this set.

In order to analyze the change of the nonzero partial multiplicities, we first need to guarantee that λ_0 remains as an eigenvalue of the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$, so that this pencil still has nonzero partial multiplicities at λ_0. It is proved in [7, Th. 3.3] that if we define

$$\rho_0 := \operatorname{rank}(B_0 + \lambda_0 B_1), \qquad \rho_1 := \operatorname{rank}(B_1), \qquad \rho := \rho_0 + \rho_1, \tag{1.1}$$

then, the condition $\rho_0 < g$, ensures that λ_0 is an eigenvalue of $A_0 + B_0 + \lambda(A_1 + B_1)$. Hence, $\rho_0 < g$ is the low rank condition in [7]. Therefore, a natural way to build up a perturbation pencil $B_0 + \lambda B_1$ in [7] is by generating $B_0 + \lambda_0 B_1$ as an arbitrary matrix with rank $\rho_0 < g$, and B_1 as an arbitrary matrix with rank ρ_1. Then, the perturbation is built up as $B_0 + \lambda B_1 = B_0 + \lambda_0 B_1 + (\lambda - \lambda_0)B_1$. This pencil has, generically, normal rank ρ, provided that $\rho < n$. Another way to generate rank-1 perturbations has been considered recently in [1, Th. 2.10]. These perturbations are of the form $B_0 + \lambda B_1 = -\alpha uv^T + \lambda\beta uv^T$, with $u, v \in \mathbb{C}^n$, and $\alpha, \beta \in \mathbb{C}$. However, none of these constructions provide generic perturbations of a given normal rank ρ

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(with $\rho = 1$ in the case of [1]), since the perturbations $B_0 + \lambda B_1$ in [1] and [7] have eigenvalues (α/β and λ_0, respectively, are eigenvalues), while generically $n \times n$ matrix pencils $B_0 + \lambda B_1$ with normal rank $\rho < n$ have no eigenvalues [6, Th. 3.2]. Hence, the problem of describing the generic change of the partial multiplicities of regular matrix pencils under low rank perturbations of a given normal rank remains open.

In this paper, we solve this problem. In particular, we consider the low-rank condition $r < g$, where $r := \text{nrank}(B_0 + \lambda B_1)$ is the normal rank of the perturbation. This condition also guarantees that λ_0 is an eigenvalue of $A_0 + B_0 + \lambda(A_1 + B_1)$, since $\text{rank}(B_0 + \lambda_0 B_1) \leq r$. An advantage in considering this condition is that the normal rank is independent of the particular eigenvalue λ_0, and allows us to deal simultaneously with all eigenvalues of $A_0+\lambda A_1$, including the infinite eigenvalue, since $\text{nrank}(A_0 + \lambda A_1) = \text{nrank}(\lambda A_0 + A_1)$. We prove that, for generic perturbations with normal rank r, the change of the nonzero partial multiplicities at any eigenvalue λ_0 (both finite and infinite) consists of just removing the largest r partial multiplicities of $A_0+\lambda A_1$ at λ_0 and leaving the remaining ones unchanged. This resembles the generic change of the partial multiplicities of matrices (i.e., of the Jordan canonical form) under low-rank perturbations [13, 19, 20], and is in contrast with the change described in [7] for generic perturbations constructed following the procedure explained in the precedent paragraph. More precisely, the generic partial multiplicities of $A_0 + B_0 + \lambda(A_1+B_1)$ at λ_0 in [7] are obtained after removing the largest ρ_0 partial multiplicities of $A_0 + \lambda A_1$ at λ_0 and turning into 1 the second ρ_1 largest partial multiplicities.

A matrix pencil $A_0+\lambda A_1$ arises naturally associated with the ordinary differential-algebraic equation

$$A_1 x'(t) + A_0 x(t) = f(t), \tag{1.2}$$

with $f(t)$ being a differentiable vector function. Low rank perturbations appear, for instance, when introducing modifications in the system associated to the equation (1.2) that only affect a small number of parameters (entries of the matrix pencil), regardless of the norm of the modification. Since the Weierstrass canonical form of $A_0 + \lambda A_1$ determines the solution of (1.2) [11, Ch. XII, §7], the description of the change of the Weierstrass canonical form under low rank perturbations becomes a relevant issue in this applied context (see, for instance, [10]).

In recent years, considerable interest has awakened in describing the change of canonical forms of structured matrices and matrix pencils under low rank perturbations, due to their connection with applications [1, 2, 3, 4, 14, 15, 16, 17, 18]. This has been one of our motivations to revisit this issue.

The paper is organized as follows. In Section 2 we introduce and recall the basic notation and definitions used along the paper. In Section 3 we present the main result of the paper (Theorem 3.4), together with its proof. This requires to introduce and prove several preliminary results, which are included in Section 3. Theorem 3.4 is rewritten in Section 4 (see Theorem 4.3) in a way that makes more explicit the genericity of the behavior described in Theorem 3.4. In Section 5 we further analyze the case of rank-1 perturbations, and we provide an interpretation of the approach followed in Section 3 in terms of the expression of the set of matrix pencils with normal rank at most 1 as the union of two irreducible components. Finally, in Section 6 we summarize the contributions of the paper.

2. Basic notation and definitions. We deal in this paper with square matrix pencils over the complex field, that is, $M_0 + \lambda M_1$, with $M_0, M_1 \in \mathbb{C}^{n\times n}$. The matrix pencil $M_0 + \lambda M_1$ is said to be *regular* if $\det(M_0 + \lambda M_1)$ is not identically zero as a polynomial in λ, and it is said to be *singular* otherwise. A *finite eigenvalue* of the regular matrix pencil $M_0+\lambda M_1$ is a complex value $\lambda_0 \in \mathbb{C}$ such that $\det(M_0+\lambda_0 M_1) = 0$. The pencil $M_0 + \lambda M_1$ has an *infinite eigenvalue* if $\lambda_0 = 0$ is an eigenvalue of the *reversal pencil* $M_1 + \lambda M_0$. The *normal rank* of a matrix pencil $M_0 + \lambda M_1$, denoted by $\operatorname{nrank}(M_0 + \lambda M_1)$, is the size of the largest non-identically zero minor of $M_0 + \lambda M_1$ (that is, the rank of $M_0 + \lambda M_1$ when considered as a matrix over the field of rational functions in λ with complex coefficients).

The *Weierstrass canonical form* (WCF) of a regular pencil $M_0 + \lambda M_1$ is a block diagonal matrix pencil, uniquely determined up to permutation of the diagonal blocks, which is strictly equivalent to $M_0 + \lambda M_1$ and reveals all its spectral information (see the classical reference [11, Ch. XII, §2] or the more recent one [7, p. 540], that follows the notation of this paper).

For the definition of *partial multiplicities* of a matrix pencil (or, more in general, of a matrix polynomial) at a finite eigenvalue λ_0 we refer the reader to [12, S1.5]. In plain words, the nonzero partial multiplicities of $M_0 + \lambda M_1$ at λ_0 are the sizes of the Jordan blocks associated with λ_0 in the WCF of $M_0 + \lambda M_1$. The number of nonzero partial multiplicities at λ_0 is the *geometric multiplicity of* λ_0 as an eigenvalue of $M_0 + \lambda M_1$, and the sum of all partial multiplicities is the *algebraic multiplicity of* λ_0 in $M_0 + \lambda M_1$. The algebraic multiplicity of λ_0 in $M_0 + \lambda M_1$ coincides with the multiplicity of λ_0 as a root of the *characteristic polynomial* of $M_0 + \lambda M_1$, namely $\det(M_0 + \lambda M_1)$. The geometric multiplicity of λ_0 at the unperturbed pencil $A_0 + \lambda A_1$ is denoted by g. For the infinite eigenvalue, the partial multiplicities and the algebraic and geometric multiplicity are the corresponding ones for the eigenvalue zero of the reversal pencil $M_1 + \lambda M_0$.

We denote by $\mathbb{P}_r$ the set of $n \times n$ matrix pencils with complex coefficients having normal rank at most r, that is:

$$\mathbb{P}_r := \{M_0 + \lambda M_1 : \; M_0, M_1 \in \mathbb{C}^{n\times n}, \; \operatorname{nrank}(M_0 + \lambda M_1) \leq r\}.$$

3. Main result. From the practical point of view, a relevant question when dealing with $n \times n$ matrix pencils with normal rank at most $r < n$ is: How are these pencils constructed? We give an answer to this question by providing a decomposition of $\mathbb{P}_r$ into $r + 1$ sets containing explicitly constructible pencils with normal rank at most r. This decomposition will be key in proving the main result of the paper (Theorem 3.4). The construction is based on Lemma 2.8 in [5], which states that any matrix pencil with normal rank at most r is a sum of r matrix pencils with normal rank at most 1 of the form:

$$v_1(\lambda)w_1(\lambda)^T + \cdots + v_r(\lambda)w_r(\lambda)^T, \tag{3.1}$$

where $v_1(\lambda), \ldots, v_r(\lambda), w_1(\lambda), \ldots, w_r(\lambda)$ are vectors with n coordinates which are polynomials of degree at most one in λ, in such a way that one of $v_i(\lambda)$ or $w_i(\lambda)$ is constant, for each $i = 1, \ldots, r$. This leads to Lemma 3.1. In the statement of this lemma, $v_i(\lambda)$ and $w_j(\lambda)$ denote vector polynomials with n coordinates, that is, vectors whose entries are polynomials in λ. The *degree* of a vector polynomial $v(\lambda)$, denoted by $\deg v$, is the maximum degree of its coordinates.

LEMMA 3.1. *Let $r \leq n$ be an integer. For each $s = 0, 1, \ldots, r$, define*

$$\mathcal{C}_s := \left\{ v_1(\lambda)w_1(\lambda)^T + \cdots + v_r(\lambda)w_r(\lambda)^T \; : \; \begin{array}{l} \deg v_i \leq 1, \text{for } i = 1, \ldots, r, \\ \deg w_j \leq 1, \text{for } j = 1, \ldots, r, \\ \deg v_1 = \cdots = \deg v_s = 0, \\ \deg w_{s+1} = \cdots = \deg w_r = 0 \end{array} \right\}.$$

Then:

$$\mathbb{P}_r = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r \,. \tag{3.2}$$

Proof. The proof is similar to the one of Lemma 2.8 in [5], and it is a consequence of the *Kronecker canonical form* (KCF) of singular matrix pencils [11, Ch. XII, §4]. Let us recall the basic features of the KCF. It is an extension of the WCF which is valid for singular pencils, and consists of diagonal blocks of the form

$$L_\varepsilon = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}_{\varepsilon\times(\varepsilon+1)}, \qquad L_\eta^T = \begin{bmatrix} \lambda & & \\ 1 & \ddots & \\ & \ddots & \lambda \\ & & 1 \end{bmatrix}_{(\eta+1)\times\eta},$$

$$J_k(\lambda-\lambda_0) = \begin{bmatrix} \lambda-\lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda-\lambda_0 \end{bmatrix}_{k\times k}, \quad N_\ell(\lambda) = \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix}_{\ell\times\ell}.$$

Any matrix pencil is strictly equivalent to a direct sum of blocks of these four types, which is unique up to permutation of the blocks [11, Ch. XII, §4]. In addition, the sum of the sizes, $\varepsilon' s$, $\eta' s$, and $k' s$ of all blocks of the KCF is precisely the normal rank of the pencil. Hence, it suffices to prove the result for matrix pencils being in KCF. Now, we consider the following decompositions of the previous blocks as in (3.1):

$$\begin{aligned}
L_\varepsilon &= e_1 \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \end{bmatrix} + e_2 \begin{bmatrix} 0 & \lambda & 1 & 0 & \dots & 0 \end{bmatrix} + \cdots \\
&\quad + e_\varepsilon \begin{bmatrix} 0 & \dots & 0 & \lambda & 1 \end{bmatrix}, \\
L_\eta^T &= \begin{bmatrix} \lambda \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_1^T + \begin{bmatrix} 0 \\ \lambda \\ 1 \\ \vdots \\ 0 \end{bmatrix} e_2^T + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda \\ 1 \end{bmatrix} e_\eta^T, \\
J_k(\lambda-\lambda_0) &= \begin{bmatrix} \lambda-\lambda_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_1^T + \begin{bmatrix} 1 \\ \lambda-\lambda_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_2^T + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda-\lambda_0 \end{bmatrix} e_k^T \\
&= e_1 \begin{bmatrix} \lambda-\lambda_0 & 1 & \dots & 0 \end{bmatrix} + \cdots + e_{k-1} \begin{bmatrix} 0 & \dots & 0 & \lambda-\lambda_0 & 1 \end{bmatrix} \\
&\quad + e_k \begin{bmatrix} 0 & \dots & 0 & \lambda-\lambda_0 \end{bmatrix}, \\
N_\ell(\lambda) &= e_1 e_1^T + \begin{bmatrix} \lambda \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_2^T + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda \\ 1 \end{bmatrix} e_\ell^T \\
&= e_1 \begin{bmatrix} 1 & \lambda & 0 & \cdots & 0 \end{bmatrix} + \cdots + e_{\ell-1} \begin{bmatrix} 0 & \cdots & 0 & 1 & \lambda \end{bmatrix} + e_\ell e_\ell^T,
\end{aligned}$$

where e_i is the ith canonical vector in $\mathbb{C}^m$ (that is, the ith column of the $m\times m$ identity matrix), with m depending on the size of each block. Combining these expressions, it is straightforward to see that $\mathbb{P}_r \subseteq \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$. The other inclusion is trivial. □

Lemma 3.1 tells us that one way to generate $n \times n$ matrix pencils with normal rank at most $r < n$ is by first fixing a number $s = 0, 1, \ldots, r$, and then generate vector polynomials $v_i(\lambda), w_j(\lambda)$, according to the degree restrictions in $\mathcal{C}_s$. Hence, Lemma 3.1 presents a constructive decomposition of $\mathbb{P}_r$.

REMARK 1. *The sets $\mathcal{C}_s$, for $s = 0, 1, \ldots, r$, are not disjoint sets, so that, for a given pencil $M(\lambda)$, with* $\operatorname{nrank}(M(\lambda)) \leq r$, *the degrees of the vectors in the decomposition* (3.1) *are not necessarily unique. From the proof of Lemma* 3.1, *it follows that those pencils whose KCF contains blocks $J_k(\lambda-\lambda_0)$ or N_ℓ belong simultaneously to different sets $\mathcal{C}_s$.*

The following definition introduces a coordinate transform for pencils in each set $\mathcal{C}_s$ introduced in Lemma 3.1.

DEFINITION 3.2. *For each $s = 0, 1, \ldots, r$, let us decompose a vector $x \in \mathbb{C}^{3rn}$ as*

$$x = \left[\begin{array}{c|c|c|c} \alpha & \beta & \gamma & \delta \end{array}\right]^T,$$

where

$$\begin{aligned}
\alpha &= \left[\begin{array}{ccc|c|ccc} \alpha_{11} & \cdots & \alpha_{n1} & \cdots & \alpha_{1r} & \cdots & \alpha_{nr} \end{array}\right] \in \mathbb{C}^{1\times rn},\\
\beta &= \left[\begin{array}{ccc|c|ccc} \beta_{1,s+1} & \cdots & \beta_{n,s+1} & \cdots & \beta_{1r} & \cdots & \beta_{nr} \end{array}\right] \in \mathbb{C}^{1\times (r-s)n},\\
\gamma &= \left[\begin{array}{ccc|c|ccc} \gamma_{11} & \cdots & \gamma_{n1} & \cdots & \gamma_{1r} & \cdots & \gamma_{nr} \end{array}\right] \in \mathbb{C}^{1\times rn},\\
\delta &= \left[\begin{array}{ccc|c|ccc} \delta_{11} & \cdots & \delta_{n1} & \cdots & \delta_{1s} & \cdots & \delta_{ns} \end{array}\right] \in \mathbb{C}^{1\times ns}.
\end{aligned}$$

Then, let us define the map $\Phi_s : \mathbb{C}^{3rn} \longrightarrow \mathcal{C}_s$ as

$$\Phi_s(x) = v_1(\lambda)w_1(\lambda)^T + \cdots + v_r(\lambda)w_r(\lambda)^T,$$

where

$$\begin{aligned}
v_i(\lambda) &= \left[\begin{array}{ccc} \alpha_{1i} & \ldots & \alpha_{ni} \end{array}\right]^T, && \text{for } i = 1, \ldots, s,\\
v_j(\lambda) &= \left[\begin{array}{ccc} \alpha_{1j} + \lambda\beta_{1j} & \ldots & \alpha_{nj} + \lambda\beta_{nj} \end{array}\right]^T, && \text{for } j = s+1, \ldots, r,\\
w_i(\lambda) &= \left[\begin{array}{ccc} \gamma_{1i} + \lambda\delta_{1i} & \ldots & \gamma_{ni} + \lambda\delta_{ni} \end{array}\right]^T, && \text{for } i = 1, \ldots, s,\\
w_j(\lambda) &= \left[\begin{array}{ccc} \gamma_{1j} & \ldots & \gamma_{nj} \end{array}\right]^T, && \text{for } j = s+1, \ldots, r.
\end{aligned}$$

REMARK 2. *The map Φ_s introduced in Definition* 3.2 *is surjective, but it is not injective, since different vectors $x \in \mathbb{C}^{3rn}$ can give the same pencil $\Phi_s(x)$.*

An *algebraic set* in $\mathbb{C}^m$ is the set of common zeroes of a finite number of multivariable polynomials with m variables and coefficients in $\mathbb{C}$. The algebraic set is *proper* if it is not the whole $\mathbb{C}^m$. This allows us to introduce the following notion of generic sets in $\mathbb{C}^m$ (see [1, Def. 2.2]).

DEFINITION 3.3. *A* generic *set of $\mathbb{C}^m$ is a subset of $\mathbb{C}^m$ whose complement is contained in a proper algebraic set.*

The main purpose of the present paper is to prove the following result.

THEOREM 3.4. *Let λ_0 be an eigenvalue (finite or infinite) of the regular complex $n \times n$ matrix pencil $A_0 + \lambda A_1$, with nonzero partial multiplicities $n_1 \geq n_2 \geq \cdots \geq n_g > 0$, and let $0 < r < g$ be an integer. For each $s = 0, 1, \ldots, r$, let Φ_s be the map in Definition* 3.2. *Then, for each $s = 0, 1, \ldots, r$, there is a generic set G_s in $\mathbb{C}^{3rn}$ such that, for all $B_0 + \lambda B_1 \in \Phi_s(G_s)$, the partial multiplicities of the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$ at λ_0 are $n_{r+1} \geq \cdots \geq n_g$.*

Proof. We first prove the result for $\lambda_0 \in \mathbb{C}$ being a finite eigenvalue of $A_0 + \lambda A_1$.

Set $\widetilde{a} := n_{r+1} + \cdots + n_g$. By [7, Lemma 2.1], if $B_0 + \lambda B_1$, with normal rank at most r, is such that $A_0 + B_0 + \lambda(A_1 + B_1)$ is regular, then there are at least $g - r$ nonzero partial multiplicities of $A_0 + B_0 + \lambda(A_1 + B_1)$ at λ_0, $m_{r+1} \geq \cdots \geq m_g > 0$, satisfying $m_i \geq n_i$, for $i = r+1, \ldots, g$. This means that the algebraic multiplicity of λ_0 in $A_0 + B_0 + \lambda(A_1 + B_1)$ is at least $\widetilde{a}$. In other words, the characteristic polynomial of $A_0 + B_0 + \lambda(A_1 + B_1)$ is of the form:

$$\det(A_0 + B_0 + \lambda(A_1 + B_1)) = (\lambda - \lambda_0)^{\widetilde{a}} q_{B_0 + \lambda B_1}(\lambda - \lambda_0),$$

for some nonzero polynomial $q_{B_0+\lambda B_1}(\lambda - \lambda_0)$. Following the reasoning in [7, p. 544], $q_{B_0+\lambda B_1}(0)$ is the coefficient of $(\lambda - \lambda_0)^{\widetilde{a}}$ in the characteristic polynomial of $A_0 + B_0 + \lambda(A_1 + B_1)$, expanded in powers of $(\lambda - \lambda_0)$. Since $A_0 + \lambda A_1$ is fixed, this coefficient is a multivariate polynomial in the entries of B_0 and B_1. Hence, the set

$$\mathcal{B}_s := \{x \in \mathbb{C}^{3rn} : q_{\Phi_s(x)}(0) = 0\}$$

is an algebraic set of $\mathbb{C}^{3rn}$. Note that if we define $G_s := \mathbb{C}^{3rn} \setminus \mathcal{B}_s$ and $x \in G_s$, then $A_0 + \lambda A_1 + \Phi_s(x)$ is a regular matrix pencil, since at least one of the coefficients of its characteristic polynomial expanded in powers of $(\lambda - \lambda_0)$ is nonzero. Moreover, if $x \in G_s$ then the algebraic multiplicity of λ_0 in $A_0 + \lambda A_1 + \Phi_s(x)$ is exactly $\widetilde{a}$ and, by [7, Lemma 2.1], the partial multiplicities of $A_0 + \lambda A_1 + \Phi_s(x)$ at λ_0 are exactly $n_{r+1} \geq \cdots \geq n_g$. Therefore, the rest of the proof reduces to prove that G_s is generic.

In order to prove that G_s is generic, it remains to prove that it is nonempty (i. e., that $\mathcal{B}_s$ is proper). For this, we are going to construct a perturbation pencil $B_0 + \lambda B_1$ such that $B_0 + \lambda B_1 = \Phi_s(x)$, for some $x \in \mathbb{C}^{3rn}$, and such that $A_0 + B_0 + \lambda(A_1 + B_1)$ has partial multiplicities $n_{r+1} \geq \cdots \geq n_g$ at λ_0, which implies $q_{\Phi_s(x)} \neq 0$. It suffices to find such a pencil for $A_0 + \lambda A_1$ being in WCF. To see this, let us assume that the result is true for the original (unperturbed) pencil being in WCF. Now, let $A_0 + \lambda A_1$ be an arbitrary pencil in the conditions of the statement such that $P(A_0 + \lambda A_1)Q = J_0 + \lambda J_1$ is in WCF, with P, Q constant invertible matrices. Assume that there is a pencil $B_0 + \lambda B_1 = \Phi_s(x)$ with normal rank at most r such that $J_0 + B_0 + \lambda(J_1 + B_1)$ has nonzero partial multiplicities $n_{r+1} \geq \cdots \geq n_g$ at λ_0. Hence, the pencil $A_0 + P^{-1}B_0Q^{-1} + \lambda(A_1 + P^{-1}B_1Q^{-1})$ is a perturbation of $A_0 + \lambda A_1$ with nonzero partial multiplicities $n_{r+1} \geq \cdots \geq n_g$ at λ_0, the normal rank of $P^{-1}B_0Q^{-1} + \lambda P^{-1}B_1Q^{-1}$ is at most r and $P^{-1}B_0Q^{-1} + \lambda P^{-1}B_1Q^{-1} \in \mathcal{C}_s$.

To prove the result for $A_0 + \lambda A_1$ being in WCF, let

$$A_0 + \lambda A_1 = \operatorname{diag}(J_{n_1}(\lambda - \lambda_0), \ldots, J_{n_g}(\lambda - \lambda_0), \widetilde{J} + \lambda I, I_\infty + \lambda N),$$

where $\widetilde{J}$ is a direct sum of Jordan blocks associated with eigenvalues other than $-\lambda_0$, I is the identity matrix with the same size as $\widetilde{J}$, N is a nilpotent matrix in Jordan canonical form, and I_∞ is the identity matrix with the same size as N (see equation (1.4) in [7]). Let $E_k(\beta)$ be the $k \times k$ matrix that is everywhere zero except for β in the $(k, 1)$ entry. The perturbation pencil

$$B_0 + \lambda B_1 = \operatorname{diag}(E_{n_1}(1), \ldots, E_{n_r}(1), 0) + \lambda\, 0_{n \times n} \tag{3.3}$$

has normal rank r and is such that $A_0 + B_0 + \lambda(A_1 + B_1)$ has nonzero partial multiplicities $n_{r+1} \geq \cdots \geq n_g$ at λ_0 (see [7, Lemma 3.1]). Moreover, $B_0 + \lambda B_1 = \Phi_s(x)$, for some $x \in \mathbb{C}^{3rn}$, for all $s = 0, 1 \ldots, r$ simultaneously. To see this, just note that, since $B_0 + \lambda B_1$ in (3.3) is a constant matrix, then $B_0 + \lambda B_1 \in \mathcal{C}_0 \cap \mathcal{C}_1 \cap \cdots \cap \mathcal{C}_r$, and then use that Φ_s is surjective (see Remark 2). This concludes the proof for the case where λ_0 is a finite eigenvalue of $A_0 + \lambda A_1$.

For $\lambda_0 = \infty$, consider the reversal pencils $A_1 + \lambda A_0$ and $B_1 + \lambda B_0$ and apply the result for the eigenvalue $\lambda_0 = 0$ of $A_1 + \lambda A_0$. □

Theorem 3.4 establishes the generic change of the WCF of $A_0+\lambda A_1$ under perturbations with normal rank at most r through the decomposition (3.2) using Definition 3.3 of genericity in $\mathbb{C}^{3rn}$. Another proof that the behavior described in Theorem 3.4 is generic, working directly on $\mathbb{P}_r$ without referring to the "parameter space" $\mathbb{C}^{3rn}$ is given in Section 4.

Another natural way to generate matrix pencils $B_0 + \lambda B_1$ with low normal rank at most r, which is different from the one suggested in Lemma 3.1, is by generating the coefficients B_0 and B_1 as low rank matrices such that $\operatorname{rank}(B_0) + \operatorname{rank}(B_1) \leq r$, since $\operatorname{nrank}(B_0 + \lambda B_1) \leq \operatorname{rank}(B_0) + \operatorname{rank}(B_1)$. This approach, however, does not give generic pencils with normal rank at most r, where by "generic pencils with normal rank at most r" we understand those pencils with KCFs as in [6, Th. 3.2]. Nonetheless this is, in some sense, the approach followed in [7]. As we have mentioned in the Introduction, the low-rank condition in [7] is $\rho_0 < g$ (see (1.1)), instead of $r < g$. This condition has the advantage of allowing for $r \geq g$ (see part (ii) in the proof of Lemma 3.2 in [7]). However, in the proof of [7, Th. 2.2], the authors consider the condition $\rho < g$, with ρ as in (1.1), and they focus on those perturbations $B_0 + \lambda B_1$ for which the inequality

$$\operatorname{nrank}(B_0 + \lambda B_1) = \operatorname{nrank}(B_0 + \lambda_0 B_1 + (\lambda - \lambda_0)B_1) \leq \operatorname{rank}(B_0 + \lambda_0 B_1) + \operatorname{rank}(B_1)$$

is an equality (see [7, p. 543]). For generic matrix pencils with normal rank at most r this inequality is strict. This is a consequence of the fact that generic matrix pencils with normal rank at most r do not have eigenvalues (neither finite nor infinite) [6, Th. 3.2]. Hence, if $B_0 + \lambda B_1$ is "generic with normal rank at most $r < n$" it holds that

$$r = \operatorname{rank}(B_0 + \lambda_0 B_1) = \operatorname{rank}(B_1),$$

for any $\lambda_0 \in \mathbb{C}$. Hence, the authors in [7] are not considering generic perturbations of normal rank at most r. To be precise, an $n \times n$ matrix pencil $B_0 + \lambda B_1$ with normal rank $\rho < n$, with ρ as in (1.1), is a singular pencil that has λ_0 as an eigenvalue with geometric multiplicity ρ_1, and an infinite eigenvalue with geometric multiplicity ρ_0.

The recent paper [1, Th. 2.10] has described the generic behavior in Theorem 3.4 for some particular rank-1 perturbations. However, the author of that paper considers perturbations of the form $-\alpha uv^T + \lambda \beta uv^T$, with $(\alpha, \beta) \in (\mathbb{C} \times \mathbb{C}) \setminus \{0\}$ and $u, v \in \mathbb{C}^n$ $u \neq 0, v \neq 0$, which are not generic perturbations with normal rank 1, since they have an eigenvalue $\lambda_0 = \alpha/\beta$.

4. Genericity in the subspace topology. Theorem 3.4 describes the change of the WCF of a pencil $A_0 + \lambda A_1$ under perturbations of low normal rank r through the following procedure:

1. Decompose the set of perturbations as the union of $r + 1$ subsets, $\mathcal{C}_0, \ldots, \mathcal{C}_r$.
2. Introduce the space $\mathbb{C}^{3rn}$ as a "parametrization" of each $\mathcal{C}_i$, for $i = 0, 1, \ldots, r$.
3. Find a generic set (complementary of an algebraic set) in each space of parameters, $\mathbb{C}^{3rn}$, for which the "generic" behavior holds.

In this Section, we are going to see that this "generic" behavior can be also established by working directly on $\mathbb{P}_r$. For this, we introduce a notion of genericity in $\mathbb{P}_r$ and then we adapt the proof of Theorem 3.4 working directly on $\mathbb{P}_r$ to state again the genericity of the behavior described in Theorem 3.4 from a different and more intrinsic perspective. The result that we achieve is Theorem 4.3.

If we denote by $\mathbb{P}$ the set of $n \times n$ matrix pencils with complex entries and we consider the natural coordinate mapping:

$$\begin{array}{cccc} \psi: & \mathbb{C}^{2n^2} & \longrightarrow & \mathbb{P} \\ & \begin{bmatrix} a \\ b \end{bmatrix} & \mapsto & A+\lambda B \end{array} \tag{4.1}$$

with

$$\begin{array}{ll} a = \left[\begin{array}{ccc|c|ccc} a_{11} & \dots & a_{1n} & \dots & a_{n1} & \dots & a_{nn} \end{array}\right]^T, & A = [a_{ij}]\,, \\ b = \left[\begin{array}{ccc|c|ccc} b_{11} & \dots & b_{1n} & \dots & b_{n1} & \dots & b_{nn} \end{array}\right]^T, & B = [b_{ij}]\,, \end{array}$$

then there is a natural induced topology, $\mathcal{T}_{\mathbb{P}}$, in $\mathbb{P}$, namely: $V \subseteq \mathbb{P}$ is an open set in $\mathcal{T}_{\mathbb{P}}$ if $\psi^{-1}(V)$ is an open set in the standard topology of $\mathbb{C}^{2n^2}$. Note that, with this topology, ψ is a continuous map. We consider in $\mathbb{P}_r$ the *subspace topology* of $\mathcal{T}_{\mathbb{P}}$, namely, an open (respectively, closed) set in $\mathbb{P}_r$ is the intersection of $\mathbb{P}_r$ with an open (resp., closed) set in $\mathcal{T}_{\mathbb{P}}$. With all these ingredients, we can state the following result.

LEMMA 4.1. *For $s = 0, 1, \dots, r$, the map $\Phi_s : \mathbb{C}^{3rn} \longrightarrow \mathcal{C}_s \subseteq \mathbb{P}_r$ introduced in Definition 3.2 is continuous in the standard topology of $\mathbb{C}^{3rn}$ and the subspace topology of $\mathcal{T}_{\mathbb{P}}$ in $\mathbb{P}_r$.*

Proof. The map Φ_s can be expressed as the composition $\Phi_s = \psi \circ \delta$, where $\psi : \mathbb{C}^{2n^2} \longrightarrow \mathbb{P}$ is the map (4.1) and $\delta : \mathbb{C}^{3rn} \longrightarrow \mathbb{C}^{2n^2}$ is the map that takes the coefficients of the polynomials $v_i(\lambda), w_j(\lambda)$ in Definition 3.2, for $i, j = 0, 1, \dots, r$, constructs via sums and multiplications the entries of the corresponding pencil, and forms with these entries a vector in $\mathbb{C}^{2n^2}$. The map ψ is continuous, as mentioned above, and the map δ is a polynomial function, so it is continuous as well. Then, Φ_s is continuous, since it is the composition of continuous functions. Notice that the precedent argument shows that Φ_s is continuous in the topology $\mathcal{T}_{\mathbb{P}}$ of $\mathbb{P}$. But this implies that it is also continuous in the subspace topology in $\mathbb{P}_r$. □

Since the codomain of the map Φ_s is $\mathcal{C}_s$, instead of $\mathbb{P}_r$, we could have used in Lemma 4.1 the subspace topology in $\mathcal{C}_s$, instead of $\mathbb{P}_r$. However, we have stated Lemma 4.1 using the subspace topology in $\mathbb{P}_r$ because this is the topology that we need in Theorem 4.3.

Given a topological space $(X, \mathcal{T})$, we say that a subset $D \subseteq X$ is *dense* in X if $\overline{D} = X$, where $\overline{D}$ denotes the closure of D in $\mathcal{T}$. This allows us to introduce the following intrinsic notion of genericity in $\mathbb{P}_r$.

DEFINITION 4.2. *A* generic *set of $\mathbb{P}_r$ is a dense open subset of $\mathbb{P}_r$ (in the subspace topology of $\mathcal{T}_{\mathbb{P}}$ in $\mathbb{P}_r$).*

Note that Definition 3.3 of genericity is more restrictive than Definition 4.2 in $\mathbb{C}^m$ (instead of $\mathbb{P}_r$). More precisely, the complementary of any proper algebraic set in $\mathbb{C}^m$ is a dense open set in the standard topology of $\mathbb{C}^m$ (we use this fact in the proof of Theorem 4.3). However, the converse is not true, that is, not any dense open set in the standard topology of $\mathbb{C}^m$ is the complementary of a proper algebraic set.

The reason to introduce Definition 4.2 for generic sets in $\mathbb{P}_r$ instead of Definition 3.3 just replacing $\mathbb{C}^m$ by $\mathbb{P}_r$ is that, though $\mathbb{P}_r$ is an algebraic set, it is not *irreducible*

[21, p. 228] unless $r = 0$. More precisely, $\mathbb{P}_r$ is an algebraic set which is the union of $r + 1$ irreducible algebraic sets (the irreducible components of $\mathbb{P}_r$) [6, Th. 3.5]. When an algebraic set $\mathcal{A}$ in $\mathbb{C}^m$ is not irreducible, it can contain a subset $\mathcal{S}$ whose complement $\mathcal{A} \setminus \mathcal{S}$ is contained in a proper algebraic subset of $\mathcal{A}$ but such that $\mathcal{S}$ can not be considered "generic" in $\mathcal{A}$ in any reasonable sense. Think, for instance of $\mathcal{A}$ being the set of zeroes in $\mathbb{C}^2$ of the polynomial $xy = 0$ (that is, the union of the coordinate axes). Then $\mathcal{S} := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 : x = 0 \right\}$ is a subset of $\mathcal{A}$ whose complement $\mathcal{A} \setminus \mathcal{S} := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 0, x \neq 0 \right\}$ is included in the proper algebraic set of $\mathcal{A}$ defined by $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 0 \right\}$. However, $\mathcal{S}$ is far from being generic in $\mathcal{A}$ in any reasonable sense.

Now, Theorem 3.4 can be restated in the following way.

Theorem 4.3. *Let λ_0 be an eigenvalue (finite or infinite) of the regular $n \times n$ complex matrix pencil $A_0 + \lambda A_1$, with nonzero partial multiplicities $n_1 \geq n_2 \geq \cdots \geq n_g > 0$, let $0 < r < g$ be an integer, and denote by $\mathbb{P}_r$ the set of $n \times n$ matrix pencils with normal rank at most r. Then, there is a generic set, G, in $\mathbb{P}_r$ such that for all $B_0 + \lambda B_1 \in G$, the partial multiplicities of the perturbed pencil $A_0 + B_0 + \lambda(A_1 + B_1)$ at λ_0 are $n_{r+1} \geq \cdots \geq n_g$.*

Proof. As in the proof of Theorem 3.4, we may focus on λ_0 being a finite eigenvalue, since for $\lambda_0 = \infty$ we can apply the result for the eigenvalue zero in the reversal pencils $A_1 + \lambda A_0$ and $B_1 + \lambda B_0$.

Let $q_{B_0+\lambda B_1}(\lambda - \lambda_0)$ be as in the proof of Theorem 3.4. Following the arguments in the first part of the proof of Theorem 3.4, it suffices to prove that if we define

$$\mathcal{C}_{q(0)} := \{B_0 + \lambda B_1 \in \mathbb{P}_r : \; q_{B_0+\lambda B_1}(0) = 0\}$$

then $\mathbb{P}_r \setminus \mathcal{C}_{q(0)}$ is a dense open subset of $\mathbb{P}_r$, because all perturbation pencils $B_0 + \lambda B_1$ in $\mathbb{P}_r \setminus \mathcal{C}_{q(0)}$ satisfy the generic behavior in the statement.

Let us first prove that $\mathcal{C}_{q(0)}$ is a closed set in the subspace topology of $\mathcal{T}_{\mathbb{P}}$ in $\mathbb{P}_r$. For this, let $\mathcal{C}_{\widetilde{a}}$ be the set of general $n \times n$ pencils $\widetilde{B}_0 + \lambda \widetilde{B}_1$, i. e., with any normal rank, such that the coefficient of the $(\lambda - \lambda_0)^{\widetilde{a}}$ term in $\det(A_0 + \widetilde{B}_0 + \lambda(A_1 + \widetilde{B}_1))$ when it is expanded in powers of $(\lambda - \lambda_0)$ is zero. This set $\mathcal{C}_{\widetilde{a}}$ is a closed set in $\mathcal{T}_{\mathbb{P}}$, because $\psi^{-1}(\mathcal{C}_{\widetilde{a}})$ is a closed set in the standard topology of $\mathbb{C}^{2n^2}$ (in fact, it is an algebraic set in $\mathbb{C}^{2n^2}$). Since $\mathcal{C}_{q(0)} = \mathbb{P}_r \cap \mathcal{C}_{\widetilde{a}}$, then $\mathcal{C}_{q(0)}$ is a closed set in the subspace topology of $\mathcal{T}_{\mathbb{P}}$ in $\mathbb{P}_r$.

Now, let us prove that $\mathbb{P}_r \setminus \mathcal{C}_{q(0)}$ is dense in $\mathbb{P}_r$. We are going to use the following basic fact, whose proof is straightforward:

(F) If X, Y are topological spaces and $f : X \to Y$ is a continuous function, then $f(\overline{S}) \subseteq \overline{f(S)}$, for all $S \subseteq X$.

Since

$$\bigcup_{s=0}^{r} \overline{(\mathcal{C}_s \setminus \mathcal{C}_{q(0)})} \subseteq \overline{\bigcup_{s=0}^{r} (\mathcal{C}_s \setminus \mathcal{C}_{q(0)})} = \overline{\mathbb{P}_r \setminus \mathcal{C}_{q(0)}},$$

then it suffices to prove that

$$\overline{\mathcal{C}_s \setminus \mathcal{C}_{q(0)}} = \mathcal{C}_s,$$

that is, $\mathcal{C}_s \setminus \mathcal{C}_{q(0)}$ is dense in $\mathcal{C}_s$, for $s = 0, 1, \ldots, r$.

Let Φ_s be as in Definition 3.2, and define

$$\mathcal{G}_s := \Phi_s^{-1}(\mathcal{C}_{q(0)}) = \{x \in \mathbb{C}^{3rn} : \Phi_s(x) \in \mathcal{C}_{q(0)}\}.$$

The set $\mathcal{G}_s$ is clearly an algebraic set in $\mathbb{C}^{3rn}$. Moreover, it is proper. To see this, recall that Φ_s is surjective and note that if $B_0 + \lambda B_1$ is as in (3.3) (multiplied adequately by the inverses of the matrices that transform $A_0 + \lambda A_1$ into its WCF), then there is some $x \in \mathbb{C}^{3rn}$ such that $\Phi_s(x) = B_0 + \lambda B_1 \notin \mathcal{C}_{q(0)}$, so $x \notin \mathcal{G}_s$. Therefore, $\mathbb{C}^{3rn} \setminus \mathcal{G}_s$ is open and dense in $\mathbb{C}^{3rn}$. Now, since Φ_s is surjective and continuous, by Lemma 4.1, fact (F) above implies that $\Phi_s(\mathbb{C}^{3rn} \setminus \mathcal{G}_s)$ is dense in $\mathcal{C}_s$. But $\Phi_s(\mathbb{C}^{3rn} \setminus \mathcal{G}_s) = \mathcal{C}_s \setminus \mathcal{C}_{q(0)}$, so $\mathcal{C}_s \setminus \mathcal{C}_{q(0)}$ is dense in $\mathcal{C}_s$, as wanted. □

5. Particular case: Matrix pencils with normal rank at most 1**.** The description of the geometry of the spaces of matrices and matrix pencils is a useful tool when analyzing the change of canonical structures under perturbations [8, 9, 21]. In particular, the geometric description of the set of perturbations may shed light in explaining the generic change of canonical structures. In the recent years, much interest has been devoted to analyze the change of canonical forms of structured matrices and matrix pencils under structured perturbations of (normal) rank 1 [1, 3, 14, 15, 16, 17]. Hence, the case of perturbations with (normal) rank 1 is of particular interest. In this section, we analyze the set of matrix pencils with normal rank at most 1 (that is, $\mathbb{P}_1$ following the notation used along the paper) and we connect its geometric description provided in [6] with the decomposition given in Lemma 3.1. In particular, we show that the two irreducible components of $\mathbb{P}_1$ described in [6] coincide with the sets $\mathcal{C}_0$ and $\mathcal{C}_1$ introduced in Lemma 3.1.

Let us recall that the *orbit under strict equivalence* of a matrix pencil $M(\lambda)$ is the set

$$\mathcal{O}(M) := \{PM(\lambda)Q \ : \ P, Q \text{ nonsingular}\}.$$

The irreducible components of $\mathbb{P}_1$ [6, Th. 3.5] are the closures, in $\mathcal{T}_{\mathbb{P}}$, of the orbits under strict equivalence, $\mathcal{O}(K_0), \mathcal{O}(K_1)$ (using the notation in [6]) of the following pencils in KCF:

$$K_0(\lambda) = \operatorname{diag}(L_1^T, 0_{(n-2)\times(n-1)}), \qquad K_1(\lambda) = \operatorname{diag}(L_1, 0_{(n-1)\times(n-2)})\,. \tag{5.1}$$

The following result relates these closures with the sets $\mathcal{C}_s$ introduced in Lemma 3.1. We denote by $\overline{\mathcal{O}}(M)$ the closure of the orbit under strict equivalence of the pencil M (in $\mathcal{T}_{\mathbb{P}}$).

PROPOSITION 5.1. *Let $\mathcal{C}_0, \mathcal{C}_1$ be the sets defined in Lemma* 3.1 *for $r = 1$, and let K_0, K_1 be the pencils in* (5.1)*. Then*

$$\overline{\mathcal{O}}(K_0) = \mathcal{C}_0 \qquad \text{and} \qquad \overline{\mathcal{O}}(K_1) = \mathcal{C}_1.$$

Proof. Any nonzero matrix pencil in $\mathbb{P}_1$ has one of the following KCF's (we drop the dependence on λ for brevity):

$$K_0, \qquad K_1, \qquad K_2 = (\lambda - \lambda_0)e_1 e_1^T, \qquad \text{or} \qquad K_3 = e_1 e_1^T,$$

for some $\lambda_0 \in \mathbb{C}$, where e_1 is the first canonical vector in $\mathbb{C}^n$. Let us prove that $\overline{\mathcal{O}}(K_0) = \mathcal{C}_0$, with

$$\mathcal{C}_0 = \left\{ \left(v_1^{(0)} + \lambda v_1^{(1)}\right)\left(w_1^{(0)}\right)^T \ : \ v_1^{(0)}, v_1^{(1)}, w_1^{(0)} \in \mathbb{C}^n \right\},$$

according to Lemma 3.1.

Let us first prove that $\mathcal{C}_0 \subseteq \overline{\mathcal{O}}(K_0)$. To this purpose, note that if $L(\lambda) \in \mathcal{O}(K_1)$ then $L(\lambda) \notin \mathcal{C}_0$ because the right minimal indices of any nonzero pencil $\left(v_1^{(0)} + \lambda v_1^{(1)}\right)\left(w_1^{(0)}\right)^T \in \mathcal{C}_0$ are the right minimal indices of $\left(w_1^{(0)}\right)^T$, which are all zero. Therefore, K_0, K_2 and K_3 are the only possible KCF's of pencils in $\mathcal{C}_0$. Then, it remains to see that $\mathcal{O}(K_i) \subseteq \overline{\mathcal{O}}(K_0)$, for $i = 0, 2, 3$.

For this, it suffices to show that $K_i \in \overline{\mathcal{O}}(K_0)$, for $i = 0, 2, 3$. For $i = 0$ the inclusion is trivial. For $i = 2$, consider the matrix pencils

$$M_k(\lambda) = \operatorname{diag}\left(\begin{bmatrix} \lambda - \lambda_0 \\ 1/k \end{bmatrix}, 0_{(n-2)\times(n-1)}\right),$$

for $k \in \mathbb{N}$. It is straightforward to see that $M_k \in \mathcal{O}(K_0)$, for all $k \in \mathbb{N}$, and that the sequence $\{M_k\}_{k\in\mathbb{N}}$ converges to K_2 in $\mathcal{T}_{\mathbb{P}}$. For $i = 3$, consider the pencils

$$\widetilde{M}_k(\lambda) = \operatorname{diag}\left(\begin{bmatrix} 1 \\ (1/k)\lambda \end{bmatrix}, 0_{(n-2)\times(n-1)}\right),$$

for $k \in \mathbb{N}$. It is again straightforward to see that $\widetilde{M}_k \in \mathcal{O}(K_0)$, for all $k \in \mathbb{N}$, and that $\{\widetilde{M}_k\}_{k\in\mathbb{N}}$ converges to K_3 in $\mathcal{T}_{\mathbb{P}}$.

Let us now prove that $\overline{\mathcal{O}}(K_0) \subseteq \mathcal{C}_0$. We proceed by contradiction. Assume that there is some $L(\lambda) \in \overline{\mathcal{O}}(K_0)$ but $L(\lambda) \notin \mathcal{C}_0$. Observe that if $M(\lambda) \in \mathcal{C}_0$, for some pencil $M(\lambda)$, then $\mathcal{O}(M) \subseteq \mathcal{C}_0$ by definition of $\mathcal{C}_0$. It is straightforward to see that $K_0, K_2, K_3 \in \mathcal{C}_0$. As a consequence, the KCF of $L(\lambda)$ must be K_1. But, since $L(\lambda) \in \overline{\mathcal{O}}(K_0)$, this would imply $K_1 \in \overline{\mathcal{O}}(K_0)$, and this in turn implies $\overline{\mathcal{O}}(K_1) \subseteq \overline{\mathcal{O}}(K_0)$. However, this is in contradiction with [6, Th. 3.2].

Using similar reasonings, we can prove that $\overline{\mathcal{O}}(K_1) = \mathcal{C}_1$. □

6. Conclusions. Let $A_0 + \lambda A_1$ be a regular matrix pencil, and λ_0 be an eigenvalue of $A_0 + \lambda A_1$ with geometric multiplicity g. For any $0 < r < g$, we have obtained the generic change of the partial multiplicities of $A_0 + \lambda A_1$ at λ_0 under perturbations with low normal rank at most r. More precisely, this generic change consists of removing the largest r partial multiplicities of $A_0 + \lambda A_1$ at λ_0 and leaving the smallest $g - r$ ones unchanged (Theorems 3.4 and 4.3). To prove this, we have provided (in Lemma 3.1) a description of the set of $n \times n$ matrix pencils with normal rank at most r as the union of $r + 1$ sets which are explicitly constructible. In the particular case $r = 1$, Proposition 5.1 shows that these two sets are precisely the irreducible components of the set of $n \times n$ matrix pencils with normal rank at most 1. It remains as an open problem to give an analogous description for the irreducible components of the set of matrix pencils with normal rank at most r, with $r > 1$.

We emphasize that we have provided, for the first time, a complete solution of the problem posed in this paper, since previous results available in the literature dealt with low rank perturbation pencils with very special properties that can be considered by no means generic. In addition, we provide the solution using two natural different definitions of genericity (Theorems 3.4 and 4.3), that are motivated by the particular structure of the set of perturbations, i. e., of the set of matrix pencils with normal rank at most r. We believe that our results show that future studies of related problems should pay close attention to the algebraic and geometric structures of the set of perturbations.

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