Uniqueness of solution of a generalized $\star$-Sylvester matrix equation[☆]

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Abstract

We present necessary and sufficient conditions for the existence of a unique solution of the matrix equation $AXB + CX^{\star}D = E$, where A, B, C, D, E are square matrices of the same size with real or complex entries, and where $\star$ stands for both the transpose or the conjugate transpose. This generalizes several previous uniqueness results for specific equations such as the $\star$-Sylvester or the $\star$-Stein equation.

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1. Introduction

Given $A, B, C, D, E \in \mathbb{F}^{n\times n}$, with $\mathbb{F}$ being $\mathbb{C}$ or $\mathbb{R}$, we consider the equation

$$AXB + CX^{\star}D = E \tag{1}$$

where $X \in \mathbb{F}^{n\times n}$ is an unknown matrix, and where, given $M \in \mathbb{F}^{n\times n}$, $M^{\star}$ stands for either the transpose M^T or the conjugate transpose M^*. In the last

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few years, this equation has been considered by several authors in the context of linear Sylvester-like equations arising in applications (see, for instance, [3, 16, 18]).

Equation (1) is a natural extension of the $\star$-Sylvester equation

$$AX + X^\star D = E, \tag{2}$$

and it is closely related to the generalized Sylvester equation

$$AXB + CXD = E. \tag{3}$$

Note that (3) contains, as a particular case, the classical Sylvester equation

$$AX + XD = E, \tag{4}$$

in the same way as (2) is a particular case of (1).

Two of the most relevant theoretical questions regarding the solvability of these matrix equations are:

(a) Find necessary and sufficient conditions for the existence of a solution.

(b) Find necessary and sufficient conditions for the existence of a *unique* solution.

These questions can be answered when considering the matrix equation as a linear system in the entries of X (or of $\mathrm{re}(X)$ and $\mathrm{im}(X)$). However, this approach is of limited interest, since it involves matrices of much larger size and difficult to be handled. For this reason, the research efforts have been focused on getting an answer to these questions in terms of matrices or pencils of the size of the matrix coefficients.

With this constraint in mind, question (a) has been already solved in the literature for all equations (1)–(4). More precisely, the characterization of consistency of the Sylvester equation (4), in terms of the matrix coefficients, was obtained back in 1952 by Roth and it is currently known as "Roth's criterion" [15]. For the $\star$-Sylvester equation (2), a similar characterization was obtained in [19], for $\mathbb{F} = \mathbb{C}$, and later in [6] for $\mathbb{F}$ being an arbitrary field with characteristic different from 2. Recently, Dmytryshyn and Kågström have obtained necessary and sufficient conditions for the consistency of general systems containing both Sylvester and $\star$-Sylvester equations, including the case where only one type of these equations is present [8]. These systems include the case of single equations such as (3) and (1).

Regarding question (b), characterizations for the uniqueness of solutions of (2)–(4) are also known. They consist of spectral properties of matrices or matrix pencils constructed in a simple way using just the coefficient matrices. In particular, (4) has a unique solution if and only if A and D have disjoint spectrum [10, Ch. 8.1]. As for (3), it has a unique solution if and only if the pencils $A + \lambda C$ and $B - \lambda D$ are regular and have disjoint spectrum [4, Thm. 1].

The characterization of the uniqueness of solution of (2) consists of exclusion conditions on the spectrum of the pencil $A + \lambda D^\star$ [1, 14] (see Theorems 4 and 5). It is interesting to note that for the equation $AX + CX^\star = E$, which is also a particular case of (1), the characterization of the uniqueness of solution is exactly the same as for (2) but replacing the pencil $A + \lambda D^\star$ by $A + \lambda C$ [5].

Necessary and sufficient conditions for the uniqueness of solution of (1), as a part of an algorithmic procedure, have been obtained in [3, Sec 4.2] (other iterative algorithms can be found in [17, 20], where uniqueness is not discussed). Nevertheless, these conditions are not given explicitly in terms of the coefficients and thus they do not give a satisfactory answer to question (b) for equation (1).

The goal of this work is to characterize the uniqueness of solution of (1) in terms of matrices and pencils constructed using the coefficients A, B, C and D and of size at most $2n$.

1.1. A pencil approach to the uniqueness problem

Equation (1) can be transformed into a linear system using: (i) the vec operator which stacks the columns of a matrix in a long vector; (ii) the Kronecker product, for which we have: $\operatorname{vec}(AXB) = (B^T \otimes A) \operatorname{vec}(X)$; and (iii) a permutation matrix Π of size $n^2 \times n^2$ such that $\Pi \operatorname{vec}(X) = \operatorname{vec}(X^T)$. In particular:

$$\operatorname{vec}(CX^T D) = (D^T \otimes C) \operatorname{vec}(X^T) = (D^T \otimes C)\Pi \operatorname{vec}(X) = \Pi(C \otimes D^T) \operatorname{vec}(X),$$

where the last identity comes from the fact that the similarity through Π inverts the order of the Kronecker product [12, Sec. 4.3].

As a consequence, equation (1) is equivalent to the system

$$\big((B^T \otimes A) + \Pi(C \otimes D^T)\big) \operatorname{vec}(X) = \operatorname{vec}(E,) \quad \text{if } \star = T, \tag{5}$$

$$(B^T \otimes A) \operatorname{vec}(X) + \Pi(C \otimes D^T) \operatorname{vec}(\overline{X}) = \operatorname{vec}(E) \quad \text{if } \star = *, \tag{6}$$

where $\overline{X}$ denotes the conjugate of the matrix X. The system of equations (5) is linear over $\mathbb{C}$, but the system (6) is not. However, if we split the real and imaginary parts of all coefficient matrices A, B, C, D, E, as well as the real and imaginary parts of the unknown matrix X, then (6) is equivalent to a linear system

$$R\,\mathrm{vec}\big(\big[\ \mathrm{re}\,X \quad \mathrm{im}\,X\ \big]\big) = \big(\big[\ \mathrm{re}\,E \quad \mathrm{im}\,E\ \big]\big), \tag{7}$$

for some matrix R with real entries and size $(2n^2) \times (2n^2)$.

As a first approach to address the uniqueness of solution of (1) note that the maps

$$\begin{array}{cccc} \mathcal{F}: & \mathbb{C}^{n^2} & \longrightarrow & \mathbb{C}^{n^2} \\ & \mathrm{vec}(X) & \mapsto & \big((B^T \otimes A) + \Pi(C \otimes D^T)\big)\,\mathrm{vec}(X), \end{array}$$

and

$$\begin{array}{cccc} \mathcal{R}: & \mathbb{R}^{2n^2} & \longrightarrow & \mathbb{R}^{2n^2} \\ & \mathrm{vec}\big(\big[\ \mathrm{re}\,X \quad \mathrm{im}\,X\ \big]\big) & \mapsto & R\,\mathrm{vec}\big(\big[\ \mathrm{re}\,X \quad \mathrm{im}\,X\ \big]\big), \end{array}$$

with R as in (7), are linear maps. Then (1) has a unique solution, for any right hand side $E \in \mathbb{C}^{n\times n}$, if and only if the homogeneous equation

$$AXB + CX^{\star}D = 0 \tag{8}$$

has only the trivial solution. As a consequence, for the uniqueness of the solution we can focus on equation (8) instead of (1).

Of course, the most interesting situation, from the point of view of applications, of the case $\star = T$ is when all coefficient matrices are real, so that the map $\mathcal{F}$ above can be seen as a real map and the solution X we are looking for is real as well. Also, when all coefficient matrices A, B, C, D, E have real entries, the unique solution of (1) with $\star = *$ has real entries as well. To see this, note that, by taking conjugates in (1), also $\overline{X}$ is a solution of (1), so it must be $X = \overline{X}$, hence X must be real. Then, in the case of real coefficient matrices, we can only consider $\star = T$. However, we cover the more general situations for the sake of completeness and the ease of statements.

Unfortunately, neither the system (5) nor the system (7) can be readily used to study equation (1), since the coefficient matrices $(B^T\otimes A)+\Pi(C\otimes D^T)$ and R, respectively, are not easy to handle.

The main result of this paper is a characterization of the uniqueness of solution of (1) in terms of elementary spectral properties of the matrix pencil $\left[\begin{smallmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{smallmatrix}\right]$. These properties are exclusion conditions of eigenvalues, similar to the one given for the pencil $A + \lambda D^\star$ in the case of the $\star$-Sylvester equation (2). In particular, our condition contains the characterization for the uniqueness of solution of equation (2), as well as the one for the equation $AX + CX^\star = E$, considered in [5]. Another relevant instance of (1) is the $\star$-Stein equation $AXB + X^\star = E$,which has been considered in the literature [2, 3, 9, 13]. Our results allow one to obtain, as a particular case, the conditions for uniqueness obtained in [13, Thm. 3] and [2, Thm. 4] for this equation and $\star = T$.

The paper is organized as follows. In Section 2 we introduce or recall the basic notions and definitions used along the paper, and we state some basic results which are used later. In Section 3 we present a couple of previous results (the characterization of the uniqueness of solution of $\star$-Sylvester and $\star$-Stein equations), together with some technical results which allow us to reduce the proof of the main theorems to the case of $\star$-Sylvester equations. The main theorems (namely, Theorems 13 and 14) are presented, and proved, in Section 4, were we show, moreover, how these results can be used to recover the characterization of the uniqueness of the equation $AX + CX^\star = E$. Finally, in Section 5 we summarize the contributions of the paper and indicate a natural continuation of this work.

2. Notation, definitions, and basic results

Given a matrix $M \in \mathbb{C}^{n\times n}$, M^{-*} denotes the inverse of M^*. The notation I_n stands for the identity matrix of size $n \times n$ though, when there is no risk of confusion, we will drop the subindex and just write I.

Our main result is a characterization of the uniqueness of solution of equation (8) in terms of spectral properties of a matrix pencil constructed from the coefficients A, B, C, D. We recall here some standard notation and results from the theory of matrix pencils that will be used along the paper. For more information on this topic, we refer the reader to [10, Ch. XII].

A matrix pencil $P(\lambda) = M - \lambda N$, with $M, N \in \mathbb{C}^{n\times n}$ is said to be *regular* if $\det(P(\lambda))$ is not identically zero. Otherwise, the pencil is said to be *singular*. A *finite eigenvalue* of a regular matrix pencil $P(\lambda)$ is a number $\lambda_0 \in \mathbb{C}$ such that $\det(P(\lambda_0)) = 0$. The regular pencil $M - \lambda N$ has an *infinite eigenvalue* if $N - \lambda M$ has 0 as eigenvalue (equivalently, if N is singular). In particular, the eigenvalues of a matrix M coincide with the eigenvalues of the pencil $M - \lambda I$. The *spectrum* of a regular matrix pencil $P(\lambda)$, denoted by $\Lambda(P)$, is the set of eigenvalues of $P(\lambda)$ (finite and infinite). Analogously, the spectrum of the matrix M is denoted by $\Lambda(M)$. The *algebraic multiplicity* of an eigenvalue λ_0 of $P(\lambda)$ is the multiplicity of λ_0 as a root of the polynomial $\det(P(\lambda))$.

A *strictly equivalent* pencil to $P(\lambda)$ is a pencil of the form $UP(\lambda)V$, with $U, V \in \mathbb{C}^{n\times n}$ invertible. Accordingly, the relation on the set of matrix pencils obtained by multiplying a given pencil on the left and/or the right by invertible matrices is called *strict equivalence*. Two strictly equivalent matrix pencils have the same eigenvalues (finite and infinite) with the same algebraic multiplicity. An eigenvalue of a pencil or a matrix is *simple* if it has algebraic multiplicity equal to 1. The algebraic multiplicity of an eigenvalue λ_0 of a pencil $P(\lambda)$ or a matrix M will be denoted by $m_{\lambda_0}(P)$ or $m_{\lambda_0}(M)$, respectively.

The characterization of the uniqueness of solution of equation (8) will strongly depend on the following notion, where we consider the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the conventions $0^{-1} = \infty$, $\infty^{-1} = 0$, $\overline{\infty} = \infty$.

Definition 1. (Reciprocal free and $*$-reciprocal free set) [14]. *Let $\mathcal{S}$ be a subset of $\overline{\mathbb{C}}$. We say that $\mathcal{S}$ is*

(a) reciprocal free *if $\lambda_i \neq \lambda_j^{-1}$, for all $\lambda_i, \lambda_j \in \mathcal{S}$;*

(b) $*$-reciprocal free *if $\lambda_i \neq (\overline{\lambda}_j)^{-1}$, for all $\lambda_i, \lambda_j \in \mathcal{S}$.*

This definition includes the values $\lambda = 0, \infty$, since for these values $\lambda^{-1} = (\overline{\lambda})^{-1} = \infty, 0$, respectively.

Remark 1. Note that if $\mathcal{S} \subseteq \overline{\mathbb{C}}$ is reciprocal free, then $\pm 1 \notin \mathcal{S}$, since λ_i, λ_j in Definition 1 can be equal. Similarly, if $\mathcal{S} \subseteq \overline{\mathbb{C}}$ is $*$-reciprocal free, then $\mathcal{S}$ cannot contain numbers of modulus 1.

In the following, the name $\star$*-reciprocal free* set stands for both reciprocal and $*$-reciprocal free sets. We will use the following basic results on $\star$-reciprocal free sets.

Lemma 2. *Let $M, N \in \mathbb{C}^{n\times n}$. Then $\Lambda(M - \lambda N)$ is $\star$-reciprocal free if and only if $\Lambda(N - \lambda M)$ is $\star$-reciprocal free.*

Proof. The result is an immediate consequence of the fact that $\lambda_0 \in \Lambda(M - \lambda N)$ if and only if $\lambda_0^{-1} \in \Lambda(N - \lambda M)$ (including $\lambda_0 = 0, \infty$). $\square$

Lemma 3. *Let $\mathcal{S}$ be a subset of $\overline{\mathbb{C}}$, and define*

$$\sqrt{\mathcal{S}} := \{z \in \overline{\mathbb{C}} : \; z^2 \in \mathcal{S}\},$$

with the convention $\infty^2 = \infty$. Then $\mathcal{S}$ is $\star$-reciprocal free if and only if $\sqrt{\mathcal{S}}$ is $\star$-reciprocal free

Proof. Let us first assume that $\mathcal{S}$ is reciprocal free, and let $z \in \sqrt{\mathcal{S}}$. Then $z^2 = s \in \mathcal{S}$. Now, if $1/z \in \sqrt{\mathcal{S}}$, we would have $1/s = (1/z)^2 \in \mathcal{S}$, which is a contradiction with the fact that $\mathcal{S}$ is reciprocal free. Conversely, if $\sqrt{\mathcal{S}}$ is reciprocal free, let $s \in \mathcal{S}$, so that $s = z^2$, for some $z \in \sqrt{S}$. If $1/s \in \mathcal{S}$, then we would have $(1/z)^2 = 1/s$, so that $1/z \in \sqrt{\mathcal{S}}$, which is a contradiction with the fact that $\sqrt{\mathcal{S}}$ is reciprocal free.

For $*$-reciprocal free sets the proof is analogous. Let us first assume that $\mathcal{S}$ is $*$-reciprocal free, and let $z \in \sqrt{\mathcal{S}}$, so that $z^2 = s \in \mathcal{S}$. Now, if $1/\overline{z} \in \sqrt{\mathcal{S}}$, we would have $1/\overline{s} = (1/\overline{z})^2 \in S$, which is in contradiction with the fact that $\mathcal{S}$ is $*$-reciprocal free Conversely, if $\sqrt{\mathcal{S}}$ is $*$-reciprocal free, let $s \in \mathcal{S}$, so that $s = z^2$, for some $z \in \sqrt{\mathcal{S}}$. If $1/\overline{s} \in \mathcal{S}$, then we would have $(1/\overline{z})^2 = 1/\overline{s}$, so that $1/\overline{z} \in \sqrt{\mathcal{S}}$, a contradiction.

Notice that these arguments hold also when $\mathcal{S}$ contains 0 or ∞. □

3. Reduction process and auxiliary results

In this section, we show that the problem of the uniqueness of solution of the general equation (8) can be reduced to the analysis of some simpler equations of the same type obtained from particular choices of the coefficient matrices. We also present some technical results that will be used in Section 4.

For the sake of completeness, and for further reference, let us recall the characterization of uniqueness of solution of $\star$-Sylvester equations.

Theorem 4. (Characterization of uniqueness of solution of T-Sylvester equations [1]). *Let $A, D \in \mathbb{C}^{n\times n}$. Then the matrix equation $AX + X^T D = 0$ has a unique solution if and only if the following conditions hold:*

(1) *The pencil $A - \lambda D^T$ is regular.*

(2) $\Lambda(A - \lambda D^T) \setminus \{1\}$ *is reciprocal free.*

(3) $m_1(A - \lambda D^T) \leqslant 1$.

Theorem 5. (Characterization of uniqueness of solution of $*$-Sylvester equations [14]). *Let $A, D \in \mathbb{C}^{n\times n}$. Then the matrix equation $AX + X^*D = 0$ has a unique solution if and only if the following conditions hold:*

(1) *The pencil $A - \lambda D^*$ is regular.*

(2) *$\Lambda(A - \lambda D^*)$ is $*$-reciprocal free.*

The proof of Theorem 5 in [14] relies on some continuity arguments. For a different proof of Theorems 4 and 5 relying only on matrix manipulations, see [7, Thm. 10-11].

If the matrices A and C in equation (8) are both singular, then there exist nonzero vectors v, w such that $Av = Cw = 0$, and thus the nonzero matrix $vw^\star$ is a solution of (8). Also, if B and D are both singular, then there exist nonzero vectors v, w such that $B^\star v = D^\star w = 0$, and thus the nonzero matrix $wv^\star$ is a solution of (8). We have thus proved the following result.

Lemma 6. *The following two conditions are necessary for the existence of a unique solution to equation* (8)*:*

(i) *At least one of the matrices A and C is invertible.*

(ii) *At least one of the matrices B and D is invertible.*

Since we are interested in the uniqueness of solution of (8), the previous conditions allows us to focus just on four cases, in any of which the equation can be reduced to a simpler form.

Case 1. The matrices A and B are invertible: then (8) is equivalent to

$$X + A^{-1}CX^\star DB^{-1} = 0. \tag{9}$$

Case 2. The matrices A and D are invertible: then (8) is equivalent to

$$XBD^{-1} + A^{-1}CX^\star = 0. \tag{10}$$

Case 3. The matrices C and B are invertible: then (8) is equivalent to

$$C^{-1}AX + X^\star DB^{-1} = 0. \tag{11}$$

Case 4. The matrices C and D are invertible: then (8) is equivalent to

$$C^{-1}AXBD^{-1} + X^\star = 0. \tag{12}$$

Note that in Cases 2 and 3, the equation is reduced to a $\star$-Sylvester equation, namely (10) and (11), respectively.

Later, in Theorems 13 and 14 we will show that the characterization of the uniqueness of solution of (8) depends on some spectral properties of the matrix pencil

$$Q(\lambda) = \begin{bmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{bmatrix}. \tag{13}$$

The following technical result deals with the determinant of this pencil when at least one of the coefficients A and C is invertible. Note that this includes Cases 1–4 above.

Lemma 7. *Let $A, B, C, D \in \mathbb{C}^{n\times n}$ and let $Q(\lambda)$ be the pencil in* (13)*.*

(a) *If C is invertible, then*

$$\det(Q(\lambda)) = \pm \det(C)\det(B^\star C^{-1}A - \lambda^2 D^\star).$$

(b) *If A is invertible, then*

$$\det(Q(\lambda)) = \pm \det(A)\det(B^\star - \lambda^2 D^\star A^{-1}C).$$

Proof. (a) When C is invertible, the following identity holds

$$\begin{bmatrix} \lambda I & B^\star C^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{bmatrix} = \begin{bmatrix} -\lambda^2 D^\star + B^\star C^{-1}A & 0 \\ A & -\lambda C \end{bmatrix}.$$

Taking determinants in both sides, we get

$$\lambda^n \det(Q(\lambda)) = \det(B^\star C^{-1}A - \lambda^2 D^\star)(-\lambda)^n \det(C),$$

which gives the result.

(b) When A is invertible,

$$\begin{bmatrix} 0 & I \\ I & \lambda D^\star A^{-1} \end{bmatrix} \begin{bmatrix} -\lambda D^* & B^\star \\ A & -\lambda C \end{bmatrix} = \begin{bmatrix} A & -\lambda C \\ 0 & B^\star - \lambda^2 D^\star A^{-1}C \end{bmatrix}.$$

Taking determinants we get

$$(-1)^{\lfloor n/2 \rfloor} \det(Q(\lambda)) = \det(A)\det(B^\star - \lambda^2 D^\star A^{-1}C),$$

and this completes the proof. □

The following result also deals with spectral properties of the pencil $Q(\lambda)$. It is again related to (the complementary of) Cases 1 and 4 above.

Lemma 8. *Let the matrix pencil* $Q(\lambda)$ *in* (13) *be regular. Then the values* 0 *and* ∞ *are eigenvalues of* $Q(\lambda)$ *if and only if* AB *and* CD *are singular.*

Proof. The matrix pencil $Q(\lambda)$ has 0 as an eigenvalue if and only if the matrix $\left[\begin{smallmatrix} 0 & B^\star \\ A & 0 \end{smallmatrix}\right]$ is singular and this happens if and only if one of the matrices A or B (and thus AB) is singular. On the other hand, $Q(\lambda)$ has ∞ as an eigenvalue if and only if the matrix $\left[\begin{smallmatrix} -D^\star & 0 \\ 0 & -C \end{smallmatrix}\right]$ is singular and this happens if and only if one of the matrices C or D (and thus CD) is singular. $\square$

Our reduction process will end up with $\star$-Stein equations of the form $AXB + X^\star = 0$, together with $\star$-Sylvester equations (see equations (9)–(12)) The characterization of uniqueness of solution of $\star$-Sylvester equations is given by Theorems 4 and 5. The following result relates the solution of $AXB + X^\star = 0$ with the solution of an associated $\star$-Sylvester equation.

Lemma 9. *Let* $A, B \in \mathbb{C}^{n\times n}$. *Then the equation* $AXB + X^\star = 0$ *has a unique solution if and only if the equation* $AB^\star Y + Y^\star = 0$ *has a unique solution.*

Proof. First assume that, for $X \neq 0$, we have

$$AXB + X^\star = 0.$$

Applying the operator $\star$ and premultiplying by A, we get

$$AB^\star(X^\star A^\star) + AX = 0,$$

so that $Y = (AX)^\star$ is a solution of $AB^\star Y + Y^\star = 0$. It remains to prove that Y is nonzero. By contradiction, $Y = 0$ implies $AX = 0$ and, since $AXB + X^\star = 0$, this would imply $X = 0$.

The opposite direction can be proved in a similary way: if we assume that $Y \neq 0$ is such that

$$AB^\star Y + Y^\star = 0,$$

then we get that $X = B^\star Y$ is a nonzero solution of $AXB + X^\star = 0$. $\square$

Lemma 9 has two important corollaries. The first one has been already obtained in [13, Thm. 3] and [2, Thm. 4] by different means.

Theorem 10. (Characterization of uniqueness of solution of T-Stein equations). *Let $A, B \in \mathbb{C}^{n\times n}$. Then the equation $AXB + X^T = 0$ has a unique solution if and only if $\Lambda(AB^T) \setminus \{1\}$ is reciprocal free and $m_1(AB^T) \leqslant 1$.*

Theorem 11. (Characterization of uniqueness of solution of $*$-Stein equations). *Let $A, B \in \mathbb{C}^{n\times n}$. Then the equation $AXB + X^* = 0$ has a unique solution if and only if $\Lambda(AB^*)$ is $*$-reciprocal free.*

Another spectral property of the pencil $Q(\lambda)$ in (13) is given in Lemma 12. This property is not going to be explicitly used in Section 4, but is implicitly used in some arguments and claims.

Lemma 12. *Assume that the matrix pencil $Q(\lambda)$ in (13) is regular. Then $\lambda_0 \in \Lambda(Q)$ if and only if $-\lambda_0 \in \Lambda(Q)$. Moreover, λ_0 is a simple eigenvalue of $Q(\lambda)$ if and only if $-\lambda_0$ is a simple eigenvalue of $Q(\lambda)$.*

Proof. Since

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} Q(\lambda) \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} = Q(-\lambda),$$

we have that the two pencils $Q(\lambda)$ and $Q(-\lambda)$ are strictly equivalent and thus they have the same eigenvalues with the same multiplicities. □

4. Characterization of the uniqueness of solution

In this section we state and prove the main results of this paper, namely, the characterization of uniqueness of solution of $AXB + CX^*D = 0$ which will be given in Theorem 14, and of $AXB + CX^TD = 0$ which is given in the following result.

Theorem 13. *Let $A, B, C, D \in \mathbb{C}^{n\times n}$, and define the pencil:*

$$Q(\lambda) := \begin{bmatrix} -\lambda D^T & B^T \\ A & -\lambda C \end{bmatrix}. \tag{14}$$

Then the equation

$$AXB + CX^TD = 0 \tag{15}$$

has a unique solution if and only if the following conditions hold:

(i) *$Q(\lambda)$ is regular;*

(ii) $\Lambda(Q)\setminus\{\pm1\}$ *is reciprocal free.*

(iii) $m_1(Q)=m_{-1}(Q)\leqslant 1$.

Proof. We first prove the following claims:

(R1) The result is true for equation (15), if and only if it is true for

$$RAXBS + RCX^TDS = 0, \tag{16}$$

for any $R, S \in \mathbb{F}^{n\times n}$ invertible.

(R2) The result is true for T-Sylvester equations

$$AX + X^TD = 0. \tag{17}$$

(R3) The result is true for T-Stein equations

$$AXB + X^T = 0. \tag{18}$$

Let us prove (R1). Note first that (15) has a unique solution if and only if (16) has a unique solution, for any R, S invertible. Also, the pencil (14) satisfies conditions (i)–(iii) in the statement if and only if the pencil

$$\widehat{Q}(\lambda) = \begin{bmatrix} -\lambda(DS)^T & (BS)^T \\ RA & -\lambda RC \end{bmatrix} = \begin{bmatrix} S^T & 0 \\ 0 & R \end{bmatrix} Q(\lambda)$$

satisfies (i)–(iii) in the statement (replacing $Q(\lambda)$ by $\widehat{Q}(\lambda)$). This proves (R1).

Now, let us prove (R2). Set

$$Q_{\rm syl}(\lambda) = \begin{bmatrix} -\lambda D^T & I \\ A & -\lambda I \end{bmatrix}.$$

Then, since, by Lemma 7,

$$\det(Q_{\rm syl}(\lambda)) = \pm\det(A-\lambda^2D^T), \tag{19}$$

condition (1) in Theorem 4 is equivalent to condition (i) in the statement for $Q_{\rm syl}(\lambda)$, which is the pencil associated with the T-Sylvester equation (17). Also, from (19) we get:

$$\sqrt{\Lambda(A-\lambda D^T)} = \Lambda(Q_{\rm syl}),$$

and $m_1(A-\lambda D^T) = m_1(Q_{\rm syl}) = m_{-1}(Q_{\rm syl})$. Moreover, since for an arbitrary $\mathcal{S} \subseteq \mathbb{C}$,

$$\sqrt{\mathcal{S} \setminus \{1\}} = \sqrt{\mathcal{S}} \setminus \{\pm 1\}, \tag{20}$$

then we conclude

$$\sqrt{\Lambda(A-\lambda D^T) \setminus \{1\}} = \Lambda(Q_{\rm syl}) \setminus \{\pm 1\}.$$

Therefore, conditions (2)–(3) in Theorem 4 are equivalent to conditions (ii)–(iii) in the statement for $Q_{\rm syl}(\lambda)$.

Finally, let us prove (R3). Assume that the T-Stein equation (18) has a unique solution. Then, by Lemma 9, the T-Sylvester equation $AB^TX+X^T=0$ has a unique solution. By (R2), the statement is true for T-Sylvester equations, so we have that $\widehat{Q}_{\rm syl}(\lambda) = \left[\begin{smallmatrix} -\lambda I & I \\ AB^T & -\lambda I \end{smallmatrix}\right]$ verifies conditions (i)-(iii) of the statement. By Lemma 7, $\det(\widehat{Q}_{\rm syl}(\lambda)) = \pm\det(AB^T - \lambda^2 I)$. Moreover, if we set

$$Q_{\rm st}(\lambda) = \begin{bmatrix} -\lambda I & B^T \\ A & -\lambda I \end{bmatrix}, \tag{21}$$

then, using Lemma 7 again, we get $\det(Q_{\rm st}(\lambda)) = \pm\det(B^TA - \lambda^2 I)$. Since $\Lambda(AB^T) = \Lambda(BA^T)$ [11, Thm. 1.3.20] we have $\det(\widehat{Q}_{\rm syl}(\lambda)) = \pm\det(Q_{\rm st}(\lambda))$ and thus (R3) is true.

As a consequence of reductions (R1)–(R3), the statement is true for any T-Sylvester and T-Stein equations which are equivalent to (15) in the sense of (16). Now, we are going to see that the proof can be reduced to one of these equations.

By Lemma 6, if A, C are both singular, or B, D are both singular, then (1) does not possess a unique solution. In both cases, either the pencil (14) is singular or it is regular but $\Lambda(Q)\setminus\{\pm 1\}$ is not reciprocal free, since both 0 and ∞ are eigenvalues of $Q(\lambda)$ (see Lemma 8). Hence, the statement is true in these two cases. Then, it suffices to consider the remaining cases, namely Cases 1–4 right after Lemma 6.

Case 1. A, B are invertible: In this case, (15) is equivalent to $C^{-1}AXBD^{-1}+X^T=0$, which a T-Stein equation, so the result follows by (R1) and (R3).

Case 2. A, D are invertible: In this case, (15) is equivalent to $XBD^{-1}+A^{-1}CX^T=0$, which is a T-Sylvester equation, so the result follows again by (R1) and (R2).

Case 3. B, C are invertible: In this case, (15) is equivalent to $XDB^{-1} + C^{-1}AX^T = 0$, which is a T-Sylvester equation.

Case 4. C, D are invertible: In this case, (15) is equivalent to $X+C^{-1}AX^TBD^{-1} = 0$, which is a T-Stein equation.

□

Now we state the corresponding result for $\star = *$.

Theorem 14. *Let $A, B, C, D \in \mathbb{C}^{n\times n}$, and define the pencil:*

$$Q(\lambda) := \begin{bmatrix} -\lambda D^* & B^* \\ A & -\lambda C \end{bmatrix}. \qquad (22)$$

Then the equation

$$AXB + CX^*D = 0 \qquad (23)$$

has a unique solution if and only if the following conditions hold:

(i) $Q(\lambda)$ *is regular, and*

(ii) $\Lambda(Q)$ *is $*$-reciprocal free.*

Proof. The proof mimics the one of Theorem 13, just replacing the transpose with the transpose conjugate and removing the condition on the eigenvalues $\lambda = \pm 1$ along the proof, according to the differences between conditions (ii)–(iii) in Theorem 13 and the condition (ii) in the statement of Theorem 14. □

Instead of using the pencil (13), we can state Theorems 13 and 14 using matrices or matrix pencils with the same size as the coefficients of the original equation (1). This is given in the following corollaries.

Corollary 15. *Equation* (15) *has a unique solution if and only if at least one of the following conditions* (a)–(d) *hold:*

(a) *A and B are invertible, $\Lambda(A^{-1}CB^{-T}D^T) \setminus \{1\}$ is reciprocal free, and $m_1(A^{-1}CB^{-T}D^T) \leqslant 1$;*

(b) *A and D are invertible, $P(\lambda) := A^{-1}C - \lambda D^{-T}B^T$ is regular, $\Lambda(P)\setminus\{1\}$ is reciprocal free, and $m_1(P) \leqslant 1$;*

(c) *C and B are invertible, $P(\lambda) := C^{-1}A - \lambda B^{-T}D^T$ is regular, $\Lambda(P)\setminus\{1\}$ is reciprocal free, and $m_1(P) \leqslant 1$;*

(d) *C and D are invertible, and $\Lambda(C^{-1}AD^{-T}B^T) \setminus \{1\}$ is reciprocal free, and $m_1(C^{-1}AD^{-T}B^T) \leqslant 1$.*

Corollary 16. *Equation* (23) *has a unique solution if and only if at least one of the following conditions* (a)–(d) *hold:*

(a) *A and B are invertible, and $\Lambda(A^{-1}CB^{-*}D^*)$ is $*$-reciprocal free.*

(b) *A and D are invertible, $P(\lambda) := A^{-1}C - \lambda D^{-*}B^*$ is regular, and $\Lambda(P)$ is $*$-reciprocal free.*

(c) *C and B are invertible, $P(\lambda) := C^{-1}A - \lambda B^{-*}D^*$ is regular, and $\Lambda(P)$ is $*$-reciprocal free.*

(d) *C and D are invertible, and $\Lambda(C^{-1}AD^{-*}B^*)$ is $*$-reciprocal free.*

4.1. Some particular cases

Apart from the $\star$-Stein equations considered in Theorems 10 and 11, Theorems 13 and 14 allow us to recover previously known results on the characterization of the uniqueness of solution of another two particular cases of (1), namely, the $\star$-Sylvester equation (2) and the equation $AX + CX^\star = E$ [5]. The characterization of uniqueness of solution of the $\star$-Sylvester equation has been used to prove Theorems 13 and 14, so it is clearly consistent with these results. It remains to consider the equation

$$AX + CX^\star = E\,. \tag{24}$$

Theorems 13 and 14 state that (24) has a unique solution if and only if the matrix pencil

$$Q_p(\lambda) = \begin{bmatrix} -\lambda I & I \\ A & -\lambda C \end{bmatrix}$$

is regular and

(i) if $\star = *$, $\Lambda(Q_p)$ is $*$-reciprocal free;

(ii) if $\star = T$, $\Lambda(Q_p) \setminus \{\pm 1\}$ is reciprocal free and $m_1(Q_p) = m_{-1}(Q_p) \leqslant 1$.

Since

$$\begin{bmatrix} I & 0 \\ \lambda C & I \end{bmatrix} Q_p(\lambda) \begin{bmatrix} 0 & I \\ I & \lambda I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A - \lambda^2 C \end{bmatrix},$$

then $\Lambda(Q_p)$ is $\star$-reciprocal free if and only if $\sqrt{\Lambda(A - \lambda C)}$ is $\star$-reciprocal free and, using Lemma 3, this happens if and only if $\Lambda(A - \lambda C)$ is $\star$-reciprocal free. Moreover, $\lambda = 1$ is an eigenvalue of $A - \lambda C$ if and only if $\lambda = \pm 1$ are eigenvalues of $Q_p(\lambda)$, with the same multiplicities. Therefore, and using (20) for the case $\star = T$, we recover the conditions for uniqueness of solution of (24) obtained in [5, Thm. 5.2].

5. Conclusions and open problems

We have obtained necessary and sufficient conditions for the uniqueness of solution of the matrix equation $AXB + CX^\star D = E$, for any right hand side, explicitly in terms of the coefficients A, B, C, D. More precisely, these conditions are given in terms of spectral properties of the pencil $\left[\begin{smallmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{smallmatrix}\right]$ which are very simple to state. Our characterization includes, as particular cases, the ones already known for the $\star$-Sylvester equation $AX + X^\star D = E$, the $\star$-Stein equation $AXB + X^\star = E$, and the equation $AX + CX^\star = E$. In view of possible applications, a subject of future work is the design and numerical analysis of an efficient algorithm for the solution of this equation.

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