The solution of the equation $AX + X^\star B = 0$

Fernando De Terán*, Froilán M. Dopico†, Nathan Guillery‡, Daniel Montealegre§, and Nicolás Reyes¶

August 11, 2011

Abstract

We describe how to find the general solution of the matrix equation $AX + X^\star B = 0$, where $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$ are arbitrary matrices, $X \in \mathbb{C}^{n\times m}$ is the unknown, and $X^\star$ denotes either the transpose or the conjugate transpose of X. We first show that the solution can be obtained in terms of the Kronecker canonical form of the matrix pencil $A + \lambda B^\star$ and the two nonsingular matrices which transform this pencil into its Kronecker canonical form. We also give a complete description of the solution provided that these two matrices and the Kronecker canonical form of $A + \lambda B^\star$ are known. As a consequence, we determine the dimension of the solution space of the equation in terms of the sizes of the blocks appearing in the Kronecker canonical form of $A+\lambda B^\star$. The general solution of the homogeneous equation $AX+X^\star B = 0$ is essential to finding the general solution of $AX+X^\star B = C$, which is related to palindromic eigenvalue problems that have attracted considerable attention recently.

Keywords: Kronecker canonical form, matrix equations, matrix pencils, palindromic eigenvalue problems, Sylvester equation for *congruence

AMS subject classification: 15A21, 15A24, 65F15

1 Introduction

Given two arbitrary complex matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, we are concerned with the general solution of the matrix equation

$$AX + X^\star B = 0\,, \tag{1}$$

where $X \in \mathbb{C}^{n\times m}$ is the unknown and $X^\star$ denotes either the transpose or the conjugate transpose of X. Equation (1) is the homogeneous version of the Sylvester equation for *congruence, $AX+X^\star B = C$, that has recently attracted the attention of several researchers due to its relationship with palindromic eigenvalue problems $(G+\lambda G^\star)\, v = 0$ [16, 3, 9, 14], and whose solution is related to block-antidiagonalization of block anti-triangular matrices via *congruence. Several results are already available for the equation $AX+X^\star B = C$: necessary and sufficient conditions for solvability have been given in [20] in the spirit of Roth's criterion (see also [9]), necessary and sufficient conditions for the existence of a unique solution for every right-hand side have been established in [3, 14], and, in this case, an efficient algorithm

*(Corresponding Author) Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (`fteran@math.uc3m.es`)

†Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (`dopico@math.uc3m.es`)

‡University of California Santa Barbara (`nguillery@umail.ucsb.edu`)

§University of California Los Angeles (`dmontealegre@ucla.edu`)

¶University of Massachusetts (`nzreyes@student.umass.edu`)

to numerically compute this unique solution has been presented in [9]. However, to the best of our knowledge, the general solution of $AX + X^\star B = C$ has not yet been studied; this problem will be solved if a particular solution can be determined and the general solution of $AX + X^\star B = 0$ is found. The purpose of this work is to find the general solution to this last equation.

Equation (1) looks similar to the homogeneous *Sylvester* equation

$$AX - XB = 0\,, \tag{2}$$

where A and B are square matrices, in general of different sizes. Equation (2) is related to block-diagonalization of block-triangular matrices via similarity and its study is a classical subject in matrix analysis that is considered in many standard references. In particular, its general solution has been known for more than 50 years [11, Ch. VIII], [12, Ch. 4]. However, the classical techniques used in [11, Ch. VIII] to solve the homogeneous Sylvester equation (2) are no longer applicable to equation (1). The (conjugate) transposition of the unknown matrix in the second summand considerably complicates the analysis, and a different approach is required.

In order to highlight the differences between equations (2) and (1), let us discuss the strategies for solving both equations. The dimension of the solution space of (2) is invariant under *independent similarities* of the coefficient matrices A and B, and there is a bijection between the solutions of (2) and the solutions of $\widetilde{A}X - X\widetilde{B} = 0$, where $\widetilde{A} = SAS^{-1}$ and $\widetilde{B} = TBT^{-1}$ are any pair of matrices similar to A and B. As a consequence, the dimension of the solution space of (2) depends only on the *Jordan canonical forms* (JCF) of A and B (denoted by $J_A = J_1^A \oplus \cdots \oplus J_p^A$ and $J_B = J_1^B \oplus \cdots \oplus J_q^B$, respectively, where $\oplus$ denotes the direct sum of matrices, that is, the block-diagonal matrix whose diagonal blocks are the summands). Moreover, the solution can be recovered from the solution of the equation $J_A X - XJ_B = 0$. If X is partitioned into blocks, $X = [X_{ij}]$, according to the blocks in J_A and J_B, then the equation $J_A X - XJ_B = 0$ is decoupled into smaller independent equations $J_i^A X_{ij} - X_{ij} J_j^B = 0$ for each block X_{ij}, $1 \leq i \leq p$ and $1 \leq j \leq q$. Thus, the problem of solving (2) reduces to solving it when the coefficients are single Jordan blocks. The key advantage of this approach is that the general solution of $J_i^A X_{ij} - X_{ij} J_j^B = 0$ can be determined very easily [11, Ch. VIII].

By contrast, a similarity transformation is not useful for solving equation (1), as the dimension of the solution space of (1) is not necessarily preserved if we replace A and B with similar matrices. However, we will see that the dimension of the solution space of (1) is invariant under *strict equivalence* of the pencil $A + \lambda B^\star$, and there is a bijection between the solutions of (1) and the solutions of $\widetilde{A}X + X^\star \widetilde{B} = 0$, where $\widetilde{A} + \lambda \widetilde{B}^\star = P(A + \lambda B^\star)Q$ is any pencil strictly equivalent to $A + \lambda B^\star$. As a consequence, we can recover the solutions of the original equation (1) from the solutions of $\widetilde{A}X + X^\star \widetilde{B} = 0$. Then, it is natural to look for $\widetilde{A}$ and $\widetilde{B}$ such that $\widetilde{A} + \lambda \widetilde{B}^\star$ is a "canonical representative" under strict equivalence of $A + \lambda B^\star$ and $\widetilde{A}X + X^\star \widetilde{B} = 0$ is easier to solve than the original equation. The natural choice for $\widetilde{A} + \lambda \widetilde{B}^\star$ is the *Kronecker canonical form* (KCF) of $A + \lambda B^\star$ [11, Ch. XII], which is a block diagonal form like the JCF, and is denoted by

$$E + \lambda F^\star = (E_1 \oplus \cdots \oplus E_d) + \lambda(F_1^\star \oplus \cdots \oplus F_d^\star).$$

Therefore, we may focus on the solution of (1) with the pencil $A + \lambda B^\star$ given in KCF, that is, we focus on $EX + X^\star F = 0$ and consider $X = [X_{ij}]$ partitioned into blocks according to the blocks in E and F. But at this point we find two relevant differences with the Sylvester equation that make the analysis much more complicated. First, the KCF involves *four* different types of blocks, instead of only one type as in the JCF, and second, the presence of the (conjugate) transpose in equation (1) implies that $EX + X^\star F = 0$ does *not* decouple into independent equations for each block X_{ij}. More precisely, one gets decoupled equations $E_i X_{ii} + X_{ii}^\star F_i = 0$ for the diagonal blocks, $1 \leq i \leq d$, while for the off-diagonal blocks one gets the following 2×2 systems of matrix equations for $1 \leq i < j \leq d$:

$$\begin{aligned} E_i X_{ij} + X_{ji}^{\star} F_j &= 0, \\ E_j X_{ji} + X_{ij}^{\star} F_i &= 0. \end{aligned}$$

This leads to 14 *different types of matrix equations or systems of matrix equations*, and the main task of this paper is to solve all of them. As a consequence of the solution of these equations/systems, we obtain a formula for the dimension of the solution space of (1) in terms of the sizes of the blocks in the KCF of $A + \lambda B^{\star}$. In this context, it is important to notice that (1) is linear over $\mathbb{C}$ when $\star = T$, but this is not the case when $\star = *$. Nonetheless, when $\star = *$, equation (1) is linear over $\mathbb{R}$. As a consequence, we will consider the solution space of (1), with $\star = T$, as a linear space over $\mathbb{C}$, whereas the solution space for $\star = *$ will be considered as a linear space over $\mathbb{R}$. Therefore, we deal with dimensions over two different fields ($\mathbb{C}$ and $\mathbb{R}$). To avoid confusion, we will refer to the dimension of a vector space over $\mathbb{C}$ simply as its *dimension* and to the dimension over $\mathbb{R}$ as its *real dimension*.

It is important to remark that the general solution of equation (1) has been determined previously in the particular but important case $A = B$, which is related to orbits of matrices and palindromic pencils under *congruence. See reference [7] for $\star = T$ and [8] for $\star = *$. However, the approach followed in [7, 8] is different than the one presented here; the transformation used in [7, 8] to reduce the original equation to a simpler form is **congruence*, more precisely, the reduction of A to its *canonical form for* **congruence (CF***C)* [13]. This approach is very natural, since *congruence preserves the structure of *palindromic pencils $A + \lambda A^{\star}$ and *congruence is the transformation used in structure-preserving numerical algorithms for computing eigenvalues of this type of pencil.

Unfortunately, for the general equation (1), *congruence fails to preserve the dimension of the solution space, as similarity does. This fact establishes another striking difference between equations (1) and the Sylvester equation (2), namely, equation (2) is solved using the JCF both in the general case and in the particular case $A = B$ [11, Ch. VIII], while the CF*C is only useful to solve (1) when $A = B$. As we have discussed above, the general solution of (1) can be obtained via the KCF of $A + \lambda B^{\star}$, which means that in the case $A = B$ the solution of (1) can be obtained in terms of the KCF of $A + \lambda A^{\star}$. We will see that the KCF of the *palindromic pencil $A + \lambda A^{\star}$ is in one-to-one correspondence with the CF*C of A (see [6] for the case $\star = T$ and Theorem 7 in this paper for the case $\star = *$). This one-to-one correspondence allows us to recover the general solution of $AX + X^{\star}A = 0$ in [7, 8] from the solution provided in the present paper for the general case of (1), as well as the dimension of the solution space.

Equation (1) does not seem to have a history as long as the Sylvester equation (2), and has not attracted the same attention either. In this paragraph we briefly discuss the most relevant references that we know for equations related to (1). Matrix equations involving both the unknown and its transpose (or its conjugate transpose) can be traced back to the 1960's [18, 1]. More recently, the non-homogenous equation

$$AX + X^{\star}B = C \tag{3}$$

has attracted the attention of several researchers. The particular case $B = \pm A^{\star}$ of (3) was solved in [15], later in [2] for $\star = T$, and recently in [10] for linear bounded operators and A of closed range (here $(\cdot)^{\star}$ stands for the adjoint operator). The general case of (3), for $\star = *$, was addressed in [20], where the author obtains necessary and sufficient conditions for the solvability of (3) over $\mathbb{C}$. This result has recently been extended in [9] to arbitrary fields with characteristic not two, for both $(\cdot)^*$ and $(\cdot)^T$. In [9], an efficient algorithm for the numerical solution of (3) is also presented in the case where this solution is unique. Equation (3), with $\star = T$, has appeared also in [3] in connection with structured condition numbers of deflating subspaces of regular palindromic matrix pencils $G + \lambda G^T$, and the same equation, with $\star = *$, arises in [14] in the context of structure-preserving QR-type algorithms for computing the eigenvalues of regular palindromic pencils.

An additional consequence of the results in this paper is that they allow us to characterize when the operator $X \mapsto AX + X^{\star}B$ is invertible. This question was already solved in [3, 14], but we provide another proof here, which is an immediate consequence of the dimension count for the solution space of (1). We stress that this operator is linear over $\mathbb{C}$ if $\star = T$ and linear over $\mathbb{R}$ if $\star = *$.

The paper is organized as follows. In Section 2 we recall the notion of strict equivalence of pencils and we show its close relationship with the solution of (1). We also recall the KCF of matrix pencils, which is the canonical form under strict equivalence. In Section 3 we outline the procedure to solve (1) by taking advantage of the KCF of the pencil $A+\lambda B^{\star}$, and we display the main results of this paper, namely, the dimension of the solution space of (1) in terms of the sizes of the blocks appearing in the KCF of $A + \lambda B^{\star}$ (Theorem 3 for $\star = T$ and Theorem 4 for $\star = *$). In Section 4 we prove Theorem 3 by solving the 14 equations or systems of equations that we have mentioned above for the case $\star = T$. In particular, Section 4.1 is devoted to solving the equations for the individual blocks and Section 4.2 is devoted to solving the 2×2 systems of equations involving pairs of blocks. We remark that the solutions of these equations and systems of equations allow us to recover the solution of the original equation (1) provided that the change matrices leading $A + \lambda B^{T}$ to its KCF are known. Section 5 is the counterpart of Section 4 for $\star = *$. In Section 6 we derive some previously known results from our main theorems stated in Section 3. More precisely, we obtain the dimension of the solution space of (1) in the particular cases $A = B$ and $A = \pm B^{\star}$, and we obtain necessary and sufficient conditions for the operator $X \mapsto AX + X^{\star}B$ to be invertible. Finally, in Section 7 we summarize the main contributions of this paper and present some lines of future research.

2 Strict equivalence of matrix pencils. The Kronecker canonical form

Throughout this paper we will use I_k and $0_{m\times n}$ to denote the $k \times k$ identity matrix and the $m \times n$ matrix of all zeros, respectively. Also, we denote the inverse of the transpose or the conjugate transpose by $(\cdot)^{-\star}$.

Definition 1 *Two matrix pencils, $A + \lambda B^{\star}$ and $\widetilde{A} + \lambda \widetilde{B}^{\star}$, are* strictly equivalent *if there exist two non-singular (constant) matrices P and Q such that $\widetilde{A} + \lambda \widetilde{B} = P(A + \lambda B)Q$.*

The strict equivalence transformation plays a crucial role in our approach to solve (1). The key to our procedure is the following result.

Theorem 1 *Let $A+\lambda B^{\star}$ and $\widetilde{A}+\lambda\widetilde{B}^{\star}$ be strictly equivalent matrix pencils, with $\widetilde{A}+\lambda\widetilde{B}^{\star} = P(A + \lambda B^{\star})Q$. Then Y is a solution of $\widetilde{A}Y + Y^{\star}\widetilde{B} = 0$ if and only if $X = QYP^{-\star}$ is a solution of $AX + X^{\star}B = 0$. As a consequence, the solution spaces of both equations are isomorphic via $Y \mapsto QYP^{-\star} = X$.*

Proof. Given the conditions of the statement, we have $\widetilde{A} = PAQ$ and $\widetilde{B}^{\star} = PB^{\star}Q$, and by applying the $\star$ operator we obtain $\widetilde{B} = Q^{\star}BP^{\star}$. Then

$$\widetilde{A}Y + Y^{\star}\widetilde{B} = PAQY + Y^{\star}Q^{\star}BP^{\star} = P(AQYP^{-\star} + P^{-1}Y^{\star}Q^{\star}B)P^{\star}$$

$$= P(AX + X^{\star}B)P^{\star}.$$

Hence, $AX + X^{\star}B = 0$ if and only if $\widetilde{A}Y + Y^{\star}\widetilde{B} = 0$. The map $Y \mapsto QYP^{-\star}$ is clearly linear and invertible, so it is an isomorphism. □

Since we want to solve (1) for arbitrary $A, B^{\star} \in \mathbb{C}^{m\times n}$, in order to take advantage of Theorem 1 we must look for a canonical representative under strict equivalence of the matrix pencil $A+\lambda B^{\star}$ and solve the equation associated with this representative (i.e. the one whose coefficients are those corresponding to this canonical representative). The canonical form for strict equivalence of matrix pencils is the Kronecker canonical form [11, Ch. XII].

Theorem 2 (Kronecker canonical form) *Each complex matrix pencil $A + \lambda B$, with $A, B \in \mathbb{C}^{m\times n}$, is strictly equivalent to a direct sum of blocks of the following types:*

(1) Right singular blocks:

$$L_\varepsilon = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{pmatrix}_{\varepsilon\times(\varepsilon+1)}.$$

(2) Left singular blocks: L_η^T*, where* L_η *is a right singular block.*

(3) Finite blocks: $J_k(\mu) + \lambda I_k$*, where* $J_k(\mu)$ *is a Jordan block of size* $k \times k$ *associated with* $\mu \in \mathbb{C}$*,*

$$J_k(\mu) = \begin{pmatrix} \mu & 1 & & & \\ & \mu & 1 & & \\ & & \ddots & \ddots & \\ & & & \mu & 1 \\ & & & & \mu \end{pmatrix}_{k\times k}.$$

(4) Infinite blocks: $N_u = I_u + \lambda J_u(0)$.

This pencil is uniquely determined, up to permutation of blocks, and is known as the Kronecker canonical form (KCF) *of* $A + \lambda B$*.*

We will denote the coefficient matrices of the right singular blocks by A_ε and B_ε, that is, $L_\varepsilon = A_\varepsilon + \lambda B_\varepsilon$ where

$$A_\varepsilon := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{pmatrix}_{\varepsilon\times(\varepsilon+1)} \quad \text{and} \quad B_\varepsilon := \begin{pmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}_{\varepsilon\times(\varepsilon+1)}.$$

We will say that $-\mu \in \mathbb{C}$ is an *eigenvalue* of $A + \lambda B$ if there is some block $J_k(\mu) + \lambda I_k$, with $k > 0$, in the KCF of $A + \lambda B$. The *right* (resp. *left*) *minimal indices* of $A + \lambda B$ are those values ε (resp. η) such that L_ε (resp. L_η^T) is a right (resp. left) singular block in the KCF of $A + \lambda B$.

3 Main results

Let the matrix pencil $E + \lambda F^\star$ be given in block-diagonal form. Then E and $F^\star$ are of the form:

$$E = \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_d \end{pmatrix}, \qquad F^\star = \begin{pmatrix} F_1^\star & & \\ & \ddots & \\ & & F_d^\star \end{pmatrix}, \tag{4}$$

with $E_i, F_i^\star \in \mathbb{C}^{m_i\times n_i}$. In order to solve equation (1), let us partition X into blocks conformally with the partition of E and $F^\star$, that is, set $X = [X_{ij}]_{i,j=1}^d$ with $X_{ij} \in \mathbb{C}^{n_i\times m_j}$. Thus, (1) becomes:

$$\begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_d \end{pmatrix}\begin{pmatrix} X_{11} & \cdots & X_{1d} \\ \vdots & \ddots & \vdots \\ X_{d1} & \cdots & X_{dd} \end{pmatrix} + \begin{pmatrix} X_{11}^\star & \cdots & X_{d1}^\star \\ \vdots & \ddots & \vdots \\ X_{1d}^\star & \cdots & X_{dd}^\star \end{pmatrix}\begin{pmatrix} F_1 & & \\ & \ddots & \\ & & F_d \end{pmatrix} = 0. \tag{5}$$

The original equation (1) can be decoupled into smaller equations after equating by blocks in (5). In this way, we obtain two different kinds of equations. On one hand, equating the (i,i) block in (5) we get a matrix equation of the form (1) but involving only the X_{ii} block of the unknown with the diagonal blocks E_i and F_i as coefficients:

$$E_i X_{ii} + X_{ii}^\star F_i = 0, \qquad i = 1, \dots, d\,. \tag{6}$$

On the other hand, if we equate both the (i,j) and the (j,i) blocks in (5) for $i \neq j$, then we get a system of two linear matrix equations involving the unknown blocks X_{ij} and X_{ji}, and with the diagonal blocks E_i, F_i, E_j, F_j as coefficients:

$$\begin{array}{l} E_i X_{ij} + X_{ji}^\star F_j = 0 \\ E_j X_{ji} + X_{ij}^\star F_i = 0 \end{array}, \qquad 1 \leq i < j \leq d\,. \tag{7}$$

The solution X of the original equation (1) with coefficients E and F is obtained from the solution of the equations (6) for each single block $E_i + \lambda F_i^\star$, and the solution of the systems (7) for all pairs of blocks $E_i + \lambda F_i^\star$ and $E_j + \lambda F_j^\star$, with $i < j$. In particular, the dimension of the solution space of (1) is equal to the sum of the dimensions of the solution spaces of all these equations and systems of two equations.

When particularizing to the KCF of $A + \lambda B^\star$,

$$K_1 + \lambda K_2^\star = P(A + \lambda B^\star)Q, \tag{8}$$

the problem of solving (1) with K_1 and K_2 as coefficient matrices reduces to solving (6) for all single blocks in the KCF and (7) for all pairs of canonical blocks.

Summarizing the previous arguments, together with those of Section 2, the solution of (1) can be obtained from the KCF of $A + \lambda B^\star$ and the nonsingular matrices P, Q in (8) in the following way:

Step 1. Solve (6) and (7) for all blocks $E_i + \lambda F_i^\star, E_j + \lambda F_j^\star$ in the KCF of $A + \lambda B^\star$ with $i < j$. This gives X_{ii}, X_{ij}, and X_{ji} for $i, j = 1, \dots, d$ and $i < j$.

Step 2. Set $X = [X_{ij}]$, where X_{ij} are the solutions obtained in **Step 1**.

Step 3. Recover the solution of (1) through the linear transformation $X \mapsto QXP^{-\star}$, where X is the matrix in **Step 2** and P, Q are as in (8).

Following this procedure, the dimension of the solution space of equation (1) depends on the sizes of the blocks in the KCF of $A + \lambda B^\star$ as stated in the following two theorems, where the cases $\star = T$ and $\star = *$ are addressed separately. For any $q \in \mathbb{R}$, we will use the standard notation $\lfloor q \rfloor$ (respectively, $\lceil q \rceil$) for the largest (resp. smallest) integer that is less (resp. greater) than or equal to q.

Theorem 3 (Breakdown of the dimension count for $AX + X^T B = 0$) *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$ be two complex matrices, and let the KCF of the pencil $A + \lambda B^T$ be*

$$\begin{aligned} K_1 + \lambda K_2^T = & \; L_{\varepsilon_1} \oplus L_{\varepsilon_2} \oplus \cdots \oplus L_{\varepsilon_p} \\ & \oplus L_{\eta_1}^T \oplus L_{\eta_2}^T \oplus \cdots \oplus L_{\eta_q}^T \\ & \oplus N_{u_1} \oplus N_{u_2} \oplus \cdots \oplus N_{u_r} \\ & \oplus J_{k_1}(\mu_1) + \lambda I_{k_1} \oplus J_{k_2}(\mu_2) + \lambda I_{k_2} \oplus \cdots \oplus J_{k_s}(\mu_s) + \lambda I_{k_s}. \end{aligned}$$

Then the dimension of the solution space of the matrix equation

$$AX + X^T B = 0$$

depends only on $K_1 + \lambda K_2^T$. It can be computed as the sum

$$d_{Total} = d_{right} + d_{fin} + d_{right,right} + d_{fin,fin} + d_{right,left} + d_{right,\infty} + d_{right,fin} + d_{\infty,fin}, \tag{9}$$

whose summands are given by:

1. *The dimension due to equation* (6) *corresponding to the right singular blocks:*

$$d_{right} = \sum_{i=1}^{p} \varepsilon_i.$$

2. *The dimension due to equation* (6) *corresponding to the finite blocks:*

$$d_{fin} = \sum_{i} \lfloor k_i/2 \rfloor + \sum_{j} \lceil k_j/2 \rceil,$$

   where the first sum is taken over all blocks in $K_1 + \lambda K_2^T$ *of the form* $J_{k_i}(1) + \lambda I_{k_i}$ *and the second sum over all blocks of the form* $J_{k_j}(-1) + \lambda I_{k_j}$.

3. *The dimension due to the systems of equations* (7) *involving a pair of right singular blocks:*

$$d_{right,right} = \sum_{\substack{i,j=1 \\ i<j}}^{p} \varepsilon_i + \varepsilon_j.$$

4. *The dimension due to the systems of equations* (7) *involving a pair of finite blocks:*

$$d_{fin,fin} = \sum_{i,j} \min\{k_i, k_j\},$$

   where the sum is taken over all pairs $J_{k_i}(\mu_i) + \lambda I_{k_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ *of blocks in* $K_1 + \lambda K_2^T$ *such that* $i < j$ *and* $\mu_i \mu_j = 1$.

5. *The dimension due to the systems of equations* (7) *involving a right singular block and a left singular block:*

$$d_{right,left} = \sum_{i,j} (\eta_j - \varepsilon_i + 1),$$

   where the sum is taken over all pairs $L_{\varepsilon_i}, L_{\eta_j}^T$ *of blocks in* $K_1 + \lambda K_2^T$ *such that* $\varepsilon_i \leq \eta_j$.

6. *The dimension due to the systems of equations* (7) *involving a right singular block and an infinite block:*

$$d_{right,\infty} = p \sum_{i=1}^{r} u_i.$$

7. *The dimension due to the systems of equations* (7) *involving a right singular block and a finite block:*

$$d_{right,fin} = p \sum_{i=1}^{s} k_i.$$

8. *The dimension due to the systems of equations* (7) *involving an infinite block and a finite block:*

$$d_{\infty,fin} = \sum_{i,j} \min\{u_i, k_j\},$$

   where the sum is taken over all pairs $N_{u_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ *of blocks in* $K_1 + \lambda K_2^T$ *with* $\mu_j = 0$.

Theorem 4 (Breakdown of the dimension count for $AX + X^*B = 0$) *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$ be two complex matrices, and let the KCF of the pencil $A + \lambda B^*$ be*

$$\begin{aligned} K_1 + \lambda K_2^* = {} & L_{\varepsilon_1} \oplus L_{\varepsilon_2} \oplus \cdots \oplus L_{\varepsilon_p} \\ & \oplus L_{\eta_1}^T \oplus L_{\eta_2}^T \oplus \cdots \oplus L_{\eta_q}^T \\ & \oplus N_{u_1} \oplus N_{u_2} \oplus \cdots \oplus N_{u_r} \\ & \oplus J_{k_1}(\mu_1) + \lambda I_{k_1} \oplus J_{k_2}(\mu_2) + \lambda I_{k_2} \oplus \cdots \oplus J_{k_s}(\mu_s) + \lambda I_{k_s}. \end{aligned}$$

Then the real dimension of the solution space of the matrix equation

$$AX + X^*B = 0$$

depends only on $K_1 + \lambda K_2^*$. It can be computed as the sum

$$d^*_{Total} = d^*_{right} + d^*_{fin} + d^*_{right,right} + d^*_{fin,fin} + d^*_{right,left} + d^*_{right,\infty} + d^*_{right,fin} + d^*_{\infty,fin}, \quad (10)$$

whose summands are given by:

1. *The real dimension due to equation (6) corresponding to the right singular blocks:*
$$d^*_{right} = 2\sum_{i=1}^{p} \varepsilon_i.$$
2. *The real dimension due to equation (6) corresponding to the finite blocks:*
$$d^*_{fin} = \sum_i k_i,$$
where the sum is taken over all blocks in $K_1 + \lambda K_2^*$ of the form $J_{k_i}(\mu) + \lambda I_{k_i}$ with $|\mu| = 1$.
3. *The real dimension due to the systems of equations (7) involving a pair of right singular blocks:*
$$d^*_{right,right} = 2\sum_{\substack{i,j=1\\ i<j}}^{p} (\varepsilon_i + \varepsilon_j).$$
4. *The real dimension due to the systems of equations (7) involving a pair of finite blocks:*
$$d^*_{fin,fin} = 2\sum_{i,j} \min\{k_i, k_j\},$$
where the sum is taken over all pairs $J_{k_i}(\mu_i) + \lambda I_{k_i}, J_{k_j}(\mu_j) + \lambda I_{k_j}$ of blocks in $K_1 + \lambda K_2^*$ such that $i < j$ and $\mu_i \overline{\mu}_j = 1$.
5. *The real dimension due to the systems of equations (7) involving a right singular block and a left singular block:*
$$d^*_{right,left} = 2\sum_{i,j} \left(\eta_j - \varepsilon_i + 1\right),$$
where the sum is taken over all pairs $L_{\varepsilon_i}, L_{\eta_j}^T$ of blocks in $K_1 + \lambda K_2^*$ such that $\varepsilon_i \le \eta_j$.
6. *The real dimension due to the systems of equations (7) involving a right singular block and an infinite block:*
$$d^*_{right,\infty} = 2p\sum_{i=1}^{r} u_i.$$

7. *The real dimension due to the systems of equations* (7) *involving a right singular block and a finite block:*

$$d^*_{right,fin} = 2p \sum_{i=1}^{s} k_i.$$

8. *The real dimension due to the systems of equations* (7) *involving an infinite block and a finite block:*

$$d^*_{\infty,fin} = 2 \sum_{i,j} \min\{u_i, k_j\},$$

where the sum is taken over all pairs $N_{u_i}, J_{k_j}(\mu_j)+\lambda I_{k_j}$ *of blocks in* $K_1+\lambda K_2^*$ *with* $\mu_j = 0$.

Remark 1 *In theorems* 3 *and* 4 *we have referred to the KCF of* $A + \lambda B^\star$ *as* $K_1 + \lambda K_2^\star$*, with the leading coefficient* $K_2^\star$ *transposed or conjugate-transposed, to highlight Theorem* 1*. However, all canonical blocks appear without transposing or conjugate-transposing the leading coefficient, as in Theorem* 2*. Hence, and according to* (4)*, the leading coefficient in the canonical blocks will appear transposed or conjugate-transposed in the corresponding equations* (6) *and* (7)*.*

Remark 2 *It is worth noticing that, though we consider real dimension instead of complex dimension in Theorem* 4*, the KCF of* $A + \lambda B^*$ *that we are using in this case is not the real, but the complex KCF.*

In the dimension count of theorems 3 and 4 the reader will notice that there are summands corresponding to the dimensions of the solution spaces of equations or systems of equations for several canonical blocks in the KCF that appear to be missing. We will see that all these dimensions are zero (lemmas 6, 7, 11, 12, 17, 18 for Theorem 3 and lemmas 22, 23, 26, 27, 32, 34 for Theorem 4). For the sake of brevity and clarity of the exposition, we have included in (9) and (10) only those cases where the dimension is not necessarily zero. The remaining cases are stated in the following two lemmas.

Lemma 1 *Let* $K_1 + \lambda K_2^T$ *be as in Theorem* 3*. The dimension due to the equations* (6) *and systems of equations* (7) *involving the remaining blocks and pairs of blocks in* $K_1 + \lambda K_2^T$ *is zero, that is,*

$$d_{left} = d_{\infty} = d_{left,left} = d_{\infty,\infty} = d_{left,\infty} = d_{left,fin} = 0.$$

Lemma 2 *Let* $K_1 + \lambda K_2^*$ *be as in Theorem* 4*. The dimension due to the equations* (6) *and systems of equations* (7) *involving the remaining blocks and pairs of blocks in* $K_1 + \lambda K_2^*$ *is zero, that is,*

$$d^*_{left} = d^*_{\infty} = d^*_{left,left} = d^*_{\infty,\infty} = d^*_{left,\infty} = d^*_{left,fin} = 0.$$

In theorems 3 and 4 we have focused on the dimension of the solution space of (1). In sections 4 and 5 we will prove these results along with those of lemmas 1 and 2 by computing the dimension of the solution spaces of equations (6) and systems of equations (7) for all blocks and pairs of blocks appearing in $K_1 + \lambda K_2^\star$. Our procedure also provides a description of the explicit solution of equation (1).

4 Solution space for canonical blocks: the transpose case

In this section we will focus on the case $\star = T$. As we have seen in sections 2 and 3, the solution of (1) is closely related to the pencil $A + \lambda B^T$. In order to emphasize this relationship, and for the sake of brevity, we will denote the (vector) space of solutions of the equation (1) by $\mathcal{S}(A+\lambda B^T)$, that is

$$\mathcal{S}(A+\lambda B^T) := \left\{X \in \mathbb{C}^{m\times n} \ : \ AX + X^T B = 0\right\}.$$

In the following, we will use the standard entry-wise notation $X = [x_{ij}]$ and $Y = [y_{ij}]$ for the matrices X and Y.

Given the KCF of the pencil $A+\lambda B^T$, some of the minimal indices ε_i or η_j, for $i = 1,\ldots,p$ or $j = 1,\ldots,q$, may be equal to zero. Right minimal indices equal to zero correspond to zero columns in the KCF, whereas zero left minimal indices correspond to zero rows. The case of zero minimal indices in Theorem 3 deserves a separate analysis, because it leads to matrix equations involving matrices with one of its dimensions equal to zero. In order to address this particular case separately, we will isolate the zero minimal indices in the KCF of $A+\lambda B^T$. More precisely, if $K_1+\lambda K_2^T$ denotes this KCF, then we may write

$$K_1 + \lambda K_2^T = (E+\lambda F^T) \oplus 0_{g\times h}, \tag{11}$$

where the pencil $E+\lambda F^T$ is in KCF and has neither left nor right minimal indices equal to zero. We use the convention that g and h are allowed to be zero. If g is zero then the pencil $K_1+\lambda K_2^T$ takes the form of $E+\lambda F^T$ with h columns of zeros appended on the right, while if h is zero then $K_1+\lambda K_2^T$ takes the form of $E+\lambda F^T$ with g rows of zeros appended on the bottom.

Lemma 3 *Let $A+\lambda B^T$ be an $m\times n$ pencil whose KCF, $K_1+\lambda K_2^T$, is of the form* (11). *Then the dimension of the solution space of the matrix equation $AX + X^T B = 0$ is*

$$\dim \mathcal{S}(A+\lambda B^T) = \dim \mathcal{S}(K_1+\lambda K_2^T) = mh + \dim \mathcal{S}(E+\lambda F^T).$$

Proof. The equation

$$K_1 X + X^T K_2 = 0$$

may be written as a block equation:

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} + \begin{pmatrix} X_{11}^T & X_{21}^T \\ X_{12}^T & X_{22}^T \end{pmatrix}\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating blocks, this is equivalent to the system of equations

$$EX_{11} + X_{11}^T F = 0 \tag{12}$$

$$F^T X_{12} = 0 \tag{13}$$

$$EX_{12} = 0 \tag{14}$$

As a consequence of (13) and (14), each column of X_{12} must lie in the kernel of both E and F^T, so that if X_{12} is nonzero, then the pencil $D+\lambda E^T$ has a constant null-vector. It follows that the pencil $E+\lambda F^T$ has a right minimal index equal to zero [11, Ch. XII, §5], contradicting our assumption about $E+\lambda F^T$. We conclude that X_{12} is identically zero. Then, since X_{22} and X_{21} may be chosen freely, we get that the dimension of $S(A+\lambda B^T)$ is equal to $mh + \dim\, \mathcal{S}(E+\lambda F^T)$, from (12). □

Notice that the proof of Lemma 3 is still valid when $g = 0$ or $h = 0$, and that Lemma 3 is in accordance with Theorem 3. More precisely, Lemma 3 states the same results as Theorem 3, even though it singles out left and right minimal indices equal to zero. In particular, if $K_1+\lambda K_2^T$ is as in (11) then it contains h right and g left minimal indices equal to zero. If we count the total dimension in Theorem 3 due to these blocks (items 1, 3, 5, 6 and 7) we get a total of mh, which is precisely the quantity stated in Lemma 3 for the zero block $0_{g\times h}$ corresponding to these minimal indices. Therefore, in order to prove Theorem 3, we may consider only nonzero minimal indices (both left and right). In this way, in all the statements and proofs in this section where left and right singular blocks appear, we will implicitly assume that the sizes of all these blocks are nonzero.

We next introduce a lemma which will help us to determine when the solution space to (6) or (7) is zero dimensional for a block in the KCF.

Lemma 4 *If $X \in \mathbb{C}^{m\times n}$ satisfies $X = AXB$, where either $A \in \mathbb{C}^{m\times m}$ or $B \in \mathbb{C}^{n\times n}$ is a nilpotent matrix, then $X = 0$.*

Proof. Since A or B is a nilpotent matrix, there exists some integer $\nu \geq 0$ such that $A^\nu = 0$ or $B^\nu = 0$. Then, iterating the identity $X = AXB$ we get

$$X = AXB = A(AXB)B = A^2XB^2 = \ldots = A^\nu XB^\nu = 0.$$

□

Another useful observation, that we will use for both $\star = T, *$ comes from the connection between equation (1) and a Sylvester equation in the particular case where A and B are nonsingular. More precisely, if A is nonsingular, then from (1) we get $X = -A^{-1}X^\star B$ and, by applying the $\star$ operator, $X^\star = -B^\star XA^{-\star}$. Replacing this expression in the original equation (1), we arrive at $AX - B^\star XA^{-\star}B = 0$, which is equivalent to

$$(B^{-\star}A)X - X(A^{-\star}B) = 0, \tag{15}$$

provided that B is nonsingular. Hence, if X is a solution of (1), then it is also a solution of the Sylvester equation (15). Though the converse is not true in general, we will take advantage of this approach (see the proof of Lemma 9 in the case $\mu \neq 0, \pm 1$).

4.1 Dimension of the solution space for single blocks

In this section we will prove all claims in Theorem 3 and Lemma 1 regarding the solution space of equations involving single blocks in the KCF of $A + \lambda B^T$.

Lemma 5 (Right singular block) *The dimension of the solution space of*

$$A_\varepsilon X + X^T B_\varepsilon^T = 0 \tag{16}$$

is

$$\dim \mathcal{S}(L_\varepsilon) = \varepsilon.$$

Proof. First, notice that the unknown matrix X in (16) is of size $(\varepsilon+1)\times\varepsilon$. Equation (16) is equivalent to

$$\begin{pmatrix} x_{21} & x_{22} & \cdots & x_{2\varepsilon} \\ x_{31} & x_{32} & \cdots & x_{3\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\varepsilon+1,1} & x_{\varepsilon+1,2} & \cdots & x_{\varepsilon+1,\varepsilon} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{\varepsilon 1} \\ x_{12} & x_{22} & \cdots & x_{\varepsilon 2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1\varepsilon} & x_{2\varepsilon} & \cdots & x_{\varepsilon\varepsilon} \end{pmatrix} = 0_{\varepsilon\times\varepsilon}.$$

This is in turn equivalent to the system of equations

$$x_{ij} = -x_{j+1,i} \quad \text{for } 1 \leq i, j \leq \varepsilon. \tag{17}$$

Iterating this once we obtain

$$x_{ij} = -x_{j+1,i} = x_{i+1,j+1} \quad \text{for } i = 1, \ldots, \varepsilon,\ j = 1, \ldots, \varepsilon - 1.$$

This implies that the matrix X is Toeplitz, and thus completely determined by its first row and column. Now, setting $i = 1$ in (17) we have

$$x_{1j} = -x_{j+1,1} \quad \text{for } j = 1, \ldots, \varepsilon,$$

which means that the first column of X is determined by the first row. Thus $x_{11}, x_{12}, \ldots, x_{1\varepsilon}$ completely determine X, so the general solution of (16) depends on at most ε free parameters. On the other hand, by direct computation it is straightforward to see that every matrix X of the following form satisfies (17):

$$X = \begin{pmatrix} c_1 & c_2 & \cdots & c_{\varepsilon-1} & c_\varepsilon \\ -c_1 & c_1 & c_2 & \cdots & c_{\varepsilon-1} \\ -c_2 & -c_1 & c_1 & \ddots & \vdots \\ \vdots & -c_2 & -c_1 & \ddots & c_2 \\ -c_{\varepsilon-1} & \vdots & \ddots & \ddots & c_1 \\ -c_\varepsilon & -c_{\varepsilon-1} & \cdots & -c_2 & -c_1 \end{pmatrix}, \tag{18}$$

for all values $c_1, c_2, \ldots, c_\varepsilon \in \mathbb{C}$. Thus, the general solution of (16) depends on exactly ε free parameters, and the result follows. Moreover, (18) is the general solution to (16). □

Lemma 6 (Left singular block) *The dimension of the solution space of*

$$A_\eta^T X + X^T B_\eta = 0 \tag{19}$$

is

$$\dim \mathcal{S}(L_\eta^T) = 0.$$

Proof. We first remark the identities

$$A_\eta A_\eta^T = I_\eta \tag{20}$$
$$B_\eta B_\eta^T = I_\eta, \tag{21}$$
$$A_\eta B_\eta^T = J_\eta(0), \tag{22}$$
$$A_\eta^T B_\eta = J_{\eta+1}(0)^T, \tag{23}$$

that will be used several times throughout the paper. Now, by multiplying (19) on the left by A_η and using (20), we obtain

$$X + A_\eta X^T B_\eta = 0, \tag{24}$$

so $X = -A_\eta X^T B_\eta$, and by transposing, $X^T = -B_\eta^T X A_\eta^T$. Replacing this in (24) we get

$$X - A_\eta B_\eta^T X A_\eta^T B_\eta = 0,$$

which is equivalent, by (21) and (23), to

$$X = J_\eta(0) X J_{\eta+1}(0)^T.$$

Since both $J_\eta(0)$ and $J_{\eta+1}(0)^T$ are nilpotent matrices, Lemma 4 implies that $X = 0$, and the result follows. □

Lemma 7 (Infinite block) *The dimension of the solution space of*

$$X + X^T J_u(0)^T = 0 \tag{25}$$

is

$$\dim \mathcal{S}(N_u) = 0.$$

Proof. From (25) we have $X = -X^T J_u(0)^T$ and, by transposition, $X^T = -J_u(0)X$. Replacing this in (25) we get

$$X = J_u(0) X J_u(0)^T,$$

and applying Lemma 4 we conclude that $X = 0$. □

In order to approach our most technical case, the case of a single Jordan block, we need the following lemma.

Lemma 8 *Let $X, Y \in \mathbb{C}^{k\times k}$ be such that $Y = RX^T R$, where*

$$R := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

is the $k \times k$ reverse identity. Then X is a solution of $J_k(\mu)X + X^T = 0$ if and only if Y is a solution of $YJ_k(\mu) + Y^T = 0$.

Proof. For simplicity, let us denote $J_k(\mu)$ by J during this proof. Then, using that $J = RJ^T R$ and the hypothesis $Y = RX^T R$, we have the following chain of identities:

$$YJ + Y^T = 0 \Leftrightarrow (RX^T R)(RJ^T R) + RXR = 0 \Leftrightarrow R(X^T J^T + X)R = 0$$

$$\Leftrightarrow X^T J^T + X = 0 \Leftrightarrow JX + X^T = 0,$$

and the result follows. $\square$

Lemma 9 (Finite block) *The dimension of the solution space of*

$$J_k(\mu)X + X^T = 0 \tag{26}$$

is

$$\dim \mathcal{S}(J_k(\mu) + \lambda I_k) = \begin{cases} 0\,, & \text{if } \mu \neq \pm 1, \\ \lfloor k/2 \rfloor\,, & \text{if } \mu = 1, \\ \lceil k/2 \rceil\,, & \text{if } \mu = -1. \end{cases}$$

Proof. Let us consider separately the following cases:

- $\mu \neq 0, \pm 1$: By transposing (26) we get $X = -X^T J_k(\mu)^T$, and replacing this expression in (26) we get
$$J_k(\mu)X^T J_k(\mu)^T - X^T = 0.$$
Since $\mu \neq 0$, the matrix $J_k(\mu)$ is invertible, so this is equivalent to $X^T J_k(\mu)^T - J_k(\mu)^{-1}X^T = 0$. This is a Sylvester equation, and since $\mu \neq \pm 1$, the matrices $J_k(\mu)^T$ and $J_k(\mu)^{-1}$ do not have common eigenvalues. Then the only solution of this equation is $X^T = 0$ [11, Ch. VIII, §1], so $X = 0$ is the only solution of (26).

- $\mu = 0$: By similar reasonings to the preceding case, we arrive at $X^T = J_k(0)X^T J_k(0)$. Since $J_k(0)$ is a nilpotent matrix, Lemma 4 implies that $X^T = 0$. Hence $X = 0$ is again the only solution of (26).

- $\mu = \pm 1$: By Lemma 8, the solutions of
$$J_k(\pm 1)X + X^T = 0$$
and the solutions of
$$YJ_k(\pm 1) + Y^T = 0$$
are related by the change of variables $Y = RX^T R$. In particular, the dimension of the solution space of both equations is the same. As a consequence, we may concentrate on the equation $XJ_k(\pm 1) + X^T = 0$. The case $\mu = (-1)^k$ has already been solved in [7, Appendix A]. The remaining case, $\mu = (-1)^{k+1}$, is rather technical and is addressed in Appendix A.

$\square$

4.2 Dimension of the solution space for pairs of blocks

In this section we will prove all claims in Theorem 3 and Lemma 1 regarding the solution space of pairs of equations involving pairs of blocks in the KCF of $A + \lambda B^T$. We first display the results for pairs of blocks of the same type and then those involving pairs of blocks of different types.

Lemma 10 (Two right singular blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^T B_\delta^T \quad = 0 \tag{27}$$
$$A_\delta Y + X^T B_\varepsilon^T \quad = 0, \tag{28}$$

is

$$\dim \mathcal{S}(L_\varepsilon, L_\delta) = \varepsilon + \delta.$$

Proof. Note X and Y have sizes $(\varepsilon+1)\times\delta$ and $(\delta+1)\times\varepsilon$ respectively. We can rewrite equations (27) and (28) entry-wise in the form:

$$\begin{pmatrix} x_{21} & x_{22} & \cdots & x_{2\delta} \\ x_{31} & x_{32} & \cdots & x_{3\delta} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\varepsilon+1,1} & x_{\varepsilon+1,2} & \cdots & x_{\varepsilon+1,\delta} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{21} & \cdots & y_{\delta 1} \\ y_{12} & y_{22} & \cdots & y_{\delta 2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1\varepsilon} & y_{2\varepsilon} & \cdots & y_{\delta\varepsilon} \end{pmatrix} = 0_{\varepsilon\times\delta} \tag{29}$$

$$\begin{pmatrix} y_{21} & y_{22} & \cdots & y_{2\varepsilon} \\ y_{31} & y_{32} & \cdots & y_{3\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ y_{\delta+1,1} & y_{\delta+1,2} & \cdots & y_{\delta+1,\varepsilon} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{\varepsilon 1} \\ x_{12} & x_{22} & \cdots & x_{\varepsilon 2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1\delta} & x_{2\delta} & \cdots & x_{\varepsilon\delta} \end{pmatrix} = 0_{\delta\times\varepsilon}. \tag{30}$$

We see in equations (29) and (30) that for any entry y_{ab} of Y, there is an entry x_{ij} of X such that $x_{ij} + y_{ab} = 0$. We conclude that Y is completely determined by X.

Equations (29) and (30) are equivalent to the system of equations:

(a) $x_{b+1,a} = -y_{ab}$ for $1 \le a \le \delta, \quad 1 \le b \le \varepsilon,$
(b) $x_{ij} = -y_{j+1,i}$ for $1 \le i \le \varepsilon, \quad 1 \le j \le \delta.$

From here we may conclude that X is a Toeplitz matrix. To see this, let x_{ij} be any entry of X such that $x_{i+1,j+1}$ is defined. Then we must have $i \le \varepsilon$ and $j \le \delta - 1$, so we may apply relation (b) to get

$$x_{ij} = -y_{j+1,i}, \text{ for } i \le \varepsilon, \quad j \le \delta - 1\,.$$

Since $j + 1 \le \delta$ and $i \le \varepsilon$, by relation (a) we have

$$-y_{j+1,i} = x_{i+1,j+1}. \text{ for } 1 \le i \le \varepsilon, \quad 1 \le j \le \delta - 1\,.$$

Hence $x_{ij} = x_{i+1,j+1}$ for $1 \le i \le \varepsilon$, and $1 \le j \le \delta - 1$, so X is indeed a Toeplitz matrix. This shows that X is of the following form:

$$X = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & \cdots \\ c_1 & d_1 & d_2 & d_3 & \ddots \\ c_2 & c_1 & d_1 & d_2 & \ddots \\ \vdots & c_2 & c_1 & d_1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}_{(\varepsilon+1)\times\delta} . \tag{31}$$

Thus X is always determined by its first row and column, a total of $\varepsilon + \delta$ parameters. Furthermore, it is straightforward to verify that any X of form (31) determines a unique matrix, Y, of the form

$$Y = \begin{pmatrix} -c_1 & -c_2 & -c_3 & \cdots & \cdots \\ -d_1 & -c_1 & -c_2 & -c_3 & \ddots \\ -d_2 & -d_1 & -c_1 & -c_2 & \ddots \\ \vdots & -d_2 & -d_1 & -c_1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}_{(\delta+1)\times\varepsilon},$$

and the pair X, Y is a solution of the system of equations (29) and (30). It follows that the dimension of the solution space to equations (27) and (28) is $\varepsilon + \delta$. $\square$

Lemma 11 (Two left singular blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + Y^T B_\gamma = 0 \tag{32}$$

$$A_\gamma^T Y + X^T B_\eta = 0, \tag{33}$$

is

$$\dim \mathcal{S}(L_\eta^T, L_\gamma^T) = 0.$$

Proof. By multiplying (32) by A_η on the left and using (20) we get $X = -A_\eta Y^T B_\gamma$, and by transposition, $X^T = -B_\gamma^T Y A_\eta^T$. Substituting this in (33) we achieve

$$A_\gamma^T Y - B_\gamma^T Y A_\eta^T B_\eta = 0,$$

and multiplying on the left by A_γ and using (20) this implies

$$Y - A_\gamma B_\gamma^T Y A_\eta^T B_\eta = 0,$$

which in turn implies, using (21) and (23),

$$Y - J_\gamma(0) Y J_{\eta+1}(0)^T = 0.$$

Now, Lemma 4 gives $Y = 0$, which implies $X = 0$ as well, so the only solution of (32) and (33) is the trivial solution. $\square$

Lemma 12 (Two infinite blocks) *The dimension of the solution space of the system of matrix equations*

$$X + Y^T J_u(0)^T = 0 \tag{34}$$

$$Y + X^T J_t(0)^T = 0, \tag{35}$$

is

$$\dim \mathcal{S}(N_u, N_t) = 0.$$

Proof. From (34) we obtain $X = -Y^T J_u(0)^T$, and transposing, $X^T = -J_u(0)Y$. Substituting this into (35) we get

$$Y = J_u(0) Y J_t(0)^T,$$

which implies $Y = 0$ by Lemma 4, so $X = 0$ as well. Then the only solution of (34) and (35) is the trivial solution, and the result follows. $\square$

Lemma 13 (Two finite blocks) *The dimension of the solution space of the system of matrix equations*

$$J_k(\mu)X + Y^T = 0 \tag{36}$$
$$J_\ell(\nu)Y + X^T = 0, \tag{37}$$

is

$$\dim \mathcal{S}(J_k(\mu) + \lambda I_k, J_\ell(\nu) + \lambda I_\ell) = \begin{cases} \min\{k, \ell\}, & \text{if } \mu\nu = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First, notice that X has size $k \times \ell$ and Y has size $\ell \times k$. We begin by solving for X in (37):

$$X = -Y^T J_\ell(\nu)^T. \tag{38}$$

Substituting back into (36), we have

$$J_k(\mu)Y^T J_\ell(\nu)^T - Y^T = 0. \tag{39}$$

It is clear by (38) that Y determines X and that the pair X, Y satisfies (36) and (37) if and only if they satisfy (38) and (39).

Now we consider the following two cases:

▶ $\mu \neq 0$ or $\nu \neq 0$: First suppose $\mu \neq 0$. In this case $J_k(\mu)$ is invertible, so we may rewrite (39) in the form

$$Y^T J_\ell(\nu)^T - J_k(\mu)^{-1} Y^T = 0, \tag{40}$$

which is a Sylvester equation in the variable Y^T. The space of solutions to (40) has dimension $\min\{k, \ell\}$ if $\nu = 1/\mu$ and 0 otherwise, and an explicit description of the solutions is available [11, Ch. VIII, §1]. The case of $\nu \neq 0$ is similar and yields the same results.

▶ $\mu = \nu = 0$: In this case (39) reads

$$J_\ell(0)Y^T J_k(0)^T - Y^T = 0.$$

Since $J_\ell(0)$ and $J_k(0)$ are nilpotent, it follows immediately from Lemma 4 that Y^T, and hence Y, must be the zero matrix. By (38) $X = 0$ as well. Thus, in this case, the dimension of the solution space is zero. $\square$

Lemma 14 (Right singular and left singular blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^T B_\eta = 0 \tag{41}$$
$$A_\eta^T Y + X^T B_\varepsilon^T = 0, \tag{42}$$

is

$$\dim \mathcal{S}(L_\varepsilon, L_\eta^T) = \begin{cases} 0, & \text{if } \eta < \varepsilon, \\ \eta - \varepsilon + 1, & \text{if } \varepsilon \leq \eta. \end{cases}$$

Proof. The sizes of X and Y are $(\varepsilon + 1) \times (\eta + 1)$ and $\eta \times \varepsilon$ respectively. The system of equations (41) and (42) is equivalent to

$$\begin{pmatrix} x_{21} & \cdots & x_{2\eta} & x_{2,\eta+1} \\ x_{31} & \cdots & x_{3\eta} & x_{3,\eta+1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{\varepsilon+1,1} & \cdots & x_{\varepsilon+1,\eta} & x_{\varepsilon+1,\eta+1} \end{pmatrix} + \begin{pmatrix} y_{11} & \cdots & y_{\eta 1} & 0 \\ y_{12} & \cdots & y_{\eta 2} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ y_{1\varepsilon} & \cdots & y_{\eta\varepsilon} & 0 \end{pmatrix} = 0_{\varepsilon \times (\eta+1)}, \tag{43}$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ y_{11} & y_{12} & \cdots & y_{1\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ y_{\eta 1} & y_{\eta 2} & \cdots & y_{\eta\varepsilon} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{\varepsilon 1} \\ x_{12} & x_{22} & \cdots & x_{\varepsilon 2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,\eta+1} & x_{2,\eta+1} & \cdots & x_{\varepsilon,\eta+1} \end{pmatrix} = 0_{(\eta+1) \times \varepsilon}. \tag{44}$$

This immediately implies that Y is completely determined by X. Also, (43) and (44) are equivalent to the system of equations

(a) $x_{2,\eta+1} = x_{3,\eta+1} = \cdots = x_{\varepsilon+1,\eta+1} = 0$,
(b) $y_{ij} = -x_{j+1,i}$ for $i = 1, \ldots, \eta, \quad j = 1, \ldots, \varepsilon$,
(c) $x_{11} = x_{21} = \cdots = x_{\varepsilon 1} = 0$,
(d) $y_{ij} = -x_{j,i+1}$ for $i = 1, \ldots, \eta, \quad j = 1, \ldots, \varepsilon$.

Combining (b) and (d), we obtain

$$x_{ij} = x_{i+1,j-1} \quad \text{for } i = 1, \ldots, \varepsilon, \quad j = 2, \ldots, \eta + 1,$$

which shows that any entries of X lying on the same anti-diagonal, $\mathcal{L}_s = \{x_{ij} : i + j = s\}$, are equal. Now, observe that (a) and (c) mean that every entry in the first and $(\eta + 1)$th columns of X are zero except for $x_{1,\eta+1}$ and $x_{\varepsilon+1,1}$. If $\eta < \varepsilon$, every anti-diagonal will contain one of these entries equal to zero. This in turn implies $X = 0 = Y$, so the trivial solution is the only solution if $\eta < \varepsilon$.

If $\varepsilon \leq \eta$, then there are anti-diagonals of X which do not contain any of these zero entries from the first and $(\eta + 1)$th columns. More precisely, these anti-diagonals are those which have an entry in both the first and last row. Since there are $\eta - \varepsilon + 1$ of these anti-diagonals, X will depend on at most $\eta - \varepsilon + 1$ free variables. On the other hand, it is immediate to realize that if

$$X = \begin{pmatrix} 0 & \cdots & 0 & c_1 & \cdots & c_{\eta-\varepsilon+1} \\ \vdots & \iddots & \iddots & & \iddots & 0 \\ 0 & \iddots & & \iddots & \iddots & \vdots \\ c_1 & \cdots & c_{\eta-\varepsilon+1} & 0 & \cdots & 0 \end{pmatrix}_{(\varepsilon+1)\times(\eta+1)} \tag{45}$$

and Y is defined by condition (b) above, then both X and Y satisfy (a–d) for all values of the parameters $c_1, \ldots, c_{\eta-\varepsilon+1}$. As a consequence, the general solution of (41) and (42) depends on exactly $\eta - \varepsilon + 1$ free variables. Moreover, (45) gives the general solution for X from which (b) gives the corresponding matrix Y. □

Lemma 15 (Right singular and infinite blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^T J(0)_u^T = 0 \tag{46}$$

$$Y + X^T B_\varepsilon^T = 0, \tag{47}$$

is

$$\dim \mathcal{S}(L_\varepsilon, N_u) = u.$$

Proof. Note that the dimensions of X and Y are $(\varepsilon + 1) \times u$ and $u \times \varepsilon$ respectively. Now, from equation (46), we obtain the following:

$$\begin{pmatrix} x_{21} & \cdots & x_{2,u-1} & x_{2u} \\ x_{31} & \cdots & x_{3,u-1} & x_{3u} \\ \vdots & \ddots & \vdots & \vdots \\ x_{\varepsilon+1,1} & \cdots & x_{\varepsilon+1,u-1} & x_{\varepsilon+1,u} \end{pmatrix} + \begin{pmatrix} y_{21} & \cdots & y_{u1} & 0 \\ y_{22} & \cdots & y_{u2} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ y_{2\varepsilon} & \cdots & y_{u\varepsilon} & 0 \end{pmatrix} = 0_{\varepsilon\times u}.$$

This implies $x_{2u} = \ldots = x_{\varepsilon+1,u} = 0$ and

$$x_{ij} = -y_{j+1,i-1} \quad \text{for } i = 2, \ldots, \varepsilon + 1, \ j = 1, \ldots, u - 1. \tag{48}$$

By equation (47) we also have

$$\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1\varepsilon} \\ y_{21} & y_{22} & \cdots & y_{2\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ y_{u1} & y_{u2} & \cdots & y_{u\varepsilon} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{\varepsilon 1} \\ x_{12} & x_{22} & \cdots & x_{\varepsilon 2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1u} & x_{2u} & \cdots & x_{\varepsilon u} \end{pmatrix} = 0_{u\times\varepsilon},$$

from which we obtain

$$x_{ij} = -y_{ji} \quad \text{for } i = 1, \dots, \varepsilon,\ j = 1, \dots, u. \tag{49}$$

This shows that Y is completely determined by X. Now, using (48) and (49) we reach a third relation, which is

$$x_{ij} = x_{i-1,j+1} \quad \text{for } i = 2, \dots, \varepsilon + 1,\ j = 1, \dots, u - 1.$$

The last relation shows that any entries of X which lie on the same anti-diagonal are equal. It follows that the anti-diagonals below the main anti-diagonal in X are equal to zero since we have already shown $x_{2u} = ... = x_{\varepsilon+1,u} = 0$. Thus we know that X depends on no more than u free parameters which correspond to $x_{11}, x_{12}, \cdots, x_{1u}$ and that it takes one of the following two forms:

$$X = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_u \\ c_2 & c_3 & \cdots & c_u & 0 \\ c_3 & \vdots & \iddots & \iddots & \vdots \\ \vdots & c_u & \iddots & & \vdots \\ c_u & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad \text{if } u < \varepsilon + 1$$

$$X = \begin{pmatrix} c_1 & c_2 & \cdots & c_{\varepsilon+1} & \cdots & c_u \\ c_2 & \cdots & c_{\varepsilon+1} & \cdots & c_u & 0 \\ \vdots & \iddots & & \iddots & \iddots & \vdots \\ c_{\varepsilon+1} & \cdots & c_u & 0 & \cdots & 0 \end{pmatrix}, \qquad \text{if } \varepsilon + 1 \leq u.$$

Moreover, given any matrix X which takes one of the above forms and setting Y as in (49), we have that the pair X, Y is a solution of (46) and (47). Then, it follows that the general solution depends on exactly u free parameters. Furthermore, the formulas above, together with (49), give the general solution of (46) and (47). □

Lemma 16 (Right singular and finite blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^T = 0 \tag{50}$$

$$J_k(\mu) Y + X^T B_\varepsilon^T = 0, \tag{51}$$

is

$$\dim \mathcal{S}(L_\varepsilon, J_k(\mu) + \lambda I_k) = k.$$

Proof. Note X has size $(\varepsilon + 1) \times k$, and Y has size $k \times \varepsilon$. Let us rewrite equations (50) and (51) entry-wise as

$$\begin{pmatrix} x_{21} & x_{22} & \cdots & x_{2k} \\ x_{31} & x_{32} & \cdots & x_{3k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\varepsilon+1,1} & x_{\varepsilon+1,2} & \cdots & x_{\varepsilon+1,k} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{21} & \cdots & y_{k1} \\ y_{12} & y_{22} & \cdots & y_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1\varepsilon} & y_{2\varepsilon} & \cdots & y_{k\varepsilon} \end{pmatrix} = 0_{\varepsilon \times k}, \tag{52}$$

$$\mu \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1\varepsilon} \\ y_{21} & y_{22} & \cdots & y_{2\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k1} & y_{k2} & \cdots & y_{k\varepsilon} \end{pmatrix} + \begin{pmatrix} y_{21} & y_{22} & \cdots & y_{2\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k1} & y_{k2} & \cdots & y_{k\varepsilon} \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{\varepsilon 1} \\ x_{12} & x_{22} & \cdots & x_{\varepsilon 2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{\varepsilon k} \end{pmatrix} = 0_{k \times \varepsilon}. \tag{53}$$

By isolating columns in (53) we obtain the following identity for $1 \leq j \leq \varepsilon$:

$$\mu \begin{pmatrix} y_{1j} \\ \vdots \\ y_{k-1,j} \\ y_{kj} \end{pmatrix} + \begin{pmatrix} y_{2j} \\ \vdots \\ y_{kj} \\ 0 \end{pmatrix} + \begin{pmatrix} x_{j1} \\ \vdots \\ x_{j,k-1} \\ x_{jk} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

We then use (52) to rewrite this identity as follows:

$$\mu \begin{pmatrix} x_{j+1,1} \\ \vdots \\ x_{j+1,k-1} \\ x_{j+1,k} \end{pmatrix} + \begin{pmatrix} x_{j+1,2} \\ \vdots \\ x_{j+1,k} \\ 0 \end{pmatrix} = \begin{pmatrix} x_{j1} \\ \vdots \\ x_{j,k-1} \\ x_{jk} \end{pmatrix}.$$

This is a recursion relation. It allows us to determine the $(j+1)$th column of X in terms of the jth column or viceversa. Hence, we can let the $x_{\varepsilon+1,1}, \ldots, x_{\varepsilon+1,k}$ be free parameters, and then the matrix X will be uniquely determined. Furthermore, by equation (52), Y is completely determined by X.

Moreover, if we set

$$c_j = x_{\varepsilon+1,j}\,, \qquad \text{for } j = 1, \ldots, k, \tag{54}$$

then the explicit solution, X, is:

$$x_{ij} = \sum_{l=0}^{\varepsilon+1-i} \mu^{\varepsilon+1-i-l} \binom{\varepsilon+1-i}{l} c_{j+l}, \tag{55}$$

where we use the convention that $c_{j+l} = 0$ for $j + l > k$. To prove (55), we can proceed by induction on $i = \varepsilon + 1, \ldots, 1$ (downwards). The initial case, $i = \varepsilon + 1$, is just (54). Now, let us assume that (55) is true for some i with $1 \leq i \leq \varepsilon + 1$, and all $j = 1, \ldots, k$. Proving (55) for $i - 1$ (and all $j = 1, \ldots, k$) reduces to showing that the recursion formula $x_{i-1,j} = \mu x_{ij} + x_{i,j+1}$ satisfies (55). The right hand side of this formula becomes:

$$\mu \sum_{l=0}^{\varepsilon+1-i} \mu^{\varepsilon+1-i-l} \binom{\varepsilon+1-i}{l} c_{j+l} + \sum_{l=0}^{\varepsilon+1-i} \mu^{\varepsilon+1-i-l} \binom{\varepsilon+1-i}{l} c_{j+1+l},$$

which is equal to

$$\sum_{l=0}^{\varepsilon+1-i} \mu^{\varepsilon+2-i-l} \binom{\varepsilon+1-i}{l} c_{j+l} + \sum_{l=1}^{\varepsilon+2-i} \mu^{\varepsilon+2-i-l} \binom{\varepsilon+1-i}{l-1} c_{j+l}. \tag{56}$$

Now, using the binomial identity

$$\binom{m-1}{n-1} + \binom{m-1}{n} = \binom{m}{n},$$

valid for all integers $m, n \geq 1$, (56) is equal to:

$$\sum_{l=0}^{\varepsilon+2-i} \mu^{\varepsilon+2-i-l} \binom{\varepsilon+2-i}{l} c_{j+l},$$

which correspond to the left hand side of the recursion formula, and thus completes the proof of (55). $\square$

Lemma 17 (Left singular and infinite blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + Y^T J_u(0)^T = 0 \tag{57}$$

$$Y + X^T B_\eta = 0, \tag{58}$$

is

$$\dim\, \mathcal{S}(L_\eta^T, N_u) = 0.$$

Proof. By transposing in equation (58) we get $Y^T = -B_\eta^T X$, and replacing this in equation (57) we achieve $A_\eta^T X - B_\eta^T X J_u(0)^T = 0$. Now, multiplying on the left by A_η and using (20) and (22) we reach

$$X - J_\eta(0) X J_u(0)^T = 0,$$

and Lemma 4 implies $X = 0$. From this, it follows that $Y = 0$ and then the only solution of (57) and (58) is the null solution. □

Lemma 18 (Left singular and finite blocks) *The dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + Y^T = 0 \tag{59}$$

$$J_k(\mu) Y + X^T B_\eta = 0, \tag{60}$$

is

$$\dim\, \mathcal{S}(L_\eta^T, J_k(\mu) + \lambda I_k) = 0.$$

Proof. By transposing in equation (59) we get $Y = -X^T A_\eta$, and replacing this in equation (60) we obtain $J_k(\mu) X^T A_\eta - X^T B_\eta = 0$. Multiplying on the right by B_η^T and using (22) and (21) we reach $J_k(\mu) X^T J_\eta(0) - X^T = 0$. Now Lemma 4 implies $X^T = 0$, which in turn gives $X = 0$ and, from this, $Y = 0$. The result follows. □

Lemma 19 (Infinite and finite blocks) *The dimension of the solution space of the system of matrix equations*

$$X + Y^T = 0 \tag{61}$$

$$J_k(\mu) Y + X^T J_u(0)^T = 0, \tag{62}$$

is

$$\dim\, \mathcal{S}(N_u, J_k(\mu) + \lambda I_k) = \begin{cases} \min\{u, k\}, & \text{if } \mu = 0, \\ 0, & \text{if } \mu \neq 0. \end{cases}$$

Proof. By transposing in equation (61) we find $X^T = -Y$, and replacing this in (62) we get the Sylvester equation $J_k(\mu) Y - Y J_u(0)^T = 0$.

If $\mu \neq 0$ then $J_\ell(\mu)$ and $J_u(0)^T$ have no common eigenvalues so that $Y = 0$ [11, Ch. VIII, §1]. From this we get $X = 0$.

If $\mu = 0$ then the solution Y of the Sylvester equation depends on $\min\{u, k\}$ free parameters and an expression for this solution is available [11, Ch. VIII, §1]. From Y the matrix X is completely determined by $X = -Y^T$, and then the result follows. □

5 Solution space for canonical blocks: the conjugate transpose case

This section is the counterpart of Section 4 for $\star = *$. We begin with the counterpart of Lemma 3, whose proof is nearly identical.

Lemma 20 *Let $A + \lambda B^*$ be an $m \times n$ pencil whose KCF is of the form*

$$K_1 + \lambda K_2^* = (E + \lambda F^*) \oplus 0_{g \times h},$$

Then the real dimension of the solution space of the matrix equation $K_1 X + X^* K_2 = 0$ is

$$\dim_{\mathbb{R}} \mathcal{S}^*(A + \lambda B^*) = \dim_{\mathbb{R}} \mathcal{S}^*(K_1 + \lambda K_2^*) = 2mh + \dim_{\mathbb{R}} \mathcal{S}^*(E + \lambda F^*).$$

5.1 Dimension of the solution space for single blocks

In this section we will prove all claims in Theorem 4 and Lemma 2 regarding the solution space of equations involving single blocks in the KCF of $A + \lambda B^*$.

Lemma 21 (Right singular block) *The real dimension of the solution space of*

$$A_\varepsilon X + X^* B_\varepsilon^* = 0 \tag{63}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon) = 2\varepsilon.$$

Proof. Equation (63) is equivalent to the system

$$x_{ij} = -\overline{x}_{j+1,i} \quad \text{for} \quad 1 \leq i, j \leq \varepsilon.$$

Iterating once yields

$$x_{ij} = -\overline{x}_{j+1,1} = x_{i+1,j+1} \quad \text{for} \quad 1 \leq i \leq \varepsilon\,, \quad 1 \leq j \leq \varepsilon - 1.$$

From here we proceed in a matter similar to the proof of Lemma 5 to conclude that X is a solution to (63) if and only if X is a Toeplitz matrix of the form

$$X = \begin{pmatrix} c_1 & c_2 & \cdots & c_{\varepsilon-1} & c_\varepsilon \\ -\overline{c}_1 & c_1 & c_2 & \cdots & c_{\varepsilon-1} \\ -\overline{c}_2 & -\overline{c}_1 & c_1 & \ddots & \vdots \\ \vdots & -\overline{c}_2 & -\overline{c}_1 & \ddots & c_2 \\ -\overline{c}_{\varepsilon-1} & \vdots & \ddots & \ddots & c_1 \\ -\overline{c}_\varepsilon & -\overline{c}_{\varepsilon-1} & \cdots & -\overline{c}_2 & -\overline{c}_1 \end{pmatrix},$$

which is determined by its first row, a total of ε complex parameters. We conclude that the real dimension of the solution space to (63) is 2ε. $\square$

Lemma 22 (Left singular block) *The real dimension of the solution space of*

$$A_\eta^T X + X^* \overline{B}_\eta = 0$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\eta^T) = 0.$$

Proof. Notice that $A_\eta^* = A_\eta^T$ and $B_\eta^* = B_\eta^T$. Using this fact and an argument identical to that of Lemma 6, the result follows. $\square$

Lemma 23 (Infinite block) *The real dimension of the solution space of*

$$X + X^* J_u(0)^* = 0$$

is

$$\dim_{\mathbb{R}} S^*(N_u) = 0.$$

Proof. Observe that $J_u(0)^* = J_u(0)^T$. The proof proceeds as the proof of Lemma 7. $\square$

Lemma 24 (Finite block) *The real dimension of the solution space of*

$$J_k(\mu) X + X^* = 0 \tag{64}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(J_k(\mu) + \lambda I_k) = \begin{cases} k, & \textit{if } |\mu| = 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. From equation (64) we have $X^* = -J_k(\mu)X$, which implies $X = -X^*J_k(\mu)^*$. Substituting this back into equation (64), we obtain

$$J_k(\mu)X^*J_k(\mu)^* - X^* = 0. \tag{65}$$

If $\mu = 0$, the result follows by Lemma 4. If $\mu \neq 0$, we know that $J_k(\mu)$ is invertible. Then (65) is equivalent to

$$J_k(\mu)X^* - X^*J_k(\mu)^{-*} = 0. \tag{66}$$

If X is a solution of (64), then X is also a solution of (66). Since (66) is a Sylvester equation, we know from [11, Ch. VIII, §1] that a nontrivial solution exists if and only if $\mu = 1/\overline{\mu}$, i.e., if $|\mu| = 1$. It follows that the dimension of the solution space of (64) is again zero if $\mu \neq 0$ and $|\mu| \neq 1$.

All that remains to consider is the case $|\mu| = 1$. Since the proof is rather technical, we will see in Appendix B that the solution depends on exactly k free variables and we will give an algorithm to obtain this solution. □

5.2 Dimension of the solution space for pairs of blocks

In this section we will prove all claims in Theorem 4 and Lemma 2 regarding the solution space of pairs of equations involving pairs of blocks in the KCF of $A + \lambda B^*$. As in Section 4.2, we first display the results for pairs of blocks of the same type and then we address the cases of pairs of blocks with different type.

Lemma 25 (Two right singular blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^*B_\delta^* = 0 \tag{67}$$
$$A_\delta Y + X^*B_\varepsilon^* = 0, \tag{68}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, L_\delta) = 2(\varepsilon + \delta).$$

Proof. Equations (67) and (68) are equivalent to the system of equations

(a) $x_{b+1,a} = -\overline{y}_{ab}$ for $1 \leq a \leq \delta$ and $1 \leq b \leq \varepsilon$,
(b) $x_{ij} = -\overline{y}_{j+1,i}$ for $1 \leq i \leq \varepsilon$ and $1 \leq j \leq \delta$,

from which it follows by an argument similar to that of the proof of Lemma 10 that X is an arbitrary Toeplitz matrix. □

Lemma 26 (Two left singular blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + Y^*\overline{B}_\gamma = 0$$
$$A_\gamma^T Y + X^*\overline{B}_\eta = 0,$$

is

$$\dim_{\mathbb{R}} S^*(L_\eta^T, L_\gamma^T) = 0.$$

Proof. Observe that $\overline{B}_\gamma = B_\gamma$ and $\overline{B}_\eta = B_\eta$. The proof follows as in the proof of Lemma 11. □

Lemma 27 (Two infinite blocks) *The real dimension of the solution space of the system of matrix equations*

$$X + Y^*J_u(0)^* = 0$$
$$Y + X^*J_t(0)^* = 0,$$

is

$$\dim_{\mathbb{R}} S^*(N_u, N_t) = 0$$

Proof. Notice $J_u(0)^* = J_u(0)^T$ and $J_t(0)^* = J_t(0)^T$. The proof proceeds as the proof of Lemma 12. $\square$

Lemma 28 (Two finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$J_k(\mu)X + Y^* = 0 \tag{69}$$

$$J_\ell(\nu)Y + X^* = 0, \tag{70}$$

is

$$\dim_{\mathbb{R}} S^*(J_k(\mu) + I_k, J_\ell(\nu)) = \begin{cases} 2\min\{k,\ell\}, & \text{if } \mu\overline{\nu} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first solve for X in (70):

$$X = -Y^* J_\ell(\nu)^*. \tag{71}$$

And substitute in (69):

$$J_k(\mu)XJ_\ell(\nu)^* - Y^* = 0. \tag{72}$$

It is clear that the system of equations (71) and (72) is equivalent to the system (69) and (70). There are now three cases to consider, just as in the proof of Lemma 13, namely, the cases $\mu \neq 0$, $\nu \neq 0$, and $\mu = \nu = 0$. We conclude via the same processes used in Lemma 13 that the set of solutions to (71) and (72) is parametrized by $\min\{k,\ell\}$ complex parameters if $\mu\overline{\nu} = 1$, and zero otherwise. The result follows. $\square$

Lemma 29 (Right singular and left singular blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^* \overline{B}_\eta = 0 \tag{73}$$

$$A_\eta^T Y + X^* B_\varepsilon^* = 0, \tag{74}$$

is

$$\dim_{\mathbb{R}} S^*(L_\varepsilon, L_\eta^T) = \begin{cases} 2(\eta - \varepsilon + 1), & \text{if } \varepsilon \leq \eta, \\ 0, & \text{if } \eta < \varepsilon. \end{cases}$$

Proof. The system (73) and (74) is equivalent to the relations

(a) $x_{2\eta+1} = x_{3,\eta+1} = \ldots = x_{\varepsilon+1,\eta+1} = 0$,
(b) $y_{ij} = -\overline{x}_{j+1,i}$ for $1 \leq i \leq \eta$, $1 \leq j \leq \varepsilon$,
(c) $x_{11} = x_{21} = \ldots = x_{\varepsilon 1} = 0$,
(d) $y_{ij} = -\overline{x}_{j,i+1}$ for $1 \leq i \leq \eta$, $1 \leq j \leq \varepsilon$.

First, notice that Y is completely determined by X, and that, by combining relations (b) and (d) we achieve:

(e) $x_{ij} = x_{i+1,j-1}$ for $1 \leq i \leq \varepsilon$, $2 \leq j \leq \eta + 1$

The result follows in the same way as in the proof of Lemma 14. $\square$

Lemma 30 (Right singular and infinite blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^* J_u(0)^* = 0 \tag{75}$$

$$Y + X^* B_\varepsilon^* = 0, \tag{76}$$

is

$$\dim_{\mathbb{R}} S^*(L_\varepsilon, N_u) = 2u.$$

Proof. The system(75) and (76) is equivalent to the relations

(a) $x_{2u} = x_{3u} = ... = x_{\varepsilon+1,u} = 0$,
(b) $x_{ij} = -\overline{y}_{j+1,i-1}$ for $2 \leq i \leq \varepsilon + 1,\ 1 \leq j \leq u - 1$,
(c) $x_{ij} = -\overline{y}_{ji}$ for $1 \leq i \leq \varepsilon,\ 1 \leq j \leq u$.

We find Y is completely determined by X, and if we combine (b) and (c) we get:

(d) $x_{ij} = x_{i-1,j+1}$ for $1 \leq i \leq \varepsilon,\ 2 \leq j \leq \eta + 1$

The result follows in the same way as in the proof of Lemma 15. □

Lemma 31 (Right singular and finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\varepsilon X + Y^* = 0 \tag{77}$$

$$J_k(\mu)Y + X^* B_\varepsilon^* = 0, \tag{78}$$

is

$$\dim_{\mathbb{R}} S^*(L_\varepsilon, J_k(\mu) + \lambda I_k) = 2k.$$

Proof. By isolating columns in equation (78) we obtain the following identity for $1 \leq j \leq \varepsilon$:

$$\mu \begin{pmatrix} y_{1j} \\ \vdots \\ y_{k-1,j} \\ y_{kj} \end{pmatrix} + \begin{pmatrix} y_{2j} \\ \vdots \\ y_{kj} \\ 0 \end{pmatrix} + \begin{pmatrix} \overline{x}_{j1} \\ \vdots \\ \overline{x}_{j,k-1} \\ \overline{x}_{jk} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Then we use equation (77) and conjugate to rewrite this as:

$$\overline{\mu} \begin{pmatrix} x_{j+1,1} \\ \vdots \\ x_{j+1,k-1} \\ x_{j+1,k} \end{pmatrix} + \begin{pmatrix} x_{j+1,2} \\ \vdots \\ x_{j+1,k} \\ 0 \end{pmatrix} = \begin{pmatrix} x_{j1} \\ \vdots \\ x_{j,k-1} \\ x_{jk} \end{pmatrix}.$$

This is a recursion relation. It allows us to determine the $(j+1)$th column of X in terms of the jth column or vice versa. Hence, we can let the $x_{\varepsilon+1,1}, \ldots, x_{\varepsilon+1,k}$ be free complex parameters, and then the matrix X will be uniquely determined. If we set

$$c_j = x_{\varepsilon+1,j}\,, \qquad \text{for } j = 1, \ldots, k,$$

then the explicit solution, X, is:

$$x_{ij} = \sum_{l=0}^{\varepsilon+1-i} \overline{\mu}^{\varepsilon+1-i-l} \binom{\varepsilon+1-i}{l} c_{j+l}, \tag{79}$$

Furthermore, by equation (77), Y is completely determined by X. The proof of (79) proceeds in the same way as in the proof of (55). □

Lemma 32 (Left singular and infinite blocks) *The real dimension of the solution space of the system of matrix equations*

$$A_\eta^T X + Y^* J_u(0)^* = 0$$

$$Y + X^* \overline{B}_\eta = 0, \tag{80}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\eta^T, N_u) = 0.$$

Proof. Note that $J_u(0)^* = J_u(0)^T$ and $Y^* = -B_\eta^T X$ by (80). The proof follows as in Lemma 17. □

Lemma 33 (Left singular and finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$\begin{aligned} A_\eta^T X + Y^* &= 0 \\ J_k(\mu) Y + X^* \overline{B}_\eta &= 0, \end{aligned} \tag{81}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(L_\eta^T, J_k(\mu) + \lambda I_k) = 0.$$

Proof. Notice that $\overline{B}_\eta = B_\eta$ and $Y = -X^* \overline{A}_\eta = -X^* A_\eta$ by (81). The proof proceeds as in the proof of Lemma 18. $\square$

Lemma 34 (Infinite and finite blocks) *The real dimension of the solution space of the system of matrix equations*

$$\begin{aligned} X + Y^* &= 0 \\ J_k(\mu) Y + X^* J_u(0)^* &= 0, \end{aligned} \tag{82}$$

is

$$\dim_{\mathbb{R}} \mathcal{S}^*(N_u, J_k(\mu) + \lambda I_k) = \begin{cases} 2\min\{u,k\}, & \text{if } \mu = 0, \\ 0, & \text{if } \mu \neq 0. \end{cases}$$

Proof. Since (82) becomes $X^* = -Y$ by taking the conjugate transpose and $J_u(0)^* = J_u(0)^T$, the result follows as in the proof of Lemma 19. $\square$

6 Corollaries of the main results

The particular cases of equation (1) with $A = \pm B^\star$ were solved in [15]. Also, the particular case $A = B$ was solved in [7] for $\star = T$ and in [8] for $\star = *$. In [3] and [14] the authors provided necessary and sufficient conditions for the operator $X \mapsto AX + X^\star B$ to be invertible. We show in this section how the main results contained in these papers can be derived from our approach for the general equation (1).

6.1 The particular case $A = B$

6.1.1 The transpose case

In [7] the authors solved the particular case of equation (1) with $A = B$, that is

$$AX + X^T A = 0. \tag{83}$$

In that case, the solution was given in terms of the *canonical form for congruence (CFC)* of A [13]. In this subsection we will show how to derive the dimension of the solution space of (83) from the dimension of the solution space of (1) given in Theorem 3 when particularizing to $A = B$. For the sake of completeness, we first recall the CFC of complex square matrices. This form consists of three types of blocks: type 0 blocks are Jordan blocks associated with 0, while type I and type II blocks (denoted by Γ_k and $H_{2k}(\mu)$, respectively), are defined as

$$\Gamma_k = \begin{bmatrix} 0 & & & & & (-1)^{k+1} \\ & & & & \iddots & (-1)^k \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix} \quad (\Gamma_1 = [1]),$$

and

$$H_{2k}(\mu) = \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix} \quad \left(H_2(\mu) = \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} \right).$$

Theorem 5 (Canonical form for congruence) [13, Theorem 1.1] *Each square complex matrix is congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the three types:*

Type 0	$J_k(0)$
Type I	Γ_k
Type II	$H_{2k}(\mu), \quad 0 \neq \mu \neq (-1)^{k+1}$ μ *is determined up to replacement by* μ^{-1}

At first sight, it seems surprising that we are using the strict equivalence transformation and the KCF of the pencil $A + \lambda B^T$ to solve equation (1) instead of the congruence transformation used in [7] to reduce the original equation (83) to the case involving CFC of A. Following this approach, when particularizing to $B = A$, we would obtain the solution of (83) in terms of the KCF of the palindromic pencil $A + \lambda A^T$. This suggests that there is some hidden connection between the CFC of A and the KCF of $A + \lambda A^T$. There is actually a one-to-one correspondence between these two canonical forms and it is based on the non-trivial fact that two matrices A and B are congruent if and only if the associated palindromic pencils $A + \lambda A^T$ and $B + \lambda B^T$ are strictly equivalent [6, Lemma 1]. In order to establish this one-to-one correspondence between the (blocks in the) CFC of A and the (blocks in the) KCF of $A + \lambda A^T$, we have to take into account that the KCF of a palindromic pencil $A + \lambda A^T$ is subject to the following restrictions (see Theorem 1 in [6]):

(i) The left and right minimal indices of $A + \lambda A^T$ coincide, that is, if ε is a right minimal index of $A + \lambda A^T$ then it is also a left minimal index and vice versa.

(ii) Each finite block with odd size associated with the eigenvalue $\mu = 1$ occurs an even number of times.

(iii) Each finite block with even size associated with the eigenvalue $\mu = -1$ occurs an even number of times.

(iv) The finite blocks associated with eigenvalues $\mu \neq \pm 1$ occur in pairs: $J_k(-\mu) + \lambda I_k$, $J_k(-1/\mu) + \lambda I_k$ (here we understand that the blocks associated with 0 are paired up with infinite blocks).

With these restrictions in mind, the one-to-one correspondence between the blocks of the CFC of A and the blocks of the KCF of $A + \lambda A^T$ is the following (see Theorem 4 in [6]):

(i) Each type 0 block with even size, $J_{2k}(0)$, corresponds to a pair of blocks associated with the zero and the infinite eigenvalues, $(J_k(0) + \lambda I_k) \oplus N_k$.

(ii) Each type 0 block with odd size, $J_{2k+1}(0)$, corresponds to a pair of left and right singular blocks, $L_k \oplus L_k^T$.

(iii) Each type I block, Γ_k, corresponds to a finite block associated with the eigenvalue $(-1)^k$, $J_k((-1)^{k+1}) + \lambda I_k$.

(iv) Each type II block, $H_{2k}(\mu)$, corresponds to a pair of finite blocks associated with inverse eigenvalues, $(J_k(\mu) + \lambda I_k) \oplus (J_k(1/\mu) + \lambda I_k)$ $\;(\mu \neq 0, (-1)^{k+1})$.

Now, by direct comparison of the dimension count stated in Theorem 3 and the corresponding result stated in [7, Theorem 2], and taking into account the previous correspondence, it is possible to obtain the main result in [7] (Theorem 2 in that reference) as a particular case of Theorem 3. In the following tables we display the correspondence between the canonical blocks in the KCF of $A + \lambda A^T$ and the canonical blocks in the CFC of A (first column in each table), and we relate the dimension of the solution spaces associated with these blocks (second column of each table). For brevity, we only display the nonzero dimensions and, to avoid repetition, we only write each dimension once, referring to previous appearances when repetitions occur. The dimensions in the case of equation (83) and the CFC of A are stated in terms of the *codimension* of individual blocks or *interaction* between pairs of blocks, as in [7, Theorem 2]. The labels for each case refer to the KCF of $A + \lambda A^T$.

Pair of finite blocks (eigenvalues $\neq 0, \pm 1$)	
Block	Dimension
KCF of $A+\lambda A^T$: $(J_k(\mu)+\lambda I_k)\oplus(J_k(1/\mu)+\lambda I_k)$	$\dim \mathcal{S}\,(J_k(\mu)+\lambda I_k, J_k(1/\mu)+\lambda I_k)=k$ $\dim \mathcal{S}(J_k(\mu)+\lambda I_k, L_\varepsilon)=k$ $\dim \mathcal{S}(J_k(1/\mu)+\lambda I_k, L_\varepsilon)=k$ For $\nu=\mu$: $\dim \mathcal{S}(J_k(\mu)+\lambda I_k, J_\ell(1/\nu)+\lambda I_\ell)=\min\{k,\ell\}$ $\dim \mathcal{S}(J_k(1/\mu)+\lambda I_k, J_\ell(\nu)+\lambda I_\ell)=\min\{k,\ell\}$
CFC of A: $H_{2k}(\mu)$	$\operatorname{codim} H_{2k}(\mu)=k$ $\operatorname{inter}\,(H_{2k}(\mu), J_{2\varepsilon+1}(0))=2k$ For $\nu=\mu$: $\operatorname{inter}\,(H_{2k}(\mu), H_{2\ell}(\nu))=2\min\{k,\ell\}$

Pair of finite block (eigenvalue 0) and infinite block	
Block	Dimension
KCF of $A+\lambda A^T$: $(J_k(0)+\lambda I_k)\oplus N_k$	$\dim \mathcal{S}\,(J_k(0)+\lambda I_k, N_k)=k$ $\dim \mathcal{S}(J_k(0)+\lambda I_k, L_\varepsilon)=k$ $\dim \mathcal{S}(N_k, L_\varepsilon)=k$ $\dim \mathcal{S}(J_k(0)+\lambda I_k, N_\ell)=\min\{k,\ell\}$ $\dim \mathcal{S}(N_k, J_\ell(0)+\lambda I_\ell)=\min\{k,\ell\}$
CFC of A: $J_{2k}(0)$	$\operatorname{codim}\,(J_{2k}(0))=k$ $\operatorname{inter}\,(J_{2k}(0), J_{2\varepsilon+1}(0))=2k$ $\operatorname{inter}\,(J_{2k}(0), J_{2\ell}(0))=2\min\{k,\ell\}$

Pair of finite blocks with odd size (eigenvalue 1)	
Block	Dimension
KCF of $A+\lambda A^T$: $(J_k(-1)+\lambda I_k)\oplus(J_k(-1)+\lambda I_k)$ (k odd)	$2\dim \mathcal{S}\,(J_k(-1)+\lambda I_k)=2\lceil k/2\rceil$ $\dim \mathcal{S}(J_k(-1)+\lambda I_k, J_k(-1)+\lambda I_k)=k$ $2\dim \mathcal{S}(J_k(-1)+\lambda I_k, L_\varepsilon)=2k$ For ℓ odd: $4\dim \mathcal{S}(J_k(-1)+\lambda I_k, J_\ell(-1)+\lambda I_\ell)=4\min\{k,\ell\}$ For ℓ even: $2\dim \mathcal{S}(J_k(-1)+\lambda I_k, J_\ell(-1)+\lambda I_\ell)=2\min\{k,\ell\}$
CFC of A: $H_{2k}(-1)$ (k odd)	$\operatorname{codim} H_{2k}(-1)=k+2\lceil k/2\rceil$ $\operatorname{inter}\,(H_{2k}(-1), J_{2\varepsilon+1}(0))=2k$ For ℓ odd: $\operatorname{inter}\,(H_{2k}(-1), H_{2\ell}(-1))=4\min\{k,\ell\}$ For ℓ even: $\operatorname{inter}\,(H_{2k}(-1), \Gamma_\ell)=2\min\{k,\ell\}$

Pair of finite blocks with even size (eigenvalue -1)	
Block	**Dimension**
KCF of $A+\lambda A^T$: $(J_k(1)+\lambda I_k)\oplus(J_k(1)+\lambda I_k)$ (k even)	$2\dim\mathcal{S}\,(J_k(1)+\lambda I_k)=2k/2$ $\dim\mathcal{S}(J_k(1)+\lambda I_k, J_k(1)+\lambda I_k)=k$ $2\dim\mathcal{S}(J_k(1)+\lambda I_k, L_\varepsilon)=2k$ For ℓ even: $4\dim\mathcal{S}(J_k(1)+\lambda I_k, J_\ell(1)+\lambda I_\ell)=4\min\{k,\ell\}$ For ℓ odd: $2\dim\mathcal{S}(J_k(1)+\lambda I_k, J_\ell(1)+\lambda I_\ell)=2\min\{k,\ell\}$
CFC of A: $H_{2k}(1)$ (k even)	$\operatorname{codim} H_{2k}(1)=k+2k/2$ $\operatorname{inter}\,(H_{2k}(1), J_{2\varepsilon+1}(0))=2k$ For ℓ even: $\operatorname{inter}\,(H_{2k}(1), H_{2\ell}(1))=4\min\{k,\ell\}$ For ℓ odd: $\operatorname{inter}\,(H_{2k}(1), \Gamma_\ell)=2\min\{k,\ell\}$

Single finite block with even size (eigenvalue 1)	
Block	**Dimension**
KCF of $A+\lambda A^T$: $J_k(-1)+\lambda I_k$ (k even)	$\dim\mathcal{S}\,(J_k(-1)+\lambda I_k)=k/2$ $\dim\mathcal{S}(J_k(-1)+\lambda I_k, L_\varepsilon)=k$ For ℓ even: $\dim\mathcal{S}(J_k(-1)+\lambda I_k, J_\ell(-1)+\lambda I_k)=\min\{k,\ell\}$ For ℓ odd: $2\dim\mathcal{S}(J_k(-1)+\lambda I_k, J_\ell(-1)+\lambda I_\ell)=$ stated above
CFC of A: Γ_k (k even)	$\operatorname{codim}\Gamma_k=k/2$ $\operatorname{inter}\,(\Gamma_k, J_{2\varepsilon+1}(0))=k$ For ℓ even: $\operatorname{inter}\,(\Gamma_k, \Gamma_\ell)=\min\{k,\ell\}$ For ℓ odd: $\operatorname{inter}\,(\Gamma_k, H_{2\ell}(-1))=$ stated above

Single finite block with odd size (eigenvalue -1)	
Block	**Dimension**
KCF of $A+\lambda A^T$: $J_k(1)+\lambda I_k$ (k odd)	$\dim\mathcal{S}\,(J_k(1)+\lambda I_k)=\lfloor k/2\rfloor$ $\dim\mathcal{S}(J_k(1)+\lambda I_k, L_\varepsilon)=k$ For ℓ odd: $\dim\mathcal{S}(J_k(1)+\lambda I_k, J_\ell(1)+\lambda I_k)=\min\{k,\ell\}$ For ℓ even: $2\dim\mathcal{S}(J_k(1)+\lambda I_k, J_\ell(1)+\lambda I_\ell)=$ stated above
CFC of A: Γ_k (k odd)	$\operatorname{codim}\Gamma_k=\lfloor k/2\rfloor$ $\operatorname{inter}\,(\Gamma_k, J_{2\varepsilon+1}(0))=k$ For ℓ odd: $\operatorname{inter}\,(\Gamma_k, \Gamma_\ell)=\min\{k,\ell\}$ For ℓ even: $\operatorname{inter}\,(\Gamma_k, H_{2\ell}(1))=$ stated above

Pair of left/right singular blocks	
Block	Dimension
KCF of $A + \lambda A^T$: $L_\varepsilon \oplus L_\varepsilon^T$	$\dim \mathcal{S}\,(L_\varepsilon) = \varepsilon$ $\dim \mathcal{S}(L_\varepsilon, L_\varepsilon^T) = 1$ $\dim \mathcal{S}(L_\varepsilon, L_\eta) + \dim \mathcal{S}(L_\varepsilon, L_\eta^T)$ $+ \dim \mathcal{S}(L_\varepsilon^T, L_\eta) = \begin{cases} \max\{2\varepsilon+1, 2\eta+1\}, & \text{if } \eta \neq \varepsilon \\ 2\varepsilon+2, & \text{if } \eta = \varepsilon \end{cases}$ The remaining cases have been stated above
CFC of A: $J_{2\varepsilon+1}(0)$	$\operatorname{codim} J_{2\varepsilon+1}(0) = \lceil (2\varepsilon+1)/2 \rceil = \varepsilon + 1$ $\operatorname{inter}\,(J_{2\varepsilon+1}(0), J_{2\eta+1}(0)) = \begin{cases} \max\{2\varepsilon+1, 2\eta+1\}, & \text{if } \eta \neq \varepsilon \\ 2\varepsilon+2, & \text{if } \eta = \varepsilon \end{cases}$ The remaining cases have been stated above

As we can see, in all cases the total dimension obtained from both the KCF and the CFC coincide.

6.1.2 The conjugate transpose case

The equation

$$AX + X^*A = 0 \tag{84}$$

has recently been solved in [8]. This section is devoted to deriving the dimension of the solution space of (84) from Theorem 4 when particularized to $A = B$. As shown in [8], the real dimension of the solution space of (84) depends on the *canonical form for* **congruence* (CF*C) of A. For the sake of completeness, we first recall this canonical form.

Theorem 6 (Canonical form for *congruence) [13, Theorem 1.1 (b)] *Each square complex matrix is* **congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the three types:*

Type 0	$J_k(0)$
Type I	$\alpha\Gamma_k, \quad \lvert\alpha\rvert = 1$
Type II	$H_{2k}(\mu), \quad \lvert\mu\rvert > 1$

In order to establish this one-to-one correspondence between the (blocks in the) CF*C of A and the (blocks in the) KCF of $A + \lambda A^*$, we have to take into account that the KCF of a *palindromic pencil $A + \lambda A^*$ is subject to the following restrictions:

(i) The left and right minimal indices of $A + \lambda A^*$ coincide, that is, if ε is a right minimal index of $A + \lambda A^*$ then it is also a left minimal index and vice versa.

(ii) The finite blocks associated with eigenvalues $|\mu| \neq 1$ occur in pairs: $J_k(-\mu) + \lambda I_k$, $J_k(-1/\overline{\mu}) + \lambda I_k$ (here we understand that the blocks associated with 0 are paired up with infinite blocks).

With these restrictions in mind, the relationship between the CF*C of A and the KCF of $A + \lambda A^*$ is given in following theorem.

Theorem 7 *The CF*C of A is related to the KCF of the* **palindromic pencil $A + \lambda A^*$ by the following one-to-one correspondence between canonical blocks:*

*(i) The CF*C of A contains a type 0 block with even size, $J_{2k}(0)$, if and only if the KCF of $A + \lambda A^*$ contains a pair consisting of an infinite block and finite block associated with 0, $(J_k(0) + \lambda I_k) \oplus N_k$.*

*(ii) The CF*C of A contains a type 0 block with odd size, $J_{2k+1}(0)$, if and only if the KCF of $A + \lambda A^*$ contains a pair consisting of a right singular block and a left singular block, $L_k \oplus L_k^T$.*

*(iii) The CF*C of A contains a type 1 block, $\alpha\Gamma_k$ with $|\alpha| = 1$, if and only if the KCF of $A + \lambda A^*$ contains a finite block associated with the eigenvalue $(-1)^k\alpha^2$, $J_k((-1)^{k+1}\alpha^2) + \lambda I_k$.*

*(iv) The CF*C of A contains a type 2 block, $H_{2k}(\mu)$ with $|\mu| > 1$, if and only if the KCF of $A + \lambda A^*$ contains a pair of finite blocks associated with the eigenvalues $-\mu$ and $-1/\overline{\mu}$, $(J_k(\mu) + \lambda I_k) \oplus (J_k(1/\overline{\mu}) + \lambda I_k)$.*

Proof. The proof mimics that of Theorem 4 in [6]. More precisely, for claims (i) and (ii) we simply note that $J_{2k}(0)$ is *congruent to $H_{2k}(0)$ [17, p. 493] and $J_{2k+1}(0)$ is *congruent to $\left[\begin{smallmatrix} 0 & A_k \\ B_k & 0 \end{smallmatrix}\right]$ [17, p. 492]. For claim (iii) we use the identity

$$(\alpha\Gamma_k)^{-*}\left((\alpha\Gamma_k) + \lambda(\alpha\Gamma_k)^*\right) = \begin{bmatrix} (-1)^{k+1}\alpha^2 & 2 & & \blacktriangle \\ & (-1)^{k+1}\alpha^2 & \ddots & \\ & & \ddots & 2 \\ 0 & & & (-1)^{k+1}\alpha^2 \end{bmatrix} + \lambda I_k,$$

where the symbol $\blacktriangle$ denotes entries with no relevance to the argument. Now (iii) follows from the uniqueness of the KCF. Claim (iv) is also an immediate consequence of the uniqueness of the KCF. $\square$

In the following tables we show the correspondence between the quantities in the main result of [8] (Theorem 3.3 in that reference) and the ones obtained from Theorem 4 for the particular case $A = B$. We follow the same conventions as the transpose case in 6.1.1. We will use the notation $\mathfrak{i} := \sqrt{-1}$.

Pair of finite blocks (eigenvalues $-\mu, -1/\overline{\mu}$ with $\lvert\mu\rvert > 1$)	
Block	Dimension
KCF of $A + \lambda A^*$: $(J_k(\mu) + \lambda I_k) \oplus (J_k(1/\overline{\mu}) + \lambda I_k)$	$\dim_{\mathbb{R}} \mathcal{S}^*(J_k(\mu) + \lambda I_k, J_k(1/\overline{\mu}) + \lambda I_k) = 2k$ $\dim_{\mathbb{R}} \mathcal{S}^*(J_k(\mu) + \lambda I_k, L_\varepsilon) = 2k$ $\dim_{\mathbb{R}} \mathcal{S}^*(J_k(1/\overline{\mu}) + \lambda I_k, L_\varepsilon) = 2k$ For $\nu = \mu$: $\dim_{\mathbb{R}} \mathcal{S}^*(J_k(\mu) + \lambda I_k, J_\ell(1/\overline{\nu}) + \lambda I_\ell) = 2\min\{k, \ell\}$ $\dim_{\mathbb{R}} \mathcal{S}^*(J_k(1/\overline{\mu}) + \lambda I_k, J_\ell(\nu) + \lambda I_\ell) = 2\min\{k, \ell\}$
CF*C of A: $H_{2k}(\mu)$	$\operatorname{codim} H_{2k}(\mu) = 2k$ $\operatorname{inter}(H_{2k}(\mu), J_{2\varepsilon+1}(0)) = 4k$ For $\nu = \mu$: $\operatorname{inter}(H_{2k}(\mu), H_{2\ell}(\nu)) = 4\min\{k, \ell\}$

Pair of finite block (eigenvalue 0) and infinite block	
Block	Dimension
KCF of $A + \lambda A^*$: $(J_k(0) + \lambda I_k) \oplus N_k$	$\dim_{\mathbb{R}} \mathcal{S}^*(N_k, L_\varepsilon) = 2k$ $\dim_{\mathbb{R}} \mathcal{S}^*(J_k(0) + \lambda I_k, L_\varepsilon) = 2k$ $\dim_{\mathbb{R}} \mathcal{S}^*(J_k(0) + \lambda I_k, N_k) = 2k$ $\dim_{\mathbb{R}} \mathcal{S}^*(J_k(0) + \lambda I_k, N_\ell) = 2\min\{k, \ell\}$ $\dim_{\mathbb{R}} \mathcal{S}^*(N_k, J_\ell(0) + \lambda I_\ell) = 2\min\{k, \ell\}$
CF*C of A: $J_{2k}(0)$	$\operatorname{codim} J_{2k}(0) = 2k$ $\operatorname{inter}(J_{2k}(0), J_{2\varepsilon+1}(0)) = 4k$ $\operatorname{inter}(J_{2k}(0), J_{2\ell}(0)) = 4\min\{k, \ell\}$

Single finite block (eigenvalue $(-1)^k\alpha^2$ with $\lvert\alpha\rvert=1$)	
Block	Dimension
KCF of $A+\lambda A^*$: $(J_k((-1)^{k+1}\alpha^2)+\lambda I_k)$	$\dim_{\mathbb{R}} \mathcal{S}^*(J_k((-1)^{k+1}\alpha^2)+\lambda I_k)=k$ $\dim_{\mathbb{R}} \mathcal{S}^*(J_k((-1)^{k+1}\alpha^2)+\lambda I_k, L_\varepsilon)=2k$ For k,ℓ with the same parity and $\beta=\pm\alpha$: $\dim_{\mathbb{R}} \mathcal{S}^*(J_k((-1)^{k+1}\alpha^2)+\lambda I_k, J_\ell((-1)^{\ell+1}\beta^2)+\lambda I_\ell)=2\min\{k,\ell\}$ For k,ℓ with different parity and $\beta=\pm\mathrm{i}\alpha$: $\dim_{\mathbb{R}} \mathcal{S}^*(J_k((-1)^{k+1}\alpha^2)+\lambda I_k, J_\ell((-1)^{\ell+1}\beta^2)+\lambda I_\ell)=2\min\{k,\ell\}$
CF*C of A: $\alpha\Gamma_k$	$\operatorname{codim}\alpha\Gamma_k=k$ $\operatorname{inter}(\alpha\Gamma_k, J_{2\varepsilon+1}(0))=2k$ For k,ℓ with the same parity and $\beta=\pm\alpha$: $\operatorname{inter}(\alpha\Gamma_k,\beta\Gamma_\ell)=2\min\{k,\ell\}$ For k,ℓ with different parity and $\beta=\pm\mathrm{i}\alpha$: $\operatorname{inter}(\alpha\Gamma_k,\beta\Gamma_\ell)=2\min\{k,\ell\}$

Pair of right/left singular blocks	
Block	Dimension
KCF of $A+\lambda A^*$: $L_\varepsilon\oplus L_\varepsilon^T$	$\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon)=2\varepsilon$ $\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, L_\varepsilon^T)=2$ $\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, N_u)=2u$ $\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, J_u(0)+\lambda I_u)=2u$ $\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, J_\ell((-1)^{\ell+1}\alpha^2)+\lambda I_\ell)=2\ell$ $\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, J_\ell(\mu)+\lambda I_\ell)+\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, J_\ell(1/\overline{\mu})+\lambda I_\ell)=2\cdot 2\ell=4\ell$ $\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, L_\eta)+\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon, L_\eta^T)$ $+\dim_{\mathbb{R}} \mathcal{S}^*(L_\varepsilon^T, L_\eta)=\begin{cases}2\max\{2\varepsilon+1,2\eta+1\}, & \text{if } \eta\neq\varepsilon,\\ 4\varepsilon+4, & \text{if } \eta=\varepsilon.\end{cases}$
CF*C of A: $J_{2\varepsilon+1}(0)$	$\operatorname{codim} J_{2\varepsilon+1}(0)=2\left\lceil\frac{2\varepsilon+1}{2}\right\rceil=2\varepsilon+2$ $\operatorname{inter}(J_{2\varepsilon+1}(0), J_{2u}(0))=4u$ $\operatorname{inter}(J_{2\varepsilon+1}(0),\alpha\Gamma_\ell)=2\ell$ $\operatorname{inter}(J_{2\varepsilon+1}(0), H_{2\ell}(\mu))=4\ell$ $\operatorname{inter}(J_{2\varepsilon+1}(0), J_{2\eta+1}(0))=\begin{cases}2\max\{2\varepsilon+1,2\eta+1\}, & \text{if } \eta\neq\varepsilon,\\ 4\varepsilon+4, & \text{if } \eta=\varepsilon.\end{cases}$

Notice that the restrictions $\beta=\pm\alpha$ and $\beta=\pm\mathrm{i}\alpha$ in the third table above arise in the following way: in order for $\dim_{\mathbb{R}} \mathcal{S}^*(J_k((-1)^{k+1}\alpha^2)+\lambda I_k, J_\ell((-1)^{\ell+1}\beta^2)+\lambda I_\ell)$ to be nonzero, we need

$$(-1)^{k+1}\alpha^2\cdot(-1)^{\ell+1}\overline{\beta}^2=1.$$

This holds if and only if $\alpha^2=(-1)^{k+\ell}\beta^2$, so that $\beta=\pm\alpha$ if k and ℓ have the same parity while $\beta=\pm\mathrm{i}\alpha$ if k and ℓ have different parity.

6.2 The particular cases $A=\pm B^\star$

The procedure described in Section 2 to solve equation (1) simplifies considerably in the case where $B=\pm A^\star$, because in this case the pencil $A+\lambda B^\star$ becomes $(1\pm\lambda)A$, so its KCF is just $(1\pm\lambda)$ times the canonical form for strict equivalence of $A\in\mathbb{C}^{m\times n}$. This canonical form reduces to

$$PAQ=I_r\oplus 0_{(m-r)\times(n-r)}, \tag{85}$$

where $r = \operatorname{rank} A$ and $P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}$ are nonsingular matrices. As a consequence, the original equation $AX \pm X^\star A^\star = 0$ can be reduced, by strict equivalence, to

$$(I_r \oplus 0_{(m-r)\times(n-r)})Y \pm Y^\star(I_r \oplus 0_{(n-r)\times(m-r)}) = 0,$$

with $Y = Q^{-1}XP^\star$. If we partition the unknown $Y = \left[\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22}\end{smallmatrix}\right]$ conformally with the partition of the coefficient matrices, that is $Y_{11} \in \mathbb{C}^{r\times r}, Y_{12} \in \mathbb{C}^{r\times(m-r)}, Y_{21} \in \mathbb{C}^{(n-r)\times r}$, and $Y_{22} \in \mathbb{C}^{(n-r)\times(m-r)}$, then we see that Y_{21} and Y_{22} can be chosen freely, and we get the following system of equations

$$\begin{aligned} &Y_{11} \pm Y_{11}^\star = 0 \\ &Y_{12} = 0\,, \end{aligned}$$

In the case $B = A^\star$ the first equation is $Y_{11} + Y_{11}^\star = 0$, so Y_{11} is an arbitrary $r \times r$ skew-symmetric matrix if $\star = T$ or an arbitrary skew-hermitian matrix if $\star = *$, whereas in the case $B = -A^\star$ we get $Y_{11} - Y_{11}^\star = 0$, so that Y_{11} is an arbitrary $r\times r$ symmetric matrix if $\star = T$ or an arbitrary hermitian matrix when $\star = *$. As a consequence, we achieve the following results.

Theorem 8 *Let $A \in \mathbb{C}^{m\times n}$ be a matrix with rank r. Then the dimension of the solution space of the equation $AX + X^TA^T = 0$ is equal to $m(n-r) + \frac{r(r-1)}{2}$, and the dimension of the solution space of $AX - X^TA^T = 0$ is equal to $m(n-r) + \frac{r(r+1)}{2}$.*

Theorem 9 *Let $A \in \mathbb{C}^{m\times n}$ be a matrix with rank r. Then the real dimension of the solution space of the equation $AX \pm X^*A^* = 0$ is equal to $2m(n-r) + r^2$.*

Notice that the previous arguments also provide a simple procedure to obtain the explicit solution of both equations in terms of the change matrices P and Q. More precisely, the solution of $AX + X^\star A^\star = 0$ (respectively, $AX - X^\star A^\star = 0$) is

$$X = Q \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{bmatrix} P^{-\star},$$

where P and Q are as in (85), Y_{11} is an arbitrary skew-symmetric (respectively, symmetric) matrix if $\star = T$ and skew-hermitian (resp. hermitian) if $\star = *$, and $Y_{21} \in \mathbb{C}^{(n-r)\times r}$ and $Y_{22} \in \mathbb{C}^{(n-r)\times(m-r)}$ are arbitrary.

The approach followed here for $AX \pm X^\star A^\star = 0$ is similar to the one in [15] and, actually, theorems 8 and 9 were already stated in that paper [15, theorems 3, 5 and 6]. The equations $AX \pm X^TA^T = 0$ were solved also in [2] using projectors and generalized inverses of A.

6.3 The operator $X \mapsto AX + X^\star B$

First, notice that, in order for this operator to be well-defined, A and $B^\star$ must have the same dimension. But, if $A, B^\star \in \mathbb{C}^{m\times n}$, then the matrix X is of size $n \times m$, whereas the matrix $AX + X^\star B$ is $m\times m$. Then, we must have $m = n$ in order for the dimension of the domain to match the dimension of the range. As a consequence, the pencil $A + \lambda B^\star$ must be square.

Notice, also, that the operator is linear in $\mathbb{C}$ only if $\star = T$, whereas if $\star = *$ then it is linear over $\mathbb{R}$. We consider separately these cases and state, in theorems 10 and 11, necessary and sufficient conditions for these operators to be invertible.

Theorem 10 (Lemma 8 in [14]) *Let $A, B \in \mathbb{C}^{m\times m}$ be given. Then the linear operator in $\mathbb{C}^{m\times m}$*

$$X \mapsto AX + X^TB$$

is invertible if and only if the following conditions hold:

(1) *The pencil $A + \lambda B^T$ is regular,*

(2) *if $\mu_j \neq -1$ is an eigenvalue of $A + \lambda B^T$, then $1/\mu_j$ is not an eigenvalue of $A + \lambda B^T$, and*

(3) *if -1 is an eigenvalue of $A+\lambda B^T$, then it has algebraic multiplicity 1.*

Note that, in particular, 1 *can not be an eigenvalue of $A+\lambda B^T$.*

Proof. Let us first prove that conditions (1–3) are necessary. The linear operator $X \mapsto AX + X^T B$ is injective if and only if the only solution to

$$AX + X^T B = 0$$

is the null matrix.

Now, looking at the dimension count in Theorem 3, in order for the dimension of the solution space of $AX + X^T B = 0$ to be zero, the KCF of $A+\lambda B^T$ must fulfill the following restrictions:

(a) It cannot contain right singular blocks.

(b) $\mu = 1$ cannot be an eigenvalue of $A+\lambda B^T$.

(c) The finite blocks associated with the eigenvalue -1, if present, must be of size 1×1.

(d) If $\mu \neq \pm 1$ is an eigenvalue of $A+\lambda B^T$ then $1/\mu$ cannot be an eigenvalue (here we understand that if $\mu = 0$ then $1/\mu$ is the infinite eigenvalue).

Notice that, since $A+\lambda B^T$ is square, the number of right minimal indices equals the number of left minimal indices so, by (a), left singular blocks can not appear either. Hence, conditions (1–3) are necessary in order for the operator $X \mapsto AX + X^T B$ to be invertible. Conversely, notice that if (a–d) hold, then the dimension of the solution space of (1) is zero, so they are also sufficient. □

Theorem 11 (Lemma 8 in [14]) *Let $A, B \in \mathbb{C}^{m\times m}$ be given. Then the $\mathbb{R}$-linear operator in $\mathbb{C}^{m\times m}$*

$$X \mapsto AX + X^* B$$

is invertible if and only if the following conditions hold:

(1) *The pencil $A+\lambda B^*$ is regular,*

(2) *if μ_j is an eigenvalue of $A+\lambda B^*$, then $1/\overline{\mu}_j$ is not an eigenvalue of $A+\lambda B^*$.*

Note that, in particular, $A+\lambda B^*$ can not contain eigenvalues on the unit circle.

Proof. The proof is similar to that of Theorem 10, but it is based on the dimension count stated in Theorem 4. We simply remark that, in order for this dimension to be zero, the KCF of $A+\lambda B^*$ can not contain neither right singular blocks nor eigenvalues μ_i, μ_j with $\mu_i \overline{\mu}_j = 1$. □

7 Conclusions and future work

We have presented a method to find the general solution of the homogeneous Sylvester equations for *congruence $AX + X^\star B = 0$ over $\mathbb{C}$ in terms of the Kronecker canonical form of the matrix pencil $A + \lambda B^\star$. In this way, this paper completes the work presented by the authors in references [7] and [8] for the equations $AX + X^\star A = 0$, where the canonical form for congruence of A was used as the main tool, and also the results in references [3, 9, 14, 20] for the nonhomogeneous equation $AX + X^\star B = C$. Several problems still remain open in this area. Among them we cite: the development of a numerical algorithm to find the general solution of $AX + X^\star B = 0$, that is, a basis of the solution space, as well as numerical algorithms for determining when $AX + X^\star B = C$ is consistent and for computing its general solution in this case. These problems will probably require the use of stable algorithms to compute the GUPTRI form of the matrix pencil $A + \lambda B^\star$ as those presented in [4, 5, 19].

Acknowledgments. This work has been partially supported by NSF grant DMS-0852065 (Fernando De Terán, Nathan Guillery, Daniel Montealegre and Nicolás Reyes) and by the

Ministerio de Ciencia e Innovación of Spain through grant MTM-2009-09281 (Fernando De Terán and Froilán M. Dopico). The work has been done while the first and the last three authors were participating at the UCSB REU program during the summer of 2011. These authors wish to thank the UCSB Mathematics Department for their hospitality and the director of the REU program, Prof. María I. Bueno, for making this meeting possible. They are also grateful to Rachel Spratt for some help during the writing of the manuscript.

A Appendix: The solution of $XJ_k((-1)^{k+1}) = -X^T$

This appendix is devoted to proving that the general solution of $XJ_k((-1)^{k+1}) = -X^T$ depends on $\lfloor k/2 \rfloor$ free parameters, a result that was used in the proof of Lemma 9. The proof of this result is just a slight modification of the one given in [7, Appendix A] for the equation $XJ_k((-1)^k) = -X^T$, though we include the proof here for the sake of completeness. We will present the case k odd in detail, which is similar to the case k even in [7, Appendix A], while we only state the main results for k even. We want to point out that our proof also provides an algorithm to obtain the explicit solution of the equation (Lemma 36 for k odd and Lemma 38 for k even).

A.1 Solution for k odd

We present first necessary and sufficient conditions in terms of entries for a matrix X to be a solution of $XJ_k(1) = -X^T$.

Lemma 35 *Let $k > 0$ be an odd number. A matrix $X = [x_{ij}]_{i,j=1}^k \in \mathbb{C}^{k\times k}$ is a solution of $XJ_k(1) = -X^T$ if and only if X satisfies the following three conditions*

$$\begin{aligned} x_{ij} &= 0 && \text{if } 2 \le i+j \le k+1, && (86)\\ x_{ij} + x_{ji} &= -x_{i,j-1} && \text{if } k+2 \le i+j \le 2k \text{ and } 2 \le j \le i \le k, && (87)\\ x_{i,j-1} &= x_{j,i-1} && \text{if } k+2 \le i+j-1 \le 2k-1 \text{ and } 2 \le j < i \le k. && (88) \end{aligned}$$

Note, in particular, that every solution of $XJ_k(1) = -X^T$ is lower anti-triangular by (86).

Proof. The arguments in the proof of Lemma 11 in [7], which are valid for both k even and odd, show that X is a solution of $XJ_k(1) = -X^T$ if and only if the following conditions on X hold

$$\begin{aligned} x_{ij} &= 0 && \text{if } 2 \le i+j \le k, && (89)\\ x_{k1} + x_{1k} &= 0, && && (90)\\ x_{ij} + x_{ji} &= -x_{i,j-1} && \text{if } k+1 \le i+j \le 2k \text{ and } 2 \le j \le i \le k, && (91)\\ x_{i,j-1} &= x_{j,i-1} && \text{if } k+1 \le i+j-1 \le 2k-1 \text{ and } 2 \le j < i \le k. && (92) \end{aligned}$$

In the present case (k odd) we can further show that $x_{ij} = 0$ for $i+j = k+1$. To see this, first notice that $x_{\frac{k+1}{2},\frac{k+1}{2}} = 0$ by (91) with $i = j = (k+1)/2$ and the fact that $x_{\frac{k+1}{2},\frac{k-1}{2}} = 0$ derived from (89). Now, from (92) we have

$$x_{\frac{k+3}{2},\frac{k-1}{2}} = x_{\frac{k+1}{2},\frac{k+1}{2}} = 0,$$

and, from (91) and using also (89) for $i+j = k$, we get

$$x_{\frac{k-1}{2},\frac{k+3}{2}} = -x_{\frac{k+3}{2},\frac{k-1}{2}} = 0.$$

Proceeding recursively in this way, applying both (92) and (91), and also (89) for $i+j = k$, we obtain:

$$0 = x_{\frac{k+5}{2},\frac{k-3}{2}} = x_{\frac{k-3}{2},\frac{k+5}{2}} = \ldots = x_{k-1,2} = x_{2,k-1} = x_{k1}.$$

Finally, using (90) we also get $x_{1k} = 0$.

As a consequence, the system of equations (89–92) is equivalent to (86–88), and the result follows. $\square$

Observe that (86) amounts to $(k^2+k)/2$ equations on the entries of X, (87) amounts to $2 \cdot (1+2+\cdots+(k-1)/2) = (k^2-1)/4$ equations, and finally (88) amounts to $2 \cdot (1+2+\cdots+(k-3)/2) = (k^2-4k+3)/4$ equations. This makes a total number of $k^2-(k-1)/2$ equations in (86–88). Therefore, the general solution of $XJ_k(1) = -X^T$ depends on *at least* $(k-1)/2$ *free parameters* (it might depend on more than $(k-1)/2$ free parameters if equations (86–88) are linearly dependent). We will show in Lemma 36 that the general solution of $XJ_k(1) = -X^T$ depends precisely on $(k-1)/2$ free parameters, because if equations (86–88) are arranged in an appropriate order, then it is evident that certain $(k-1)/2$ entries of X determine uniquely the remaining ones. This appropriate order consists in ordering equations (87–88) by anti-diagonals in such a way that every anti-diagonal $\mathcal{L}_s = \{x_{ij} : i+j = s\}$ is obtained from $\mathcal{L}_{s-1}$.

Lemma 36 *If $k > 0$ is an odd number, then the general solution X of $XJ_k(1) = -X^T$ depends on $(k-1)/2$ free variables. In particular, the entries*

$$x_{\frac{k+3}{2},\frac{k+1}{2}},\ x_{\frac{k+5}{2},\frac{k+3}{2}},\ x_{\frac{k+7}{2},\frac{k+5}{2}},\ldots,\ x_{k,k-1},$$

can be taken as free variables and then the remaining entries of X are uniquely determined by the following algorithm:

set $x_{ij} = 0$ *if* $2 \le i+j \le k$
for $s = k+2 : 2k$
    if s *is odd*
        $h = \frac{s+1}{2}$
        $x_{h,h-1}$ *is a free variable*
        $x_{h-1,h} = -x_{h,h-1} - x_{h,h-2}$
    else
        $h = \frac{s}{2}$
        $x_{hh} = -(x_{h,h-1})/2$
    endif
    for $i = h+1 : k$
        $x_{i,s-i} = x_{s-(i-1),i-1}$
        $x_{s-i,i} = -x_{i,s-i} - x_{i,s-i-1}$
    endfor
endfor

For simplicity, in this algorithm we define $x_{k0} \equiv 0$ and it is understood that the inner loop "for $i = h+1 : k$" is not performed if $h+1 > k$.

Proof. Note that the algorithm arranges all the equations in (86–88) in an order that allows each entry to be computed from entries that are already known. We only remark that $x_{h,h} = -(x_{h,h-1})/2$ is (87) with $i = j = h$, that $x_{h-1,h} = -x_{h,h-1} - x_{h,h-2}$ and $x_{s-i,i} = -x_{i,s-i} - x_{i,s-i-1}$ are special cases of (87) with appropriate indices, and that $x_{i,s-i} = x_{s-(i-1),i-1}$ is a particular case of (88). Since we have already established that the general solution of $XJ_k(1) = -X^T$ depends on at least $(k-1)/2$ free parameters, and all the equations in (86–88) are satisfied in the algorithm in a unique way for any selection of arbitrary values of the $(k-1)/2$ entries $x_{h,h-1}$, for $h = (k+3)/2, (k+5)/2, (k+7)/2, \ldots, k$, then the number of free variables is precisely $(k-1)/2$. $\square$

A.2 Solution for k even

We state without proofs counterparts of Lemmas 35 and 36. The proofs are similar to those of Lemmas 35 and 36 with the corresponding variations.

Lemma 37 *Let $k > 0$ be an even number. A matrix $X = [x_{ij}]_{i,j=1}^{k} \in \mathbb{C}^{k\times k}$ is a solution of $XJ_k(-1) = -X^T$ if and only if X satisfies the following four conditions*

$$\begin{aligned} x_{ij} &= 0 && \text{if } \ 2 \leq i+j \leq k+1, \\ x_{i,i-1} &= 0 && \text{if } i = \tfrac{k}{2}+2, \tfrac{k}{2}+3, \ldots, k, \\ x_{ij} - x_{ji} &= x_{i,j-1} && \text{if } \ k+2 \leq i+j \leq 2k-1 \ \text{ and } 2 \leq j < i \leq k, \\ x_{i,j-1} &= -x_{j,i-1} && \text{if } \ k+2 \leq i+j-1 \leq 2k-1 \ \text{ and } 2 \leq j < i \leq k. \end{aligned}$$

Note, in particular, that every solution of $XJ_k(-1) = -X^T$ is lower anti-triangular.

A similar count to the one for Lemma 35 gives that the total amount of equations in the statement of Lemma 37 is

$$\left(\frac{k(k+1)}{2}\right) + \left(\frac{k}{2} - 1\right) + \left(2\cdot(1+2+\cdots+\frac{k}{2}-1)\right) + \left(2\cdot(1+2+\cdots+\frac{k}{2}-2) + \frac{k}{2} - 1\right)$$
$$= \frac{k^2+k}{2} + \frac{k}{2} - 1 + \frac{k^2}{4} - \frac{k}{2} + \frac{k^2}{4} - k + 1 = k^2 - \frac{k}{2}.$$

Then the solution of $XJ_k(-1) = -X^T$ depends on at most $k/2$ free variables for k even. The following lemma shows that it actually depends on exactly $k/2$ variables, and provides these variables.

Lemma 38 *Let $k > 0$ be an even number, then the general solution X of $XJ_k(-1) = -X^T$ depends on $k/2$ free variables. In particular, the entries*

$$x_{\frac{k+2}{2},\frac{k+2}{2}},\ x_{\frac{k+4}{2},\frac{k+4}{2}},\ x_{\frac{k+6}{2},\frac{k+6}{2}}, \ldots, x_{k,k},$$

can be taken as free variables and then the remaining entries of X are uniquely determined by the following algorithm:

> *set* $x_{ij} = 0$ *if* $2 \leq i+j \leq k$
> *for* $s = k+2 : 2k$
>   *if s is odd*
> $h = \frac{s+1}{2}$
> $x_{h,h-1} = 0$
> $x_{h-1,h} = x_{h,h-1} - x_{h,h-2}$
>   *else*
> $h = \frac{s}{2}$
> x_{hh} *is a free variable*
>   *endif*
>   *for* $i = h+1 : k$
> $x_{i,s-i} = -x_{s-(i-1),i-1}$
> $x_{s-i,i} = x_{i,s-i} - x_{i,s-i-1}$
>   *endfor*
> *endfor*

For simplicity, in this algorithm we define $x_{k0} \equiv 0$ and it is understood that the inner loop "for $i = h+1 : k$" is not performed if $h+1 > k$.

B Appendix: The solution to $J_k(\mu)X = -X^*$ with $|\mu| = 1$

This appendix is devoted to proving that the general solution of $J_k(\mu)X = -X^*$ for $|\mu| = 1$ depends on k free parameters, a result that was used in the proof of Lemma 24. This is proved using techniques similar to those used in Appendix A; we present the case k even in detail, while we only state the main results for k odd.

B.1 Solution for k even

Lemma 39 *Let $k > 0$ be an even number. A matrix $X = [x_{ij}]_{i,j=1}^k \in \mathbb{C}^{k\times k}$ is a solution of $J_k(\mu)X = -X^*$ for $|\mu| = 1$ if and only if X satisfies the following three conditions:*

$$x_{ij} = 0 \qquad \text{if} \quad k+1 < i+j \leq 2k, \tag{93}$$

$$\mu x_{ij} + \overline{x}_{ji} = -x_{i+1,j} \qquad \text{if} \quad 2 \leq i+j \leq k+1 \quad \text{for} \quad 1 \leq i \leq j \leq k, \tag{94}$$

$$x_{i+1,j} = \mu \overline{x}_{j+1,i} \qquad \text{if} \quad 3 \leq i+j \leq k \quad \text{for} \quad 1 \leq i < j \leq k-1. \tag{95}$$

Note, in particular, that every solution of $J_k(\mu)X = -X^*$ for $|\mu| = 1$ is upper anti-triangular by (93).

Proof. When written entry-wise, (64) is equivalent to the system of equations

$$\mu x_{ij} + x_{i+1,j} + \overline{x}_{ji} = 0 \quad \text{for } 1 \leq i \leq k-1,\ 1 \leq j \leq k, \tag{96}$$

$$\mu x_{kj} + \overline{x}_{jk} = 0 \quad \text{for } 1 \leq j \leq k. \tag{97}$$

We first show that (96) and (97) imply (93). This is proven by induction on the nth row and column starting from the kth pair and moving downwards. Multiplying (97) by $\overline{\mu}$ and taking conjugates, we get that (97) is equivalent to $\mu x_{jk} + \overline{x}_{kj} = 0$. Now, by setting $j = k$ in (96) and adding these equations, it follows that $x_{i+1,k} = 0$ for $i = 1, \ldots, k-1$, i.e., $x_{ik} = 0$ for $i = 2, \ldots, k$, which also gives $x_{ki} = 0$ for $i = 2, \ldots, k$ by (97). This proves the base case. Suppose the claim holds for some $n \leq k$, that is, $x_{in} = x_{ni} = 0$ when $k+1 < i+n \leq 2k$. If we set $\ell = k-n+2$, then this means $x_{in} = x_{ni} = 0$ for $i = \ell, \ldots, k$. Now, by setting $i = n-1$ and $j = n-1$ separately in (96), we obtain

$$\mu x_{n-1,j} + x_{nj} + \overline{x}_{j,n-1} = 0 \quad \text{for } 1 \leq j \leq k, \tag{98}$$

$$\text{and} \quad \mu x_{i,n-1} + x_{i+1,n-1} + \overline{x}_{n-1,i} = 0 \quad \text{for } 1 \leq i \leq k-1. \tag{99}$$

By the inductive hypothesis, (98) implies $\mu x_{n-1,j} + \overline{x}_{j,n-1} = 0$ for $j = \ell, \ldots, k$, which is in turn equivalent to $\mu x_{j,n-1} + \overline{x}_{n-1,j} = 0$ (multiply by $\overline{\mu}$ and take conjugates). Subtracting this from (99), we have $x_{i+1,n-1} = 0$ for $i = \ell, \ldots, k-1$, i.e., $x_{i,n-1} = 0$ for $i = \ell+1, \ldots, k$. This implies $x_{n-1,i} = 0$ for $i = \ell+1, \ldots, k$ as well by (98) and the inductive hypothesis.
Therefore (96) and (97) imply $x_{ij} = 0$ for $k+1 < i+j \leq 2k$, and an elementary argument allows us to conclude that (96) and (97) are in fact equivalent to

$$x_{ij} = 0 \qquad \text{if} \quad k+1 < i+j \leq 2k, \tag{100}$$

$$\mu x_{ij} + \overline{x}_{ji} = -x_{i+1,j} \qquad \text{if} \quad 2 \leq i+j \leq k+1 \quad \text{for} \quad 1 \leq i \leq k-1,\ 1 \leq j \leq k. \tag{101}$$

Now, consider any pair i, j with $i \neq j$. By the two equations from (101) involving this pair we have $x_{i+1,j} = \mu \overline{x}_{j+1,i}$. Thus (100) and (101) imply (93–95).
On the other hand, to see that (93–95) are actually equivalent to (100) and (101) (and consequently (96) and (97)), notice that we only need to show that (93–95) imply (101) when $j < i$. Since $j < i$, by (94) and (95) we have

$$\mu x_{ji} + \overline{x}_{ij} = -x_{j+1,i} \quad \text{and} \quad x_{j+1,i} = \mu \overline{x}_{i+1,j}.$$

Combining these equalities, then multiplying by $\overline{\mu}$ and taking conjugates, it follows that (101) holds. This completes the proof. □

Observe that (93) amounts to $(k^2-k)/2$ equations on the entries of X, (94) amounts to $(k^2/4)+(k/2)$ equations, and (95) amounts to $(k^2/4)-(k/2)$. This gives a total number of $k^2-(k/2)$ equations in (93–95). Therefore, the general solution of $J_k(\mu)X=-X^*$ for $|\mu|=1$ depends on *at least k free real parameters* (it may depend on more than k free parameters if equations (93–95) are linearly dependent). We will show in Lemma 40 that the general solution of $J_k(\mu)X=-X^*$ for $|\mu|=1$ depends on precisely k free real parameters, since if equations (93–95) are arranged in an appropriate order, then it is evident that $(k/2)$ particular entries of X uniquely determine the remaining ones. Each of these $(k/2)$ entries gives 2 free real variables. At each step we take a diagonal entry x_{ii} as a free variable, starting from $x_{\frac{k}{2}\frac{k}{2}}$, and we set the entries in the ith row and the ith column above the main anti-diagonal as linear combinations of $x_{ii},\dots,x_{\frac{k}{2}\frac{k}{2}}$. We state Lemma 40 without proof since the proof is nearly identical to that of Lemma 12 in [7].

Lemma 40 *If $k>0$ is an even number, then the general solution of $J_k(\mu)X=-X^*$ for $|\mu|=1$ depends on k free real variables. In particular, the complex and imaginary parts of the entries*

$$x_{11},x_{22},\dots,x_{\frac{k}{2}\frac{k}{2}},$$

can be taken as free variables and then the remaining entries of X are uniquely determined by the following algorithm:

set $x_{ij}=0$ *for* $k+2\le i+j\le 2k$
for $s=1:\frac{k}{2}$
    $h=\frac{k}{2}-s+1$
    $x_{hh}=a_{hh}+\mathrm{i}b_{hh}$ *is a free variable*
    $x_{h+1,h}=-\left(\mu x_{hh}+\overline{x}_{hh}\right)$
    $x_{h,h+1}=-\overline{\mu}\left(\overline{x}_{h+1,h}+x_{h+1,h+1}\right)$
    for $i=2:2s-1$
        $x_{h+i,h}=\mu\overline{x}_{h+1,h+i-1}$
        $x_{h,h+i}=-\overline{\mu}\left(\overline{x}_{h+i,h}+x_{h+1,h+i}\right)$
    endfor
endfor

B.2 Solution for k odd

The following two lemmas are the counterparts of lemmas 39 and 40. The proofs are nearly identical to those of lemmas 39 and 40 with the corresponding variations.

Lemma 41 *Let $k>0$ be an odd number. A matrix $X=[x_{ij}]_{i,j=1}^{k}\in\mathbb{C}^{k\times k}$ is a solution of $J_k(\mu)X=-X^*$ for $|\mu|=1$ if and only if X satisfies the following five conditions:*

$$\begin{array}{l}
x_{ij}=0 \qquad \text{if} \quad k+1<i+j\le 2k,\\
\mu x_{ij}+\overline{x_{ji}}=-x_{i+1,j} \qquad \text{if} \quad 2\le i+j\le k \quad \text{for} \quad 1\le i\le j\le k,\\
x_{i+1,j}=\mu\overline{x_{j+1,i}} \qquad \text{if} \quad 3\le i+j\le k \quad \text{for} \quad 1\le i<j\le k-1,\\
\mu x_{ij}+\overline{x_{ji}}=0 \qquad \text{if} \quad i+j=k+1,\ i<j,\\
\mu x_{\frac{k+1}{2},\frac{k+1}{2}}+\overline{x_{\frac{k+1}{2},\frac{k+1}{2}}}=0.
\end{array}$$

Note, in particular, that every solution of $J_k(\mu)X=-X^*$ for $|\mu|=1$ is upper anti-triangular.

Lemma 42 *If $k>0$ is an odd number, then the general solution of $J_k(\mu)X=-X^*$ for $|\mu|=1$ depends on k free real variables. In particular, the complex and imaginary parts of the entries*

$$x_{11},x_{22},\dots,x_{\frac{k-1}{2},\frac{k-1}{2}},$$

can be taken as free variables while the entry $x_{\frac{k+1}{2},\frac{k+1}{2}}$ *contributes one free real variable, so that the remaining entries of* X *are uniquely determined by the following algorithm:*

set $x_{ij} = 0$ *for* $k+2 \leq i+j \leq 2k$
If $\mu \neq 1$,
  $a_{\frac{k+1}{2},\frac{k+1}{2}}$ *is a free variable (real),*
  $b_{\frac{k+1}{2},\frac{k+1}{2}} = -\frac{\mathfrak{i}(\mu+1)}{\mu-1} \cdot a_{\frac{k+1}{2},\frac{k+1}{2}}$
else
  $b_{\frac{k+1}{2},\frac{k+1}{2}}$ *is a free variable*
  $a_{\frac{k+1}{2},\frac{k+1}{2}} = 0$
endif
$x_{\frac{k+1}{2},\frac{k+1}{2}} = a_{\frac{k+1}{2},\frac{k+1}{2}} + \mathfrak{i} b_{\frac{k+1}{2},\frac{k+1}{2}}$
for $s = 1 : \frac{k-1}{2}$
  $h = \frac{k-1}{2} - s + 1$
  $x_{hh} = a_{hh} + \mathfrak{i} b_{hh}$ *is a free variable*
  $x_{h+1,h} = -(\mu x_{hh} + \overline{x}_{hh})$
  $x_{h,h+1} = -\overline{\mu}(\overline{x}_{h+1,h} + x_{h+1,h+1})$
  for $i = 2 : 2s$
    $x_{h+i,h} = \mu \overline{x}_{h+1,h+i-1}$
    $x_{h,h+i} = -\overline{\mu}(\overline{x}_{h+i,h} + x_{h+1,h+i})$
  endfor
endfor

Proof. The only difference with the case "k even" is in lines 2-9, in the definition of $x_{\frac{k+1}{2},\frac{k+1}{2}}$. If we set $x_{\frac{k+1}{2},\frac{k+1}{2}} = a_{\frac{k+1}{2},\frac{k+1}{2}} + \mathfrak{i} b_{\frac{k+1}{2},\frac{k+1}{2}}$, then we know

$$(\mu+1) a_{\frac{k+1}{2},\frac{k+1}{2}} + \mathfrak{i}(\mu-1) b_{\frac{k+1}{2},\frac{k+1}{2}} = 0.$$

Notice that, because $|\mu| = 1$, the quotient $\frac{\mathfrak{i}(\mu+1)}{\mu-1}$ in line 4 gives a real number for $\mu \neq 1$. $\square$

References

[1] C. S. Ballantine, A note on the matrix equation $H = AP + PA^*$, Linear Algebra Appl., 2:37–47, 1969.

[2] H. W. Braden, The equations $A^T X \pm X^T A = B$, SIAM J. Matrix Anal. Appl., 20:295–302, 1998.

[3] R. Byers, D. Kressner, Structured condition numbers for invariant subspaces, SIAM J. Matrix Anal. Appl., 28(2):326–347, 2006.

[4] J. W. Demmel, B. Kågström, The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: Robust software with error bounds and applications. Part I: Theory and Algorithms, ACM Trans. Math. Software, 19:160–174, 1993.

[5] J. W. Demmel, B. Kågström, The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: Robust software with error bounds and applications. Part II: Software and Applications, ACM Trans. Math. Software, 19:175–201, 1993.

[6] F. De Terán, Canonical forms for congruence of matrices: a tribute to H. W. Turnbull and A. C. Aitken, Actas del II congreso de la red ALAMA, Valencia, June 2–4, 2010. Preprint available at:
`http://gauss.uc3m.es/web/personal_web/fteran/papers/2010-1.pdf`

[7] F. De Terán, F. M. Dopico, The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits, Linear Algebra Appl., 434:44–67, 2011.

[8] F. De Terán, F. M. Dopico, The equation $XA + AX^* = 0$ and the dimension of *congruence orbits, Electron. J. Linear Algebra, 22:448–465, 2011.

[9] F. De Terán, F. M. Dopico, Consistency and efficient solution of the Sylvester equation for $\star-$congruence: $AX + X^\star B = C$, submitted to Electron. J. Linear Algebra. Preprint available at:

`http://gauss.uc3m.es/web/personal_web/fdopico/papers/ela2011.pdf`

[10] D. S. Djordević, Explicit solution of the operator equation $A^*X+X^*A = B$, J. Comput. Appl. Math., 200:701–704, 2007.

[11] F. R. Gantmacher, The Theory of Matrices, Chelsea, New York, 1959.

[12] R. A. Horn, C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.

[13] R. A. Horn, V. V. Sergeichuk, Canonical forms for complex matrix congruence and *congruence, Linear Algebra Appl., 416:1010–1032, 2006.

[14] D. Kressner, C. Schröder, D. S. Watkins, Implicit QR algorithms for palindromic and even eigenvalue problems, Numerical Algorithms, 51(2):209–238, 2009.

[15] P. Lancaster, P. Rozsa, On the matrix equation $AX + X^*A^* = C$, SIAM J. Alg. Disc. Meth., 4:432–436, 1983.

[16] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Structured polynomial eigenvalue problems: Good vibrations from good linearizations, SIAM J. Matrix Anal. Appl., 28:1029–1051, 2006.

[17] V. V. Sergeichuk, Classification problems for system of forms and linear mappings, Math USSR, Izvestiya 31(3):481–501, 1988.

[18] O. Taussky, H. Wielandt, On the matrix function $AX + X'A'$, Arch. Rational Mech. Anal., 9:93-96, 1962.

[19] P. Van Dooren, The computation of Kronecker's canonical form of a singular pencil, Linear Algebra Appl., 27:103–140, 1979.

[20] H. K. Wimmer, Roth's theorems for matrix equations with symmetry constraints, Linear Algebra Appl., 199:357–362, 1994.